Harmonische Analysis — Blatt 2

Tu as voulu de l'algèbre, et tu en auras jusqu'au menton!
(Jules Verne; 1828-1905)

Problems

2.1. Let $\mathcal{A}$ be the set of matrices

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \ : \ a, b \in \mathbb{C} \right\} \subset \mathbb{C}^{2\times 2}.$$

Show that $\mathcal{A}$ is a unital Banach algebra and compute its Gelfand transform.

2.2. Let $\mathcal{B} \subset \mathcal{L}(H)$ be a closed *-subalgebra with $I \in \mathcal{B}$.

(a) Assume that for $A \in \mathcal{B}$ the resolvent set $\rho(A) = \{\lambda \in \mathbb{C} \ : \ (\lambda - A)^{-1} \in \mathcal{L}(H)\}$ is path connected. Show that for any $\lambda \in \rho(A)$ the resolvent satisfies $(\lambda - A)^{-1} \in \mathcal{B}$.

(b) Let $A \in \mathcal{B}$ be arbitrary. Show that A is invertible in $\mathcal{B}$ if and only if A is invertible in $\mathcal{L}(H)$. (Hint: Consider the selfadjoint element A^*A and use **(a)**.)

(c) Conclude $\sigma_{\mathcal{B}}(A) = \sigma(A)$ for any $A \in \mathcal{B}$.

2.3. Let $\mathcal{A}$ be a unital commutative Banach *-algebra, H a Hilbert space and $\varphi \in \operatorname{Hom}_*(\mathcal{A}, \mathcal{L}(H))$. Assume further that $\varphi(1) = I$.

(a) Show, that the norm closure of the image $\mathcal{B} = \operatorname{clos} \varphi(\mathcal{A}) \subset \mathcal{L}(H)$ is a commutative C^*-algebra.

(b) Show that $\varphi^*(\Phi) = \Phi \circ \varphi$ for $\Phi \in \sigma(\mathcal{B})$ defines an injective continuous map $\varphi^* : \sigma(\mathcal{B}) \to \sigma(\mathcal{A})$.

(c) Let $\boldsymbol{\mu}$ be the spectral measureof $\mathcal{B}$ and denote by $\boldsymbol{\nu} = \varphi^*(\boldsymbol{\mu})$ the pushforward measure under φ^*. Show that for all $x \in \mathcal{A}$

$$\varphi(x) = \int_{\sigma(\mathcal{A})} \widehat{x}(\zeta) \mathrm{d}\boldsymbol{\nu}(\zeta).$$

Topics as preparation

2.4. The Gelfand theory was only developed for unital commutative C^*-algebras. We already showed that any C^*-algebra without unit element can be embedded into a C^*-algebra with unit element in an essentially unique way. Use this to develop a Gelfand theory for C^*-algebras without unit. What are the major differences?

Source: Section 1.3 of G.B. Folland: *A Course on Abstract Harmonic Analysis*