Cycle indices from the exponential formula and subset / multiset sums divisible by a parameter

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The exponential formula is a classic result from combinatorics which we apply here to answer the question of the probability that a set or a multiset drawn from $[n]$ sums to a multiple of n. An algorithm is provided to compute formulae for sets that sum to a multiple of a parameter k. We count unordered rooted trees by leaves where the outdegree must be more than one. These are from posts to math.stackexchange.com (MSE) and have retained the question answer format from that site. Maple code for the algorithm and for verification purposes can be found at the included links.

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The exponential formula tells us that the cycle index $Z(P_k)$ of the unlabeled set operator SET on k slots has OGF

$$Z(P_k) = [w^k] \exp\left(\sum_{l\ge 1} (-1)^{l-1} a_l \frac{w^l}{l}\right).$$

The desired probability is given by

$${\binom{n}{k}}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} Z(P_k)\left(\sum_{q=1}^{n} z^q\right)\Bigg|_{z=\exp(2\pi i p/n)}.$$

This is

$${\binom{n}{k}}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} [w^k] \exp\left(\sum_{l\ge 1} (-1)^{l-1} \left(\sum_{q=1}^{n} z^{ql}\right) \frac{w^l}{l}\right)\Bigg|_{z=\exp(2\pi i p/n)}.$$

Evaluating the contribution for $p = 0$ first we get

$${\binom{n}{k}}^{-1} \frac{1}{n} [w^k] \exp\left(\sum_{l\ge 1} (-1)^{l-1} n \frac{w^l}{l}\right)$$

$$= {\binom{n}{k}}^{-1} \frac{1}{n} [w^k] \exp\left(n \log(1+w)\right)$$

$$= {\binom{n}{k}}^{-1} \frac{1}{n} [w^k] (1+w)^n = {\binom{n}{k}}^{-1} \frac{1}{n} \binom{n}{k} = \frac{1}{n}.$$

It remains to evaluate the contribution from $1 \le p \le n-1$. Now for these p if l is a multiple of $m = n/\gcd(p,n)$ we have

$$\sum_{q=1}^{n} \exp(2\pi i p/n)^{ql} = n.$$

We get zero otherwise. This yields for the remaining terms without the scalar in front

$$\sum_{p=1}^{n-1} [w^k] \exp\left(\sum_{l\ge 1} (-1)^{ml-1} n \frac{w^{ml}}{ml}\right) = \sum_{p=1}^{n-1} [w^k] \exp\left(-\frac{n}{m} \sum_{l\ge 1} \frac{(-w)^{ml}}{l}\right)$$

$$= \sum_{p=1}^{n-1} [w^k] \exp\left(-\frac{n}{m}\log\frac{1}{1-(-w)^m}\right) = \sum_{p=1}^{n-1} [w^k](1-(-w)^{n/\gcd(p,n)})^{\gcd(p,n)}$$

$$= \sum_{p=1}^{n-1} [w^k](1+(-1)^{1+n/\gcd(p,n)} w^{n/\gcd(p,n)})^{\gcd(p,n)}.$$

Using an Iverson bracket this becomes

$$\sum_{p=1}^{n-1} [[n/\gcd(p,n)|k]] \times \binom{\gcd(p,n)}{k\gcd(p,n)/n} (-1)^{k\gcd(p,n)/n+k}.$$

Putting it all together we thus obtain

$$\frac{1}{n} + (-1)^k \binom{n}{k}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} [[n/\gcd(p,n)|k]] \times \binom{\gcd(p,n)}{k\gcd(p,n)/n} (-1)^{k\gcd(p,n)/n}.$$

Note that $n/\gcd(p,n)$ is a divisor of n that is at least two (we would need $p = n$ to get $n/\gcd(p,n) = 1$ but $p < n$). This means when $\gcd(k,n) = 1$ the Iverson bracket fails for all p and only the first term remains, which is what we wanted to prove.

As a sanity check we evaluate the formula when $k = n$. The sum of the one subset is $n(n+1)/2$ so we should get for the probability one when n is odd and zero when it is even. We find

$$\frac{1}{n} + (-1)^n \binom{n}{n}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} \binom{\gcd(p,n)}{\gcd(p,n)} (-1)^{\gcd(p,n)}$$

$$= \frac{1}{n} + (-1)^n \frac{1}{n} \sum_{p=1}^{n-1} (-1)^{\gcd(p,n)} = (-1)^n \frac{1}{n} \sum_{p=1}^{n} (-1)^{\gcd(p,n)}$$

$$= (-1)^n \frac{1}{n} \sum_{d|n} \sum_{\gcd(p,n)=d} (-1)^d = (-1)^n \frac{1}{n} \sum_{d|n} (-1)^d \sum_{\gcd(q,n/d)=1} 1$$

$$= (-1)^n \frac{1}{n} \sum_{d|n} (-1)^d \varphi(n/d).$$

We evaluate this using formal Dirichlet series, starting from $\sum_{d|n} \varphi(d) = n$ which yields

$$\sum_{n\ge1} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

Furthermore we have

$$\sum_{n\ge1} \frac{(-1)^n}{n^s} = -\left(1 - \frac{2}{2^s}\right)\zeta(s).$$

This means that

$$\sum_{n\ge 1} \frac{1}{n^s} \sum_{d|n} (-1)^d \varphi(n/d) = -\left(1-\frac{2}{2^s}\right)\zeta(s-1).$$

