Harmonische Analysis — Blatt 3

Müßiggang ist der Feind der Seele.
(Benedikt von Nursia)

Topics as preparation
Topics covered in the following problem classes are listed here. Each participant should choose one and prepare it for presentation at a agreed problem class later in the term. The level of the topics is different and the list is incomplete as yet, but volunteers are already welcome.
The description of the topics is kept as short as possible. The order is not final.

3.1. The Gelfand theory was only developed for unital commutative C^*-algebras. We already showed that any C^*-algebra without unit element can be embedded into a C^*-algebra with unit element in an essentially unique way. Use this to develop a Gelfand theory for C^*-algebras without unit. What are the major differences?

Keywords: Gelfand transform, multiplicative functionals, one-point compactifications

3.2. Summability properties of Fourier and Laplace series.

To any continuous and periodic function $f \in \mathrm{C}(\mathbb{R}/\mathbb{Z})$ we can associate a formal Fourier series

$$\sum_{k\in\mathbb{Z}} \alpha_k \mathrm{e}^{2\pi\mathrm{i}kt} \qquad \text{with} \qquad \alpha_k = \int_0^1 f(t)\mathrm{e}^{-2\pi\mathrm{i}kt}\mathrm{d}t.$$

In general, this series does not converge in the Banach space $\mathrm{C}(\mathbb{R}/\mathbb{Z})$. The aim of this topic is to discuss summability properties of these series and similar series in terms of spherical functions on spheres $\mathbb{S}^n$.

Keywords: Cesaro summability, Abel summability, positive kernels

3.3. Construction of Haar measures for some matrix groups.

In the lectures we will show that all locally compact groups are endowed with a unique left-invariant measure, the so-called Haar measure. The aim of this topic is to construct some Haar measures explicitly.

Keywords: matrix Lie groups, Radon–Nikodym theorem, functional equations

3.4. Continuous wavelet transforms.

Wavelet transforms provide information about functions from $\mathrm{L}^2(\mathbb{R})$ (or more general $\mathrm{L}^2(\mathbb{R}^n)$) localised in position and scale. For an admissible function $\varphi \in \mathrm{L}^2(\mathbb{R})$ the associated wavelet transform is given by

$$\mathscr{W}_\varphi f(a,b) = \frac{1}{\sqrt{a}} \int f(x)\varphi\left(\frac{x-b}{a}\right)\mathrm{d}x$$

the aim of this topic is to discuss mapping properties of this transform, conditions of admissibility and inversion formulas.

Keywords: partial isometries, coherent state transforms

3.5. p-adic Fourier transforms.

The field $\mathbb{Q}_p$ of p-adic numbers is the completion of rational numbers with respect to a different metric. With respect to addition $\mathbb{Q}_p$ forms a locally compact Abelian group and therefore a Fourier transform can be defined for square integrable functions on $\mathbb{Q}_p$. The aim of the topic is to explicitly compute this Fourier transform and its inversion formula.

Keywords: group characters, Pontrjagin duality, number theory

3.6. Irreducible representations of the affine group of $\mathbb{R}$.

The group of affine-linear functions $x \mapsto ax + b$ with $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ is a locally compact non-abelian group. The aim of this topic is to characterise all irreducible unitary representations of this group.

Keywords: matrix Lie groups, Radon–Nikodym theorem, functional equations

3.7. Spectral properties of periodic Schrödinger operators.

Operators on $\mathrm{L}^2(\mathbb{R}^n)$ which are invariant under a discrete co-compact lattice $\Lambda \subset \mathbb{R}^n$ of translations decompose by Fourier–Bloch transform into direct integrals of simpler operators. The aim of this topic is to use this abstract result to characterise the spectrum of particular Schrödinger operators.

Keywords: Floquet–Bloch decomposition, spectral bands, differential operators

3.8. Induced representations.

Unitary representations of a group can (obviously) be restricted to unitary representations of (closed) subgroups. More interesting is the construction of representations of a group starting from representations of a subgroup. This induction process is important to generate all irreducible representations of a given group. The aim of this topic is to discuss the construction of representations by induction for some examples.

Keywords: matrix Lie groups, Radon–Nikodym theorem, functional equations

3.9. Discretisation of continuous transforms: sampling and frames

The aim of this topic is to discuss for some particular examples how continuous transforms give rise to discrete transforms and the notion of frames for Banach- and Hilbert spaces. Frames generalise the notion of a basis by removing uniqueness requirements.

Keywords: Riesz bases, Riesz frames, Beurling density