# Hardy Type Inequalities and Virtual Bound States for Semi-Bounded Operators

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# Plan of the Talk

0. Introduction

1. The Abstract Setting

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## 0.1.

Consider the Schrödinger operators

$$H(\alpha) = -\Delta - \alpha V \qquad \text{on} \qquad L_2(\mathbb{R}^d),$$

with a real-valued potential $V$ coupled by the positive constant $\alpha > 0$. If

$$V(x) \longrightarrow 0 \qquad \text{as} \qquad |x| \to 0.$$

in a suitable sense, then $\sigma_{\text{ess}}(H(\alpha)) = [0, \infty)$ and the negative spectrum is discrete:

Define the counting function

$$N(\alpha) = \operatorname{tr} \chi_-(H(\alpha)),$$

where

$$\chi_-(x) = \begin{cases} 0 & \text{for } x \geq 0 \\ 1 & \text{for } x < 0 \end{cases}.$$

Then

$$N(\alpha) \le C\alpha^{d/2} \int V^{d/2} dx$$

and hence $N(\alpha) = 0$ as $\alpha \to +0$.

Assume $d = 1$ or $d = 2$, $V \ge 0$, $V \not\equiv 0$.

Then

$$N(\alpha) \ge 1 \quad \text{for all} \quad \alpha > 0$$

and

$$N(\alpha) = 1 \quad \text{as} \quad \alpha \to +0.$$

We call this negative eigenvalue a *virtual bound state*.

### 0.2.

We have $N(\alpha) = 0$, if and only if

$$h(\alpha)[u] = \int |\nabla u|^2 dx - \alpha \int V|u|^2 dx \ge 0$$

holds for all $u \in C_0^\infty(\mathbb{R}^d)$;
or equivalently, iff the Hardy type inequality

$$\int V|u|^2 dx \le C \int |\nabla u|^2 dx, \qquad u \in C_0^\infty(\mathbb{R}^d),$$

holds with $C = \alpha^{-1}$.

Hence $\lim_{\alpha \to +0} N(\alpha) = 0$, if and only if the previous bound holds for some finite $C = C(V)$.

For $d \geq 3$ the classical Hardy inequality holds:

$$\int \frac{|u|^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \int |\nabla u|^2 dx, \qquad u \in C_0^\infty(\mathbb{R}^d)$$

For $d = 1$ or $d = 2$, $V \geq 0$ and $V \not\equiv 0$, the bound

$$\int V|u|^2 dx \leq C(V) \int |\nabla u|^2 dx, \qquad u \in C_0^\infty(\mathbb{R}^d),$$

*fails* for arbitrary $V$ and $C(V)$.

Indeed, for $d = 1$ fix some function $u \in C_0^\infty(\mathbb{R})$, for which

$$0 \leq u \leq 1, \qquad u(x) = 1 \quad \text{for} \quad |x| \leq 1.$$

For $u_n(x) := u(xn^{-1})$ we find

$$\int V|u_n|^2 dx \to \int V dx > 0 \qquad \text{as} \qquad n \to \infty,$$

$$\int \left|\frac{du_n}{dx}\right|^2 dx = \frac{1}{n} \int \left|\frac{du}{dx}\right|^2 dx \to 0 \qquad \text{as} \qquad n \to \infty.$$

The completion of $C_0^\infty(\mathbb{R}^d)$, $d = 1, 2$, with respect to the Dirichlet metric $\int |\nabla u|^2 dx$ *cannot be realized as a function space in the usual way*.

We observe that

Existence of a virtual bound state
$\lim_{\alpha \to +0} N(\alpha) > 0$
$\Longleftrightarrow$
Hardy's inequality $\int V|u|^2 dx \le C \int |\nabla u|^2 dx$ fails
$\Longleftrightarrow$
The topology induced by the form $\int |\nabla u|^2 dx$
is *not* compatible with the topology on $W_{2,1}^{\mathrm{loc}}$
$\Longleftrightarrow$
$(-\Delta + \lambda)^{-1/2} V (-\Delta + \lambda)^{-1/2}$
does not converge to a compact operator as $\lambda \to +0$

## 0.5.

Indefinite perturbations in the case of virtual bound states.
Assume $d = 1$ or $d = 2$, $V \not\equiv 0$,

$$(1 + |x|)V(x) \in L_1(\mathbb{R}) \quad \text{if} \quad d = 1,$$
$$(1 + |x|)^{\epsilon} V(x) \in L_1(\mathbb{R}^2) \quad \text{if} \quad d = 2 \quad \text{for some} \quad \epsilon > 0.$$

