Harmonische Analysis — Blatt 10

Gleichungen sind wichtiger für mich, weil die Politik für die Gegenwart ist,
aber eine Gleichung etwas für die Ewigkeit.
(Albert Einstein; 1879–1955)

Problems

10.1. Let $\delta \in (-1,1)$ be fixed. Then we consider for continuous functions $f : \mathbb{S}^n \to \mathbb{C}$ the integral transform

$$\mathfrak{F}_\delta f(x) = c_\delta \int_{\{y\,:\,x\cdot y=\delta\}} f(y)\mathrm{d}y,$$

where integration is carried out with respect to the induced surface measure on $\{y \mid x \cdot y = \delta\}$ and c_δ is chosen in such a way that $\mathfrak{F}_\delta 1 = 1$.

(a) Show that $\mathfrak{F}_\delta : \mathrm{C}(\mathbb{S}^n) \to \mathrm{C}(\mathbb{S}^n)$ is continous.

(b) Is it possible to choose δ in such a way that $\mathfrak{F}_\delta$ becomes injective?

(c) Show that the operator extends by continuity to $\mathfrak{F}_\delta : \mathrm{L}^2(\mathbb{S}^n) \to \mathrm{L}^2(\mathbb{S}^n)$ and that the resulting operator is self-adjoint and compact.

10.2. Let $\mathbb{H}_n$ denote the n-dimensional (symmetric) Heisenberg group,

$$\mathbb{H}_n = \{(x,\xi,\tau)\ :\ x,\xi \in \mathbb{R}^n,\ \ \tau \in \mathbb{R}\}, \qquad (x,\xi,\tau) \boxplus (y,\eta,\sigma) = (x+y, \xi+\eta, \tau+\sigma+\tfrac{1}{2}(x\cdot\eta - y\cdot\xi)).$$

(a) Show that the $2n+1$ dimensional Lebesgue measure is left and right invariant on $\mathbb{H}_n$.

(b) For $f, g \in \mathrm{C}_c(\mathbb{H}_n)$ we define the Heisenberg convolution

$$f \star g(X) = \int_{\mathbb{H}_n} f(X \boxplus Y) g(\boxminus Y)\mathrm{d}Y,$$

where $\boxminus Y$ is the inverse element of Y, $Y \boxplus (\boxminus Y) = (\boxminus Y) \boxplus Y = 0$. Prove the inequalities

$$\|f \star g\|_1 \le \|f\|_1 \|g\|_1, \qquad \|f \star g\|_\infty \le \|f\|_2 \|g\|_2, \qquad \|f \star g\|_\infty \le \|f\|_1 \|g\|_\infty.$$

Topics as preparation

10.3. Let G be a compact group and $H \subset G$ be a closed subgroup. Then to each representation $\boldsymbol{\pi} : G \to \mathcal{L}(V)$, V Hilbert space, the restriction $\mathrm{Res}^G_H \boldsymbol{\pi} = \boldsymbol{\pi}|_H$ to the subgroup H gives a representation of H.

Let now $\boldsymbol{\pi} : H \to \mathcal{L}(V)$ be a representation of the subgroup H. Then it is possible to construct an induced representation $\mathrm{Ind}^G_H \pi$ of the group G in the Hilbert space $W = \mathrm{Ind}^G_\pi V$.

Lemma. *Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}$ be representations of the subgroup H of G. Then*

(a) $\boldsymbol{\pi}_1 \sim \boldsymbol{\pi}_2$ *implies* $\mathrm{Ind}^G_H \boldsymbol{\pi}_1 \sim \mathrm{Ind}^G_H \boldsymbol{\pi}_2$.

(b) $\mathrm{Ind}^G_H(\boldsymbol{\pi}_1 \oplus \boldsymbol{\pi}_2) = (\mathrm{Ind}^G_H \boldsymbol{\pi}_1) \oplus (\mathrm{Ind}^G_H \boldsymbol{\pi}_2)$.

(c) $\mathrm{Ind}^G_H \boldsymbol{\pi}$ *is irreducible if and only if* $\boldsymbol{\pi}$ *is irreducible.*

Theorem (Frobenius-Reciprocity). *If $[\xi] \in \widehat{G}$ and $[\eta] \in \widehat{H}$. Then*

$$\mathrm{mult}([\xi], \mathrm{Ind}^G_H \eta) = \mathrm{mult}([\eta], \mathrm{Res}^G_H \xi).$$

Source: M. Ruzhansky, V. Turunen *Pseudo-differential operators and symmetries*, Chapter 7.9