Now we have for n even

$$-(-1)^n \frac{1}{n}[n^{-s}]\left(1-\frac{2}{2^s}\right)\zeta(s-1) = -(-1)^n \frac{1}{n}(n - 2[(n/2)^{-s}]\zeta(s-1))$$

$$= -(-1)^n \frac{1}{n}(n - 2\times n/2) = 0.$$

We obtain zero as required. On the other hand with n odd we find

$$-(-1)^n \frac{1}{n}[n^{-s}]\left(1-\frac{2}{2^s}\right)\zeta(s-1) = -(-1)^n \frac{1}{n}[n^{-s}]\zeta(s-1)$$

$$= -(-1)^n \frac{1}{n} \times n = 1,$$

again as required. This concludes the sanity check. and indeed the entire argument.

This was math.stackexchange.com problem 2417918

2 Probability that a subset of n elements of $[kn]$ sums to a multiple of n

We ask about the probability that a set of size n drawn from $[kn]$ has sum divisible by n. The exponential formula tells us that the cycle index $Z(P_n)$ of the unlabeled set operator

$$\text{SET}$$

on n slots has OGF

$$Z(P_n) = [w^n] \exp\left(\sum_{l\ge 1} (-1)^{l-1} a_l \frac{w^l}{l}\right).$$

The desired probability is given by

$${\binom{kn}{n}}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} Z\left(P_n; \sum_{q=1}^{kn} z^q\right)\Bigg|_{z=\exp(2\pi i p/n)}.$$

This is

$$\binom{kn}{n}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} [w^n] \exp\left(\sum_{l\ge 1} (-1)^{l-1} \left(\sum_{q=1}^{kn} z^{ql} \right) \frac{w^l}{l} \right) \Bigg|_{z=\exp(2\pi i p/n)}.$$

Evaluating the contribution for $p = 0$ first we get

$$\binom{kn}{n}^{-1} \frac{1}{n} [w^n] \exp\left(\sum_{l\ge 1} (-1)^{l-1} kn \frac{w^l}{l} \right)$$
$$= \binom{kn}{n}^{-1} \frac{1}{n} [w^n] \exp\left(kn \log(1+w) \right)$$
$$= \binom{kn}{n}^{-1} \frac{1}{n} [w^n] (1+w)^{kn} = \binom{kn}{n}^{-1} \frac{1}{n} \binom{kn}{n} = \frac{1}{n}.$$

It remains to evaluate the contribution from $1 \le p \le n-1$. Now for these p if l is a multiple of $m = n/\gcd(p,n)$ we have

$$\sum_{q=1}^{kn} \exp(2\pi i p/n)^{ql} = kn.$$

We get zero otherwise. This yields for the remaining terms without the scalar in front

$$\sum_{p=1}^{n-1} [w^n] \exp\left(\sum_{l\ge 1} (-1)^{ml-1} kn \frac{w^{ml}}{ml} \right) = \sum_{p=1}^{n-1} [w^n] \exp\left(-\frac{kn}{m} \sum_{l\ge 1} \frac{(-w)^{ml}}{l} \right)$$
$$= \sum_{p=1}^{n-1} [w^n] \exp\left(-\frac{kn}{m} \log \frac{1}{1-(-w)^m} \right) = \sum_{p=1}^{n-1} [w^n] (1-(-w)^{n/\gcd(p,n)})^{k \gcd(p,n)}$$
$$= \sum_{p=1}^{n-1} [w^n] (1 + (-1)^{1+n/\gcd(p,n)} w^{n/\gcd(p,n)})^{k \gcd(p,n)}.$$

This is

$$\sum_{p=1}^{n-1} \binom{k \gcd(p,n)}{\gcd(p,n)} (-1)^{(1+n/\gcd(p,n)) \gcd(p,n)}.$$

Putting it all together we thus obtain

$$\frac{1}{n} + (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} \binom{k \gcd(p,n)}{\gcd(p,n)} (-1)^{\gcd(p,n)}.$$

While this formula will produce results it may perhaps be simplified. Write

$$\frac{1}{n} = (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \binom{k\,\mathrm{gcd}(n,n)}{\mathrm{gcd}(n,n)} (-1)^{\mathrm{gcd}(n,n)}$$

to get

$$(-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{p=1}^{n} \binom{k\,\mathrm{gcd}(p,n)}{\mathrm{gcd}(p,n)} (-1)^{\mathrm{gcd}(p,n)}$$

$$= (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{d|n} \sum_{\mathrm{gcd}(p,n)=d} \binom{kd}{d} (-1)^d$$

$$= (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{d|n} \binom{kd}{d} (-1)^d \sum_{\mathrm{gcd}(q,n/d)=1} 1.$$

We find the alternate closed form

$$(-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{d|n} \binom{kd}{d} (-1)^d \varphi(n/d).$$

These two formulae were verified by simple enumeration for $1 \le n \le 7$ and $1 \le k \le 6$, see below.

This was math.stackexchange.com problem 2894653.

3 Probability that a multiset of k elements of $[n]$ sums to a multiple of n

We ask about the probability that a multiset of size k drawn from $[n]$ has sum divisible by n.