Then [Simon]

$$\int V dx < 0 \Longleftrightarrow \lim_{\alpha \to +0} N(\alpha) = 0,$$
$$\int V dx \ge 0 \Longleftrightarrow \lim_{\alpha \to +0} N(\alpha) = 1.$$

In particular, $\int V dx < 0$ implies

$$\int V|u|^2 dx \le C(V) \int |\nabla u|^2 dx, \qquad u \in C_0^\infty(\mathbb{R}^d).$$

1.1.

Consider

$$A = A^* \geq 0 \qquad \text{with} \qquad \min \sigma(A) = 0.$$

Let

$$V = V_+ - V_-, \qquad V_+ \geq 0,\ V_- \geq 0,$$

where $V_\pm$ are $(A + \mathbb{I})$-bounded. The respective quadratic forms are $a$, $v$, $v_\pm$.

Set

$$\begin{aligned} A(\alpha) &= A - \alpha(V_+ - V_-) = A - \alpha V, \\ \tilde{A}(\alpha) &= A - \alpha(V_+ + V_-) = A - \alpha \tilde{V}, \end{aligned}$$

and

$$\begin{aligned} N(\alpha) &= \operatorname{tr} \chi_-(A(\alpha)), \qquad & \tilde{N}(\alpha) &= \operatorname{tr} \chi_-(\tilde{A}(\alpha)), \\ N &= \lim_{\alpha \to +0} N(\alpha), \qquad & \tilde{N} &= \lim_{\alpha \to +0} \tilde{N}(\alpha). \end{aligned}$$

**Condition 1.** $\tilde{N}(\alpha) < \infty$ for some $\alpha > 0$.

**Condition 2.** $\tilde{N} \geq 1$.

Consider the special case, when 0 is an isolated eigenvalue of finite multiplicities of $A = A(0)$ with the eigenspace $\Lambda = \ker A$.

Then analytic perturbation theory is applicable.

Let $\{\mu_k\}_{k=1}^n$ be the non-decreasing sequence of the eigenvalues of $v|_\Lambda$, $n = \dim \Lambda$.

$$\underbrace{\mu_1, \cdots, \mu_{n_-}}_{n_- \text{ neg eigv}}, \underbrace{\mu_{n_-+1}, \cdots, \mu_{n_-+n_0}}_{n_0 \text{ zero eigv}}, \underbrace{\mu_{n_-+n_0+1}, \cdots, \mu_n}_{n_+ \text{ pos eigv}}$$

Then the eigenvalue 0 splits as follows:

$$\lambda_k(\alpha) = 0 - \alpha\mu_k + \mathcal{O}_k(\alpha^2) \quad \text{as} \quad \alpha \to 0.$$

We perturb the lower edge of the spectrum of $A$, hence

$$\mathcal{O}_k(\alpha^2) \le 0, \qquad k = 1, \cdots, n.$$

The indices $k$ with $\mu_k = 0$ and $\mathcal{O}_k(\alpha^2) = 0$ correspond to $\ker A \cap \ker V$. Put

$$\begin{aligned} n_{0,0} &= \dim(\ker A \cap \ker V) \\ &= \dim\{\phi \in \Lambda | v[\phi, u] = 0 \quad \forall u\}. \end{aligned}$$

Then

$$N = n_+ + n_0 - n_{0,0}.$$

In general we do *not* put forward any conditions on the spectral structure of $A$ at the point 0. Analytic perturbation theory is *not* applicable.

As out main result we adapt the formula

$$N = n_+ + n_0 - n_{0,0}$$

to the general abstract case.

