Harmonische Analysis — Blatt 10

Gleichungen sind wichtiger für mich, weil die Politik für die Gegenwart ist, aber eine Gleichung etwas für die Ewigkeit.

(Albert Einstein; 1879–1955)

Problems

10.1. Let $\delta \in (-1,1)$ be fixed. Then we consider for continuous functions $f : \mathbb{S}^n \to \mathbb{C}$ the integral transform

$$\mathfrak{F}_\delta f(x) = c_\delta \int_{\{y \mid x \cdot y = \delta\}} f(y) dy,$$

where integration is carried out with respect to the induced surface measure on $\{y \mid x \cdot y = \delta\}$ and $c_\delta$ is chosen in such a way that $\mathfrak{F}_\delta 1 = 1$.

(a) Show that $\mathfrak{F}_\delta : \mathcal{C}(\mathbb{S}^n) \to \mathcal{C}(\mathbb{S}^n)$ is continuous.
(b) Is it possible to choose $\delta$ in such a way that $\mathfrak{F}_\delta$ becomes injective?
(c) Show that the operator extends by continuity to $\mathfrak{F}_\delta : \mathcal{L}^2(\mathbb{S}^n) \to \mathcal{L}^2(\mathbb{S}^n)$ and that the resulting operator is self-adjoint and compact.

10.2. Let $\mathbb{H}_n$ denote the $n$-dimensional (symmetric) Heisenberg group,

$$\mathbb{H}_n = \{(x, \xi, \tau) : x, \xi \in \mathbb{R}^n, \tau \in \mathbb{R}\}, \quad (x, \xi, \tau) \boxplus (y, \eta, \sigma) = (x + y, \xi + \eta, \tau + \sigma + \frac{1}{2}(x \cdot \eta - y \cdot \xi)).$$

(a) Show that the $2n + 1$ dimensional Lebesgue measure is left and right invariant on $\mathbb{H}_n$.
(b) For $f, g \in \mathcal{C}_c(\mathbb{H}_n)$ we define the Heisenberg convolution

$$f \ast g(X) = \int_{\mathbb{H}_n} f(X \boxplus Y) g(Y) dY,$$

where $\boxplus Y$ is the inverse element of $Y$, $Y \boxplus (\boxplus Y) = (\boxplus Y) \boxplus Y = 0$. Prove the inequalities

$$\|f \ast g\|_1 \leq \|f\|_1\|g\|_1, \quad \|f \ast g\|_\infty \leq \|f\|_2\|g\|_2, \quad \|f \ast g\|_\infty \leq \|f\|_1\|g\|_\infty.$$

Topics as preparation

10.3. Let $G$ be a compact group and $H \subset G$ be a closed subgroup. Then to each representation $\pi : G \to \mathcal{L}(V)$, $V$ Hilbert space, the restriction $\text{Res}_H^G \pi = \pi|_H$ to the subgroup $H$ gives a representation of $H$.

Let now $\pi : H \to \mathcal{L}(V)$ be a representation of the subgroup $H$. Then it is possible to construct an induced representation $\text{Ind}_H^G \pi$ of the group $G$ in the Hilbert space $W = \text{Ind}_H^G V$.

Lemma. Let $\pi_1, \pi_2, \pi$ be representations of the subgroup $H$ of $G$. Then

(a) $\pi_1 \sim \pi_2$ implies $\text{Ind}_H^G \pi_1 \sim \text{Ind}_H^G \pi_2$.
(b) $\text{Ind}_H^G (\pi_1 \oplus \pi_2) = (\text{Ind}_H^G \pi_1) \oplus (\text{Ind}_H^G \pi_2)$.
(c) $\text{Ind}_H^G \pi$ is irreducible if and only if $\pi$ is irreducible.

Theorem (Frobenius-Reciprocity). If $[\xi] \in \tilde{G}$ and $[\eta] \in \tilde{H}$. Then

$$\text{mult}([\xi], \text{Ind}_H^G \eta) = \text{mult}([\eta], \text{Res}_H^G \xi).$$

Source: M. Ruzhansky, V. Turunen Pseudo-differential operators and symmetries, Chapter 7.9