The exponential formula tells us that the cycle index $Z(S_k)$ of the unlabeled multiset operator MSET on k slots has OGF

$$Z(S_k) = [w^k] \exp\left(\sum_{l\ge 1} a_l \frac{w^l}{l}\right).$$

The number of these multisets is thus given by

$$Z(S_k)\left(\sum_{q=1}^{n} z^q\right)\Bigg|_{z=1} = [w^k] \exp\left(\sum_{l\ge 1} \left(\sum_{q=1}^{n} z^{ql}\right) \frac{w^l}{l}\right)\Bigg|_{z=1}$$

$$= [w^k] \exp\left(n \sum_{l\ge 1} \frac{w^l}{l}\right) = [w^k] \frac{1}{(1-z)^n} = \binom{n+k-1}{k}.$$

The desired probability is then given by

$$\binom{n+k-1}{k}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} Z(S_k) \left(\sum_{q=1}^{n} z^q \right) \Bigg|_{z=\exp(2\pi i p/n)}.$$

This is

$$\binom{n+k-1}{k}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} [w^k] \exp \left(\sum_{l\ge 1} \left(\sum_{q=1}^{n} z^{ql} \right) \frac{w^l}{l} \right) \Bigg|_{z=\exp(2\pi i p/n)}.$$

Evaluating the contribution for $p = 0$ first we get

$$\binom{n+k-1}{k}^{-1} \frac{1}{n} [w^k] \exp \left(\sum_{l\ge 1} n \frac{w^l}{l} \right) = \frac{1}{n}.$$

This was the same as when we counted. It remains to evaluate the contribution from $1 \le p \le n-1$. Now for these p if l is a multiple of $m = n/\gcd(p,n)$ we have

$$\sum_{q=1}^{n} \exp(2\pi i p/n)^{ql} = n.$$

We get zero otherwise. This yields for the remaining terms without the scalar in front

$$\sum_{p=1}^{n-1} [w^k] \exp \left(\sum_{l\ge 1} n \frac{w^{ml}}{ml} \right) = \sum_{p=1}^{n-1} [w^k] \exp \left(\frac{n}{m} \sum_{l\ge 1} \frac{w^{ml}}{l} \right)$$

$$= \sum_{p=1}^{n-1} [w^k] \exp \left(\frac{n}{m} \log \frac{1}{1-w^m} \right) = \sum_{p=1}^{n-1} [w^k] \frac{1}{(1-w^{n/\gcd(p,n)})^{\gcd(p,n)}}.$$

Using an Iverson bracket this becomes

$$\sum_{p=1}^{n-1} [[n/\gcd(p,n) | k]] \times \binom{k\gcd(p,n)/n + \gcd(p,n) - 1}{k\gcd(p,n)/n}.$$

Putting it all together we thus obtain

$$\frac{1}{n} + \binom{n+k-1}{k}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} [[n/\gcd(p,n) | k]] \times \binom{k\gcd(p,n)/n + \gcd(p,n) - 1}{k\gcd(p,n)/n}.$$

Note that $n/\gcd(p,n)$ is a divisor of n that is at least two (we would need $p = n$ to get $n/\gcd(p,n) = 1$ but $p < n$). This means when $\gcd(k,n) = 1$ the Iverson bracket fails for all p and only the first term remains.

We also have a Maple routine to compute and verify these data including two routines for enumeration, the second one optimized.

This was math.stackexchange.com problem 2533814

4 Generic algorithm for counting subsets of n elements of $[q]$ whose sum is divisible by some k

This is a straightforward application of the Polya Enumeration Theorem. We treat the problem of subsets with n elements of the set $\{1, 2, \ldots, q\}$ whose sum is divisible by k. Suppose $Z(P_n)$ is the cycle index of the set operator $\mathrm{SET}_{=n}$ given by the recurrence by Lovasz which is

$$Z(P_n) = \frac{1}{n}\sum_{l=1}^{n}(-1)^{l+1} a_l Z(P_{n-l}) \quad \text{where} \quad Z(P_0) = 1.$$

We obtain by PET the following formula for the OGF of ordinary sets

$$Z(P_n)\left(w + w^2 + \cdots + w^q\right) = Z(P_n)\left(\sum_{m=1}^{q} w^m\right).$$

With ρ a root of unity namely

$$\rho = \exp(2\pi i/k)$$

we get for the desired count the value

$$\frac{1}{k}\sum_{p=0}^{k-1} Z(P_n)\left(\sum_{m=1}^{q} w^m\right)\Bigg|_{w=\rho^p}.$$

We will compute the value for $p = 0$ separately and to do this recall the OGF of the set operator $\mathfrak{P}_{=n}$ which is

$$Z(P_n) = [z^n]\exp\left(a_1 z - a_2\frac{z^2}{2} + a_3\frac{z^3}{3} - a_4\frac{z^4}{4} + \cdots\right).$$

or

$$Z(P_n) = [z^n]\exp\left(\sum_{r\ge 1}(-1)^{r+1} a_r \frac{z^r}{r}\right).$$

On substituting this into our formula we get

$$\frac{1}{k}\sum_{p=0}^{k-1} [z^n]\exp\left(\sum_{r\ge 1}(-1)^{r+1}\frac{z^r}{r}\sum_{m=1}^{q} w^{rm}\right)\Bigg|_{w=\rho^p}.$$

When $p = 0$ we obtain

$$\frac{1}{k}[z^n]\exp\left(q\sum_{r\ge 1}(-1)^{r+1}\frac{z^r}{r}\right) = \frac{1}{k}[z^n]\exp\left(q\log(1+z)\right)$$

$$= \frac{1}{k}[z^n](1+z)^q = \frac{1}{k}\binom{q}{n}.$$

We switch to *algorithmics* for the remainder of this discussion.