## 2. Applications I

**2.1.**

Let $q_0$, $q_1$ be continuous functions on $\mathbb{R}^d$,

$$0 < q_0(x) \le q_1(x) < \infty \qquad \text{for all} \qquad x \in \mathbb{R}^d.$$

For $l \in \mathbb{N}$ the symbol $\nabla^l$ denotes the $\kappa = \binom{d-1+l}{l}$-vector of all partial derivatives

$$\frac{\partial^l}{\partial x_1^{l_1} \cdots \partial x_d^{l_d}}, \qquad l = l_1 + \cdots + l_d.$$

Let the $\kappa \times \kappa$ matrix function $a(x)$ satisfy

$$q_0(x)\mathbb{I} \le a(x) \le q_1(x)\mathbb{I} \qquad \text{for all} \qquad x \in \mathbb{R}^d.$$

Put

$$a[u] = \int \left\langle a(x)\nabla^l u, \nabla^l u \right\rangle dx, \qquad u \in C_0^\infty(\mathbb{R}^d).$$

The function $\phi \in W_{2,l}^{\text{loc}}(\mathbb{R}^d)$ is said to be a *limit element of* $a$, iff there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$, such that

$$u_n \to \phi \quad \text{in} \quad W_{2,l}^{\text{loc}} \quad \text{and} \quad a[u_n] \to 0 \quad \text{as} \quad n \to \infty.$$

Let the *limit space* $\Lambda(a)$ be the set of all limit elements. This is a linear subspace of $\Omega_{d,l-1}$, the set of all polynomials on $\mathbb{R}^d$ of degree up to $l-1$.

**2.3.**

Let $V(x) \geq 0$, $V \not\equiv 0$ and $A = (-1)^l(\nabla^l)^T a(x)\nabla^l$,

$$A(\alpha) = (-1)^l(\nabla^l)^T a(x)\nabla^l - \alpha V(x).$$

**Theorem 1.** *If* $N(\alpha) < \infty$ *for some* $\alpha > 0$*, then*

$$\lim_{\alpha\to+0} N(\alpha) = \dim \Lambda(a),$$

$$\int V|p|^2 dx < \infty \quad \textit{for all} \quad p \in \Lambda(a).$$

**Corollary.** *The inequality*

$$\int_{|x|\leq 1} |u|^2 dx \leq C(a,d,l) \int \left\langle a(x)\nabla^l u, \nabla^l u\right\rangle dx$$

*holds on all* $u \in C_0^\infty(\mathbb{R}^d)$*, iff* $\dim \Lambda(a) = 0$*.*

Let $v = v_+ - v_-$, $v_\pm \geq 0$, be some quadratic form defined on $C_0^\infty(\mathbb{R}^d)$.

**Theorem 2.** *Assume that* $v \geq 0$ *and that the topology induced by*

$$a[\cdot] + \int_{|x|\leq 1} |\cdot|^2 dx + v[\cdot]$$

*is compatible with the topology on* $W_{2,l}^{loc}$*. Then*

$$N(\alpha) = \infty \quad \textit{for all} \quad \alpha > 0$$

*or*

$$N = \lim_{\alpha \to +0} N(\alpha) \leq \dim \Lambda(a).$$

**Theorem 3.** *Assume that the topology induced by*

$$a[\cdot] + \int_{|x|\leq 1} |u|^2 dx + v_+[\cdot] + v_-[\cdot]$$

*is compatible with the topology on* $W_{2,l}^{loc}$ *and that* $\tilde{N}(\alpha) < \infty$ *for some* $\alpha > 0$*. Then*

$$N = \lim_{\alpha \to +0} N(\alpha) = n_+ + n_0 - n_{0,0},$$

*where* $n_-$, $n_0$, $n_+$, $n_{0,0}$ *are defined as above for* $v|_{\Lambda(a)}$*.*

## 3.1.

Assume that

$$c_0(1+|x|)^r \le a(x) \le c_1(1+|x|)^r, \qquad x \in \mathbb{R}^d.$$

Put $m = \left[l - \frac{d+r}{2}\right]$. Then

$$\begin{array}{rll} 2l-d<r & \text{implies} & \Lambda(a)=\{0\}\ , \\ 2-d<r\le 2l-d & \text{implies} & \Lambda(a)=\Omega_{d,m}\ , \\ r\le 2-d & \text{implies} & \Lambda(a)=\Omega_{d,l-1}\ . \end{array}$$

## 3.2.

A typical example is

$$A(\alpha)u = u'''' - \alpha\left\{V_0 + \frac{1}{i}\left(\frac{d}{dx}V_1 + V_1\frac{d}{dx}\right) - \frac{d}{dx}V_2\frac{d}{dx}\right\}.$$

The functions $V_0$, $V_1$, $V_2$ are real, bounded and of compact support.

