Harmonische Analysis — Blatt 5

Die Mathematik ist die Wissenschaft von dem, was sich von selbst versteht.
(Heinrich Heine; 1797 – 1856)

Problems

5.1. For $f \in \mathrm{L}^1(\mathbb{R})$ the Fourier transform is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} x\xi} f(x) \mathrm{d}x.$$

(a) The function $\widehat{f}$ defined in this way is uniformly continuous on the real line $\mathbb{R}$.

(b) Assume that $k_\Lambda \in \mathrm{L}^1(\mathbb{R})$ satisfies

$$k_\Lambda(x) \geq 0, \qquad \int_{\mathbb{R}} k_\Lambda(x)\mathrm{d}x = 1, \qquad \lim_{\Lambda\to\infty} \int_{|x|>\delta} k_\Lambda(x)\mathrm{d}x = 0$$

for any $\delta > 0$. Prove that for any $f \in \mathrm{L}^1(\mathbb{R})$

$$\lim_{\Lambda\to\infty} \|f - k_\Lambda * f\|_1 = 0.$$

(c) Prove that for any $f \in \mathrm{L}^1(\mathbb{R})$

$$f(x) = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \left(1 - \frac{|\xi|}{\Lambda}\right) \mathrm{e}^{2\pi \mathrm{i} x\xi} \widehat{f}(\xi) \mathrm{d}\xi$$

with convergence in $\mathrm{L}^1(\mathbb{R})$.

Topics as preparation

5.2. Wavelet transforms provide information about functions from $\mathrm{L}^2(\mathbb{R})$ localised in position and scale. For an admissible function $\varphi \in \mathrm{L}^2(\mathbb{R})$ the associated wavelet transform can be defined as

$$\mathscr{W}_\varphi f(u,v) = \mathrm{e}^{-u/2} \int_{\mathbb{R}} f(x) \varphi\left(\mathrm{e}^{-u} x - v\right) \mathrm{d}x$$

and assigns a function $\mathscr{W}_\varphi f$ on the affine group $\mathbf{A} = \{x \mapsto \mathrm{e}^{-u}x - v : u, v \in \mathbb{R}\} \simeq \mathbb{R} \ltimes \mathbb{R}$. It has some remarkable properties:

(a) Let H denote any of the Hilbert spaces $\mathrm{L}^2(\mathbb{R})$ or $\mathcal{H}^2(\mathbb{C}_\pm) = \{f \in \mathrm{L}^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset \mathbb{R}_\pm\}$. The representation $\mathscr{U} : \mathbf{A} \to \mathcal{L}(H)$ defined as

$$\mathscr{U}(u,v) f(x) = \mathrm{e}^{-u/2} f(\mathrm{e}^{-u}x - v)$$

is unitary.

(b) For $f, g \in \mathrm{L}^2(\mathbb{R})$ with g satisfying the admissibility condition

$$\int_{\mathbb{R}} \frac{|\widehat{g}(\xi)|^2}{|\xi|} \mathrm{d}\xi < \infty$$

the function

$$(u,v) \mapsto \big(f, \mathscr{U}(u,v) g\big)$$

belongs to $\mathrm{L}^2(\mathbf{A}, \mathrm{d}u\mathrm{d}v)$.

(c) $\mathscr{U}$ is irreducible on $\mathcal{H}^2(\mathbb{C}_+)$ and $\mathcal{H}^2(\mathbb{C}_-)$. It is cyclic on $\mathrm{L}^2(\mathbb{R})$.

(d) For admissible $\varphi \in \mathcal{H}^2(\mathbb{C}_+)$ the wavelet transform

$$\mathscr{W}_\varphi f(u,v) = \big(f, \mathscr{U}(u,v)\varphi\big)$$

is inverted by

$$f = \int_{\mathbf{A}} \mathscr{W}_\varphi f(u,v)\,\mathscr{U}(u,v)\varphi \mathrm{d}u\mathrm{d}v$$

(understood in an appropriate weak sense).

<u>Source:</u> The results are based on parts of the overview paper

C. E. Heil and D. F. Walnut *Continuous and Discrete Wavelet Transforms*, SIAM Review Vol. 31, No. 4 (1989) pp. 628–666.