In treating the case $p \ge 1$ we make the following observation. When substituting into the terms of the cycle index those a_r from the product where pr is a multiple of k produce the value q while the remaining a_r create a sequence of period k that depends only on the remainder b when q is divided by k where we take $1 \le b \le k$.

This yields an algorithm where we iterate over the cycle index, extract eventual powers of q from the terms and interpolate the rest in terms of b. The algorithm can be used to compute formulae for fixed combinations of n and k like the ones at this MSE link automatically.

We obtain for $(n,k) = (3,3)$

$$1/18\, q^3 + 1/3\, b^2 - 1/6\, q^2 - \frac{11\, b}{9} + q/3 + 2/3$$

and sure enough comparing it to the link these are the right values.

Supposing now that we are interested in divisibility by five of three-element subsets i.e. the pair $(n,k) = (3,5)$ we find

$$1/12\, b^4 - \frac{13\, b^3}{15} + 1/30\, q^3 + \frac{181\, b^2}{60} - 1/10\, q^2 - \frac{127\, b}{30} + q/15 + 2$$

which gives the sequence (starting at $q = 3$)

$$0, 0, 2, 4, 7, 11, 16, 24, 33, 44, 57, 72, 91, \ldots$$

For the pair $(11,5)$ we obtain

$$\frac{q^{11}}{199584000} - \frac{q^{10}}{3628800} + \frac{q^9}{151200} - \frac{11\, q^8}{120960} + \frac{683\, q^7}{864000} + \frac{b^4 q^2}{1200}$$
$$- \frac{781\, q^6}{172800} - \frac{b^4 q}{80} - \frac{b^3 q^2}{120} + \frac{31063\, q^5}{1814400} + 1/24\, b^4 + 1/8\, b^3 q + \frac{7\, b^2 q^2}{240}$$
$$- \frac{1529\, q^4}{36288} - \frac{631\, b^3}{1500} - \frac{859\, b^2 q}{2000} - \frac{137\, b q^2}{3000} + \frac{16103\, q^3}{252000} + \frac{863\, b^2}{600}$$
$$+ \frac{129\, bq}{200} - \frac{419\, q^2}{12600} - \frac{25\, b}{12} - \frac{31\, q}{110} + 1$$

which gives the sequence (starting at $q = 11$)

$$0, 2, 15, 72, 273, 873, 2474, 6363, 15114, 33592, \ldots.$$

Another interesting pair is $(3,6)$ which gives

$$-\frac{b^5 q}{90} - \frac{b^5}{90} + \frac{7\,b^4 q}{36} + 1/4\,b^4 - \frac{23\,b^3 q}{18} - 2\,b^3 + \frac{35\,b^2 q}{9}$$
$$+1/36\,q^3 + 7\,b^2 - \frac{242\,bq}{45} - 1/12\,q^2 - \frac{313\,b}{30} + \frac{17\,q}{6} + 5$$

and $(4,7)$ which produces

$$\frac{b^6}{360} - \frac{3\,b^5}{40} + \frac{389\,b^4}{504} + \frac{q^4}{168} - \frac{215\,b^3}{56} - 1/28\,q^3$$
$$+\frac{3041\,b^2}{315} + \frac{11\,q^2}{168} - \frac{403\,b}{35} - q/28 + 5.$$

The Maple code for this including a total enumeration routine for verification and some code to prettify the formulae for k small was as follows.

The reader is invited to contribute a better simplification routine making more effective use of the mathematical givens. The Maple code should be considered betaware.

Remark Sat Jan 23 2016. I present one of the special cases where radical simplification is possible. Start from the formula

$$\frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\sum_{r\ge 1} (-1)^{r+1} \frac{z^r}{r} \sum_{m=1}^{q} w^{rm}\right)\Bigg|_{w=\rho^p}.$$

Now suppose that q is a multiple of k and k is an odd prime. Observe that the sum

$$\sum_{m=1}^{q} w^{rm}$$

is equal to $q/k \times k = q$ if pr is a multiple of k and zero otherwise. But pr can only be a multiple of k if r is a multiple of k. This yields

$$\frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\sum_{r\ge 1} (-1)^{kr+1} \frac{z^{kr}}{kr} \sum_{m=1}^{q} w^{krm}\right)\Bigg|_{w=\rho^p}$$
$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\sum_{r\ge 1} (-1)^{kr+1} \frac{z^{kr}}{kr} \frac{q}{k} \times k\right)\Bigg|_{w=\rho^p}$$
$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\frac{q}{k} \sum_{r\ge 1} (-1)^{kr+1} \frac{z^{kr}}{r}\right)$$

$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1}[z^n]\exp\left(-\frac{q}{k}\sum_{r\ge 1}\frac{(-z)^{kr}}{r}\right)$$

$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1}[z^n]\exp\left(-\frac{q}{k}\log\frac{1}{1-(-z)^k}\right)$$

$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1}[z^n](1+z^k)^{q/k}.$$

Therefore if n is coprime to k we obtain

$$\frac{1}{k}\binom{q}{n}$$

and if it is a multiple of k

$$\frac{1}{k}\binom{q}{n} + \frac{k-1}{k}\binom{q/k}{n/k}.$$

This was math.stackexchange.com problem 1618420

5 Symmetric Polynomials

With $f_X(m) = [u^m]\prod_{k=1}^{n}(1+ux_k)$ and $S_\ell = \sum_{k=1}^{n} x_k^\ell$ we seek to express the former in terms of the latter where the number of permutations having cycle structure $\lambda \vdash m$ appears (coefficients of the cycle index of the symmetric group).