We have $r=0$, $d=1$, $l=2$, $\Lambda(a)=\Omega_{1,1}$ and

$$v|_{\Lambda(a)} \sim \begin{pmatrix} \int V_0 dx & \int xV_0 dx - i\int V_1 dx \\ \int xV_0 dx + i\int V_1 dx & \int x^2 V_0 dx + \int V_2 dx \end{pmatrix}.$$

$$v|_{\Lambda(a)} \sim \begin{pmatrix} 0 & 0 \\ 0 & \int V_2 dx \end{pmatrix},$$

and $\mu_1 = 0$, $\mu_2 = \int V_2 dx$. We have $n_{0,0} = 1$;

$$\int V_2 dx < 0 \quad \text{implies} \quad N = 0\,,$$
$$\int V_2 dx \geq 0 \quad \text{implies} \quad N = 1\,.$$

**Special case $V_0 = V_2 = 0$, $V_1 \not\equiv 0$**

$$v|_{\Lambda(a)} \sim \begin{pmatrix} 0 & -i\int V_1 dx \\ i\int V_1 dx & 0 \end{pmatrix},$$

and $\mu_1 = -|\int V_1 dx|$, $\mu_2 = |\int V_1 dx|$. We have $n_{0,0} = 0$;

$$\int V_1 dx \neq 0 \quad \text{implies} \quad N = 1\,,$$
$$\int V_1 dx = 0 \quad \text{implies} \quad N = 2\,.$$

**Special case $V_1 = V_2 = 0$, $V_0 \not\equiv 0$**

$$2\mu_{1,2} = \int (1+x^2) V_0 dx \\ \pm \sqrt{\left\{\int (1-x^2) V_0 dx\right\}^2 + 4\left\{\int x V_0 dx\right\}^2}.$$

Moreover, $n_{0,0} = 0$ and $N = 0, 1, 2$ are possible.

# 4. Applications III

**4.1.**

Let $W$ be a bounded non-trivial function of compact support. Let $\beta$ be the maximal coupling, for which

$$A = -\Delta - \beta W$$

does not have negative spectrum. Assume that $\beta > 0$.

The problem $(-\Delta - \beta W)\psi = 0$ has a positive distributional solution (principal eigenvalue). Due to Harnack's inequality $\psi + \psi^{-1}$ is locally bounded.

**4.2.**

Let $V$ be a bounded function of compact support. Set

$$A(\alpha) = A - \alpha V = -\Delta - \beta W - \alpha V.$$

Due to the identity

$$\begin{aligned} a(\alpha)[u] &= \int |\nabla u|^2 dx - \beta \int W|u|^2 dx - \alpha \int V|u|^2 dx \\ &= \int \psi^2 |\nabla(u\psi^{-1})|^2 dx - \alpha \int V\psi^2 |u\psi^{-1}|^2 dx \\ &= a_\psi[\eta] - \alpha v_\psi[\eta], \qquad \eta = u\psi^{-1}, \end{aligned}$$

$$N(\alpha) = \text{tr}\, \chi_-(-\Delta - \beta W - \alpha V)$$
$$= \text{tr}\, \chi_-(-\nabla^T \psi^2 \nabla - \alpha V \psi^2).$$

If $V \not\equiv 0$ then [Pinchover]

$$\int V\psi^2 dx < 0 \quad \text{implies} \quad \lim_{\alpha \to +0} N(\alpha) = 0,$$
$$\int V\psi^2 dx \geq 0 \quad \text{implies} \quad \lim_{\alpha \to +0} N(\alpha) = 1.$$

## 5. Hardy type inequalities for magnetic operators

**5.1.**

Let $q_0$, $q_1$ be continuous functions on $\mathbb{R}^d$, $d \geq 2$,

$$0 < q_0(x) \leq q_1(x) < \infty \qquad \text{for all} \qquad x \in \mathbb{R}^d.$$

Let $a(x)$ be a $d \times d$-matrix, such that

$$\rho_0(x)\mathbb{I} \leq a(x) \leq \rho_1(x)\mathbb{I}, \qquad x \in \mathbb{R}^d.$$

Let $\mathcal{A}(x) = (\mathcal{A}_1(x), \ldots, \mathcal{A}_d(x))$ be a real vector field,

$$\mathcal{A} \in L_d^{\text{loc}}(\mathbb{R}^d) \qquad \text{if} \quad d \geq 3,$$
$$\mathcal{A} \in L_{2+\epsilon}^{\text{loc}}(\mathbb{R}^d) \quad \text{for some} \quad \epsilon > 0 \quad \text{if} \quad d = 2.$$

