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Harmonische Analysis — Blatt 1

Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.
(Richard Dedekind; 1831-1916)

Problems

1.1. Let H be a Hilbert space and denote by $\mathcal{L}(H)$ the space of all bounded linear operators on H and by $\mathcal{K}(H) \subset \mathcal{L}(H)$ the subspace of compact operators. Show that

- **(a)** $\mathcal{L}(H)$ is a C^*-algebra;
- **(b)** $\mathcal{K}(H)$ is a closed, twosided *-ideal in the C^*-algebra $\mathcal{L}(H)$;
- **(c)** $\mathcal{K}(H)$ is the smallest non-trivial closed twosided ideal of $\mathcal{L}(H)$.

1.2. We consider the space

$$\mathcal{H}^\infty(\mathbb{D}) = \left\{ f \in \mathfrak{A}(\mathbb{D}) \ : \ \sup_{\zeta\in\mathbb{D}} |f(\zeta)| < \infty \right\}$$

of bounded analytic functions on the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} \ : \ |\zeta| < 1\}$. Show that

- **(a)** it forms a Banach algebra if endowed with pointwise multiplication and supremum norm;
- **(b)** it becomes a Banach *-algebra with involution $f^*(\zeta) := \overline{f(\overline{\zeta})}$;
- **(c)** this Banach *-algebra is not C^*.

1.3. Let $\mathcal{A}$ be a non-unital Banach *-algebra and let $\mathcal{A} \times \mathbb{C}$ be the product vector space endowed with the multiplication

$$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

and involution

$$(x, \alpha)^* = (x^*, \overline{\alpha}).$$

- **(a)** Denote $\|(x, \alpha)\| = \|x\| + |\alpha|$. Show that $\mathcal{A} \times \mathbb{C}$ becomes a unital Banach *-algebra.
- **(b)** Assume now that $\mathcal{A}$ is a C^*-algebra. Show that the norm

$$\|(x, \alpha)\| = \sup_{y\in\mathcal{A},\, \|y\|\le 1} \|xy + \alpha y\|$$

makes $\mathcal{A} \times \mathbb{C}$ into a unital C^*-algebra and the canonical embedding $\mathcal{A} \to \mathcal{A} \times \mathbb{C}$ into an isometry.

Topics as preparation

A part of the problem classes will be used to present topics not covered by the course or by previous lectures, which are useful (or even necessary) for us. The topics will be presented by one of the participants based on original literature or text book material.

1.4. Let V be a locally convex topological vector space and $K \subset V$ an arbitrary set. A boundary point $x \in \partial K$ is called *extremal point*, if we can **not** find two points $y, z \in K \cup \partial K$ with $x = \frac{1}{2}(y + z)$. The set of all extremal points of K will be denoted as $\partial_{\text{ext}} K$.

All nonempty compact subsets of V possess extremal points. This follows from

Theorem (Krein–Milman)**.** *Let $K \subseteq V$ be compact. Then $K \subseteq \overline{\operatorname{conv}}\, \partial_{\text{ext}} K$.*

One application of the theorem of Krein–Milman is the theorem Stone–Weierstrass. It characterises dense sub-algebras of $C(X)$ for a compact Hausdorff space X.

Theorem (Stone–Weierstrass)**.** *Let X be compact and Hausdorff. Let further $\mathcal{A} \subset \mathrm{C}(X)$ be a unital $*$-subalgebra, such that the elements of $\mathcal{A}$ separate points in X. Then $\mathcal{A}$ is dense in $\mathrm{C}(X)$.*

<u>Source:</u> Section 5.6 of Sh. Kantorovitz: *Introduction to Modern Analysis*

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