To find a closed form for $f_X(m)$ in terms of the S_ℓ we first require the exponential formula for the cycle index of the unlabeled set operator

$$\text{SET}.$$

Let A be a generating function in some number of variables and let $B = c_B X_B$ a contributing monomial term where c_B is the leading coefficient (positive) and X_B the product of the variables to their respective powers, so that $A = \sum_{B\in A} c_B X_B$. We then have from first principles that the generating function of sets drawn from A containing m elements is

$$[z^m]\prod_{B\in A}(1+zX_B)^{c_B}.$$

Manipulate this to obtain

$$[z^m]\prod_{B\in A}\exp\log(1+zX_B)^{c_B}$$

$$= [z^m]\prod_{B\in A}\exp\left(-c_B\log\frac{1}{1+zX_B}\right)$$

$$= [z^m] \exp \sum_{B \in A} \left(-c_B \log \frac{1}{1 + zX_B} \right)$$

$$= [z^m] \exp \sum_{B \in A} \left(-c_B \sum_{\ell \geq 1} (-1)^\ell X_B^\ell \frac{z^\ell}{\ell} \right)$$

$$= [z^m] \exp \left(\sum_{\ell \geq 1} (-1)^{\ell - 1} \left(\sum_{B \in A} c_B X_B^\ell \right) \frac{z^\ell}{\ell} \right).$$

But $\sum_{B \in A} c_B X_B^\ell$ is by definition the Polya substitution applied to A through the cycle index variable a_ℓ and we have proved the exponential formula for the unlabeled set operator, which says that

$$Z(P_m) = [z^m] \exp \left(\sum_{\ell \geq 1} (-1)^{\ell - 1} a_\ell \frac{z^\ell}{\ell} \right).$$

Now we have from the definition applying PET that

$$f_X(m) = Z(P_m; x_1 + x_2 + \cdots + x_n).$$

Hence

$$f_X(m) = [z^m] \exp \left(\sum_{\ell \geq 1} (-1)^{\ell - 1} S_\ell \frac{z^\ell}{\ell} \right)$$

$$= [z^m] \prod_{\ell \geq 1} \exp \left((-1)^{\ell - 1} S_\ell \frac{z^\ell}{\ell} \right)$$

$$= [z^m] \prod_{\ell \geq 1} \sum_{q \geq 0} \frac{1}{q!} (-1)^{q(\ell - 1)} S_\ell^q \frac{z^{\ell q}}{\ell^q}.$$

We want to expand this product. We are interested in the coefficient on $[z^m]$ so we consider integer partitions $\lambda \vdash m$. Let the partition be $1^{q_1} 2^{q_2} 3^{q_3} \cdots$ where all but a finite number of exponents are zero. We now obtain

$$[z^m] \sum_{\lambda \vdash m} \prod_{\ell \geq 1} \frac{1}{q_\ell!} (-1)^{(\ell - 1) q_\ell} S_\ell^{q_\ell} \frac{z^{\ell q_\ell}}{\ell^{q_\ell}}.$$

Since $\sum_{\ell \geq 1} \ell q_\ell = m$ this becomes

$$\sum_{\lambda \vdash m} \prod_{\ell \geq 1} \frac{1}{q_\ell!} (-1)^{(\ell - 1) q_\ell} S_\ell^{q_\ell} \frac{1}{\ell^{q_\ell}}$$

$$= (-1)^m \sum_{\lambda \vdash m} (-1)^{\sum_{\ell \geq 1} q_\ell} \prod_{\ell \geq 1} S_\ell^{q_\ell} \frac{1}{q_\ell! \times \ell^{q_\ell}}.$$

We thus get the closed form

$$\boxed{f_X(m) = \frac{(-1)^m}{m!} \sum_{\lambda \vdash m} (-1)^{\sum_{\ell\ge 1} q_\ell} \left(\prod_{\ell\ge 1} S_\ell^{q_\ell}\right) \left(m! \prod_{\ell \ge 1} \frac{1}{q_\ell! \times \ell^{q_\ell}}\right).}$$

This yields for example

$$f_X(4) = \frac{1}{4!} \left({S_1}^4 - 6\,{S_1}^2 S_2 + 8\,S_1 S_3 + 3\,{S_2}^2 - 6\,S_4\right)$$

and

$$f_X(5) = \frac{1}{5!} \left({S_1}^5 - 10\,{S_1}^3 S_2 + 20\,{S_1}^2 S_3 + 15\,S_1 {S_2}^2 - 30\,S_1 S_4 - 20\,S_2 S_3 + 24\,S_5\right).$$

We now show that

$$m! \prod_{\ell\ge 1} \frac{1}{q_\ell! \times \ell^{q_\ell}}$$

counts the number of permutations with cycle structure λ.