For $u \in C_0^\infty(\mathbb{R}^d)$ define

$$a_{\mathcal{A}}[u] = \int \langle a(x)(i\nabla + \mathcal{A}(x))u, (i\nabla + \mathcal{A}(x))u \rangle \, dx \, .$$

Assume that $V(x) \geq 0$. By Kato's inequality

$$\int V|u|^2 dx \leq C \int \langle a\nabla u, \nabla u\rangle \, dx$$

for all $u \in C_0^\infty(\mathbb{R}^d)$ implies

$$\int V|u|^2 dx \leq C \int \langle a(i\nabla + \mathcal{A})u, (i\nabla + \mathcal{A})u\rangle \, dx$$

for all $u \in C_0^\infty(\mathbb{R}^d)$ [Kato, Simon].

## 5.3.

Magnetic fields induce Hardy's inequality:

**Theorem 4.** *Let $V(x) \geq 0$ be bounded and of compact support. Assume that $\mathcal{A}$ cannot be removed by gauge transformation, that means*

$$(i\nabla + \mathcal{A})\phi = 0$$

*does not have a non-trivial global solution $\phi \in W_{2,1}^{loc}$. Then*

$$\int V|u|^2 dx \leq C(V, \mathcal{A}, a, d) \int \langle a(i\nabla + \mathcal{A})u, (i\nabla + \mathcal{A})u\rangle \, dx$$

*holds for all $u \in C_0^\infty(\mathbb{R}^d)$.*

**Corollary.** *In the setting of Th. 4 the operator*

$$(i\nabla + \mathcal{A})^T a(x)(i\nabla + \mathcal{A}) - \alpha V$$

*does not have negative spectrum for sufficiently small positive $\alpha$.*

Let $W$ be a bounded non-trivial function of compact support. Let $\beta$ be the maximal coupling, for which

$$A = -\Delta - \beta W$$

does not have negative spectrum. Assume that $\beta > 0$.

The problem $(-\Delta - \beta W)\psi = 0$ has a positive distributional solution (principal eigenvalue). Due to Harnack's inequality $\psi + \psi^{-1}$ is locally bounded.

Then for $u \in C_0^\infty(\mathbb{R}^d)$ and non-trivial $\mathcal{A}$ it holds

$$\begin{aligned} \int |(i\nabla + \mathcal{A})u|^2 dx - \beta \int W|u|^2 dx \\ &= \int \psi^2 |(i\nabla + \mathcal{A})(u\psi^{-1})|^2 dx \\ &\geq C^{-1}(V, \mathcal{A}, \psi^2, d) \int V|u\psi^{-1}|^2 dx \\ &\geq c(V, \mathcal{A}, W, d) \int V|u|^2 dx\ . \end{aligned}$$

A particular interesting case is

$$\begin{aligned} &\int_{|x|\leq 1} |u|^2 dx \\ &\leq C(d, \mathcal{A}) \left\{ \int |(i\nabla + \mathcal{A})u|^2 dx - \frac{(d-2)^2}{4} \int \frac{|u|^2}{|x|^2} dx \right\} \end{aligned}$$

for $u \in C_0^\infty(\mathbb{R}^d)$, $d \geq 3$, $\mathcal{A}$ non-trivial.

In the dimension $d = 2$ the classical Hardy inequality fails. It can be replaced by

$$\int \frac{|u|^2 dx}{|x|^2(1+\log^2|x|)} \leq C \int |\nabla u|^2 dx,$$

which holds on all functions

$$u \in C_0^\infty(\mathbb{R}^2), \qquad \oint_{|x|=1} u(x)dx = 0.$$

Let $\mathcal{A}$ be a continuously differentiable real vector field on $\mathbb{R}^2$, such that

$$B = \frac{\partial \mathcal{A}_1}{\partial x_2} - \frac{\partial \mathcal{A}_2}{\partial x_1}$$

is integrable. Then

$$\Phi = \frac{1}{2\pi} \int B dx$$

is the (regularized) magnetic flux through the plane $\mathbb{R}^2$.

**Theorem 5.** [with A. Laptev] *Assume that* $\Phi \notin \mathbb{Z}$. *Then the bound*

$$\int \frac{|u|^2 dx}{1+|x|^2} \leq C(\mathcal{A}) \int |(i\nabla + \mathcal{A})u|^2 dx$$

*holds for all* $u \in C_0^\infty(\mathbb{R}^2)$.