First, selecting the values to go on the cycles yields the multinomial coefficient

$$\frac{m!}{\prod_{\ell \ge 1} (\ell!)^{q_\ell}}.$$

A set of ℓ values gives $\frac{\ell!}{\ell}$ cycles:

$$\prod_{\ell\ge 1} \left(\frac{\ell!}{\ell}\right)^{q_\ell}.$$

Any permutation of the cycles of length ℓ yields the same permutation:

$$\prod_{\ell\ge 1} \frac{1}{q_\ell!}.$$

Multiply these to obtain

$$\frac{m!}{\prod_{\ell\ge 1} (\ell!)^{q_\ell}} \prod_{\ell\ge 1} \left(\frac{\ell!}{\ell}\right)^{q_\ell} \prod_{\ell\ge 1} \frac{1}{q_\ell!} = m! \prod_{\ell\ge 1} \frac{1}{q_\ell! \times \ell^{q_\ell}}.$$

This is the claim and concludes the argument.

As an addendum we have by inspection for the boxed closed form that it is given by a substitution into $Z(Q_m)$, the cycle index of the symmetric group (the variable S is in use already), which is

$$Z(Q_m; a_\ell = (-1)^{\ell-1} S_\ell).$$

With $Z(Q_m)$ being the average of all $m!$ permutations factorized into cycles, where a_ℓ stands for a cycle of ℓ elements we have

$$Z(Q_m) = \frac{1}{m!} \sum_{\lambda \vdash m} \left(\prod_{\ell \ge 1} a_\ell^{q_\ell} \right) \left(m! \prod_{\ell \ge 1} \frac{1}{q_\ell! \times \ell^{q_\ell}} \right).$$

Now put $a_\ell = (-1)^{\ell-1} S_\ell$ to get the boxed form. The substitution $a_\ell := (-1)^{\ell-1} a_\ell$ converts the unlabeled multiset operator into the unlabeled set operator through their cycle indices.

This was math.stackexchange.com problem 3983165.

6 Unordered trees

Here we count unordered trees classified by the number of leaves, where outdegree one is not allowed, as well as the number of such trees with a given number of leaves. This is asking about the combinatorial class $\mathcal{F}$ where

$$\mathcal{F} = \mathcal{U} \times \mathcal{Z} + \mathcal{Z} \times \mathrm{MSET}_{\ge 2}(\mathcal{F}).$$

We first compute these trees by the number of nodes represented by $\mathcal{Z}$ classified by the number of leaves and then extract the coefficient on the number of leaves, which are marked here with $\mathcal{U}$. Translating to generating functions and using the exponential formula for the multiset operator we find

$$F(z,u) = uz + z \left(\exp\left(\sum_{\ell \ge 1} \frac{F(z^\ell, u^\ell)}{\ell} \right) - 1 - F(z,u) \right).$$

We introduce $F_n(u) = [z^n] F(z,u)$ to get

$$F(z,u) = uz + z \left(\exp\left(\sum_{\ell \ge 1} \frac{1}{\ell} \sum_{q \ge 0} F_q(u^\ell) z^{q\ell} \right) - 1 - F(z,u) \right).$$

Differentiating with respect to z we obtain

$$F'(z,u) = u + (F(z,u) - uz)/z$$

$$+z \left(\exp\left(\sum_{\ell \ge 1} \frac{F(z^\ell, u^\ell)}{\ell} \right) \left(\sum_{\ell \ge 1} \sum_{q \ge 1} q F_q(u^\ell) z^{q\ell - 1} \right) - F'(z,u) \right).$$

This is

$$F'(z,u) = F(z,u)/z$$

$$+z\left(((F(z,u)-uz)/z+1+F(z,u))\times\left(\sum_{\ell\ge 1}\sum_{q\ge 1} qF_q(u^\ell)z^{q\ell-1}\right)-F'(z,u)\right)$$

or alternatively

$$\begin{aligned}&F'(z,u)/z-F(z,u)/z^2+F'(z,u)\\&=(1-u+F(z,u)+F(z,u)/z)\times\left(\sum_{\ell\ge 1}\sum_{q\ge 1} qF_q(u^\ell)z^{q\ell-1}\right).\end{aligned}$$

Extracting the coefficient on z^{n-2} we obtain for the LHS

$$nF_n(u)-F_n(u)+(n-1)F_{n-1}(u)=(n-1)(F_n(u)+F_{n-1}(u)).$$

We get for the first piece on the RHS

$$(1-u)[z^{n-2}]\left(\sum_{\ell=1}^{n-1}\sum_{q\ge 1} qF_q(u^\ell)z^{q\ell-1}\right).$$

Here we must have $n-2=q\ell-1$ or $n-1=q\ell$. We find

$$(n-1)(1-u)\sum_{\ell|(n-1)}\frac{1}{\ell}F_{(n-1)/\ell}(u^\ell).$$

We get for the second piece

$$\begin{aligned}&[z^{n-2}]\left(\sum_{\ell=1}^{n-1}\sum_{q\ge 1} qF_q(u^\ell)z^{q\ell-1}\right)(F(z,u)+F(z,u)/z)\\&=\sum_{\ell=1}^{n-1}\sum_{q\ge 1} qF_q(u^\ell)[z^{n-1-q\ell}](F(z,u)+F(z,u)/z)\\&=\sum_{\ell=1}^{n-1}\sum_{q=1}^{\lfloor (n-1)/\ell\rfloor} qF_q(u^\ell)[z^{n-1-q\ell}](F(z,u)+F(z,u)/z)\\&=\sum_{\ell=1}^{n-1}\sum_{q=1}^{\lfloor (n-1)/\ell\rfloor} qF_q(u^\ell)(F_{n-1-q\ell}(u)+F_{n-q\ell}(u)).\end{aligned}$$

This gives the recurrence for $n\ge 2$ where $F_0(u)=0$ and $F_1(u)=u$:

$$\boxed{\begin{aligned}F_n(u)&=-F_{n-1}(u)+(1-u)\sum_{\ell|(n-1)}\frac{1}{\ell}F_{(n-1)/\ell}(u^\ell)\\&\quad+\frac{1}{n-1}\sum_{\ell=1}^{n-1}\sum_{q=1}^{\lfloor (n-1)/\ell\rfloor} qF_q(u^\ell)(F_{n-1-\ell q}(u)+F_{n-\ell q}(u)).\end{aligned}}$$

As an example we have

$$F_5(u) = u^4 + u^3.$$

The reader is invited to replicate these trees from the number of leaves. We also have as another example

$$F_{10}(u) = u^9 + 6\,u^8 + 16\,u^7 + 12\,u^6.$$

This says that e.g. there are sixteen of these trees on ten nodes having seven leaves.

We get for the count of our trees the sequence $\{F_n(1)\}_{n\geq 0}$ which is

$$0, 1, 0, 1, 1, 2, 3, 6, 10, 19, 35, 67, 127, 248, 482, 952, \ldots$$

which points us to OEIS A001678 where a considerable amount of material both theoretic and applied can be found. We quote from the definition of the sequence which says it is the "number of unordered rooted trees with n nodes where nodes cannot have out-degree 1." This is precisely the meaning of the combinatorial class we have used so we know we have the correct answer.

Now to collect the trees with m leaves on some number of nodes we first observe that we need at least $m+1$ nodes (this gives the star graph). On the other hand the maximum number of nodes happens in a full binary tree on $2m-1$ nodes. Therefore the number of trees with m leaves where $m \geq 2$ is given by

$$G_m = \sum_{q=m+1}^{2m-1} [u^m] F_q(u).$$

We also have $G_1 = 1$. If we desire a single formula for $m \geq 1$ and don't mind a zero term for $m \geq 2$ we may also use

$$G_m = \sum_{q=m}^{2m-1} [u^m] F_q(u).$$

This gives the sequence $\{G_m\}_{m\geq 1}$ which is

$$1, 1, 2, 5, 12, 33, 90, 261, 766, 2312, 7068, 21965, \ldots$$

Note that the diagram by OP for $m = 5$ contains a duplicate (trees number two and three). The sequence points us to OEIS A000669 where we learn that this object is known as a "series-reduced planted tree with n leaves" and there is much more.

This was math.stackexchange.com problem 4080821.

7 Inequivalent colorings of a 2xN board

We seek to count inequivalent colorings of a $2 \times N$ board under row and column permutations. For didactic purposes we first examine the case of a 2×4 board. In trying to compute $Z(S_2 \times S_4)$ we use the two possibilities from

$$Z(S_2) = \frac{1}{2}(s_1^2 + s_2^1).$$

We also have

$$Z(S_4) = \frac{1}{24}(t_1^4 + 6t_1^2 t_2 + 8t_1 t_3 + 3t_2^2 + 6t_4).$$

Now to compute the product of s_p^n and t_q^m we consider two cycles of length p and q. These yield $p \times q$ pairs and we seek the factorization into cycles of the joint action on these pairs of s_p and t_q, with s_p acting on the first component of the pair and t_q the second. The order of such a pair is $\mathrm{LCM}(p, q)$ which divides the pq pairs into $pq/\mathrm{LCM}(p, q) = \mathrm{GCD}(p, q)$ cycles. For the first cycle we have n possibilities and for the second, m possibilities, for a total of $m \times n$ pairs of cycles s_p and t_q. Therefore the product operation is given by

$$s_p^n \cdot t_q^m = a_{\mathrm{LCM}(p,q)}^{mn \times \mathrm{GCD}(p,q)}.$$

Now for s_1^2 we are fixing the two rows of the board and permuting the columns, hence it suffices to square terms from $Z(S_4)$ to get

$$\frac{1}{48}(a_1^8 + 6a_1^4 a_2^2 + 8a_1^2 a_3^2 + 3a_2^4 + 6a_4^2).$$

For s_2 we have the rule that pairing with t_q (we have $2q$ pairs) where q is odd has order $2q$ and hence yields a_{2q}, and order q when q is even and hence yields a_q^2. Processing these in sequence we obtain

- t_1^4: $q = 1$, total is a_2^4
- $6t_1^2 t_2$: $q = 1$, a_2^2 and $q = 2$, a_2^2, total is $6a_2^4$.
- $8t_1 t_3$: $q = 1$, a_2 and $q = 3$, a_6, total is $8a_2 a_6$
- $3t_2^2$: $q = 2$, a_2^2, total is $3a_2^4$
- $6t_4$: $q = 4$, total is $6a_4^2$.

This contribution is

$$\frac{1}{48}(10a_2^4 + 8a_2 a_6 + 6a_4^2).$$

Collecting the two pieces we get

$$\boxed{Z(S_2 \times S_4) = \frac{1}{48}(a_1^8 + 6a_1^4 a_2^2 + 8a_1^2 a_3^2 + 13a_2^4 + 8a_2 a_6 + 12a_4^2).}$$

We normally would not do this computation with pen and paper but use a CAS instead. With Maple as shown below we can treat the case of a $2 \times N$ board and get e.g. for $N = 6$:

$$Z(S_2 \times S_6) = \frac{{a_1}^{12}}{1440} + \frac{{a_1}^8{a_2}^2}{96} + 1/36\,{a_1}^6{a_3}^2 + 1/32\,{a_1}^4{a_2}^4$$
$$+1/16\,{a_1}^4{a_4}^2 + 1/12\,{a_1}^2{a_2}^2{a_3}^2 + \frac{91\,{a_2}^6}{1440} + 1/10\,{a_1}^2{a_5}^2$$
$$+1/9\,{a_2}^3a_6 + 3/16\,{a_2}^2{a_4}^2 + 1/36\,{a_3}^4 + 1/10\,a_2a_{10} + \frac{7\,{a_6}^2}{36}.$$

We obtain for two colors with a $2 \times N$ board the following sequence of inequivalent colorings:

$$3, 7, 13, 22, 34, 50, 70, 95, 125, 161, \ldots$$

which points us to OEIS A002623 where the problem statement by OP is cited as a definition of the sequence, so we have the correct result.

Note that we can compute a closed form of the count of colorings using a $2 \times N$ board and C colors with the exponential formula which gives the OGF of the cycle index $Z(S_N)$ of the symmetric group. The formula states

$$Z(S_N) = [w^N] \exp\left(\sum_{\ell\ge1} a_\ell \frac{w^\ell}{\ell}\right).$$

Now for the first term we square the a_ℓ and replace by the count C of the number of colors to get

$$\frac{1}{2}[w^N] \exp\left(\sum_{\ell\ge1} C^2 \frac{w^\ell}{\ell}\right) = \frac{1}{2}[w^N] \exp\left(C^2 \log\frac{1}{1-w}\right)$$
$$= \frac{1}{2}[w^N] \frac{1}{(1-w)^{C^2}} = \frac{1}{2}\binom{N+C^2-1}{N}.$$

For the second term we double the length of odd cycles and square even ones:

$$\frac{1}{2}[w^N] \exp\left(\sum_{\ell\ge1} a_{2\ell}^2 \frac{w^{2\ell}}{2\ell} + \sum_{\ell\ge0} a_{4\ell+2} \frac{w^{2\ell+1}}{2\ell+1}\right)$$
$$= \frac{1}{2}[w^N] \exp\left(\sum_{\ell\ge1} C^2 \frac{w^{2\ell}}{2\ell} + \sum_{\ell\ge0} C \frac{w^{2\ell+1}}{2\ell+1}\right)$$
$$= \frac{1}{2}[w^N] \frac{1}{(1-w^2)^{C^2/2}} \exp\left(C\sum_{\ell\ge0}\frac{w^\ell}{\ell} - C\sum_{\ell\ge0}\frac{w^{2\ell}}{2\ell}\right)$$

$$= \frac{1}{2}[w^N] \frac{1}{(1-w^2)^{C^2/2}} \frac{1}{(1-w)^C} (1-w^2)^{C/2}$$
$$= \frac{1}{2}[w^N] \frac{1}{(1-w^2)^{C(C-1)/2}} \frac{1}{(1-w)^C}.$$

Collecting the two pieces we get the following closed form:

$$\boxed{\frac{1}{2}[w^N] \left[\frac{1}{(1-w)^{C^2}} + \frac{1}{(1+w)^{C(C-1)/2}} \frac{1}{(1-w)^{C(C+1)/2}} \right].}$$

Extracting coefficients yields

$$\frac{1}{2} \binom{N+C^2-1}{N}$$
$$+ \frac{1}{2} \sum_{q=0}^{N} \binom{C(C-1)/2-1+q}{q} \times (-1)^q \times \binom{C(C+1)/2-1+N-q}{N-q}.$$

We get in particular for two colors the generating function is the following:

$$\frac{1}{2} \frac{1}{(1-w)^4} + \frac{1}{2} \frac{1}{(1+w)(1-w)^3}$$
$$= \frac{1}{2} \frac{1+w}{(1+w)(1-w)^4} + \frac{1}{2} \frac{1-w}{(1+w)(1-w)^4} = \frac{1}{(1+w)(1-w)^4}.$$

Converting to partial fractions, we get

$$\frac{1}{16} \frac{1}{1+w} + \frac{1}{16} \frac{1}{1-w} + \frac{1}{8} \frac{1}{(1-w)^2} + \frac{1}{4} \frac{1}{(1-w)^3} + \frac{1}{2} \frac{1}{(1-w)^4}.$$

Extracting the coefficient we have

$$\frac{1}{16}(-1)^N + \frac{1}{16} + \frac{1}{8}\binom{N+1}{1} + \frac{1}{4}\binom{N+2}{2} + \frac{1}{2}\binom{N+3}{3}$$

for a total of

$$\frac{1}{12}N^3 + \frac{5}{8}N^2 + \frac{17}{12}N + \frac{(-1)^N}{16} + \frac{15}{16}.$$

This was math.stackexchange.com problem 4085977.