Cycle indices from the exponential formula and subset / multiset sums divisible by a parameter

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The exponential formula is a classic result from combinatorics which we apply here to answer the question of the probability that a set or a multiset drawn from $[n]$ sums to a multiple of n. An algorithm is provided to compute formulae for sets that sum to a multiple of a parameter k. These are from posts to math.stackexchange.com (MSE) and have retained the question answer format from that site. Maple code for the algorithm and for verification purposes can be found at the included links.

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1 Probability that a subset of k elements of $[n]$ sums to a multiple of n

The exponential formula tells us that the cycle index $Z(P_k)$ of the unlabeled set operator SET on k slots has OGF

$$Z(P_k) = [w^k] \exp\left(\sum_{l\ge 1} (-1)^{l-1} a_l \frac{w^l}{l}\right).$$

The desired probability is given by

$${n \choose k}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} Z(P_k)\left(\sum_{q=1}^{n} z^q\right)\Bigg|_{z=\exp(2\pi i p/n)}.$$

This is

$${n \choose k}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} [w^k] \exp\left(\sum_{l\ge 1} (-1)^{l-1} \left(\sum_{q=1}^{n} z^{ql}\right) \frac{w^l}{l}\right)\Bigg|_{z=\exp(2\pi i p/n)}.$$

Evaluating the contribution for $p = 0$ first we get

$$\begin{aligned} &{n \choose k}^{-1} \frac{1}{n} [w^k] \exp\left(\sum_{l\ge 1} (-1)^{l-1} n \frac{w^l}{l}\right) \\ &= {n \choose k}^{-1} \frac{1}{n} [w^k] \exp\left(n \log(1+w)\right) \\ &= {n \choose k}^{-1} \frac{1}{n} [w^k] (1+w)^n = {n \choose k}^{-1} \frac{1}{n} {n \choose k} = \frac{1}{n}. \end{aligned}$$

It remains to evaluate the contribution from $1 \le p \le n-1$. Now for these p if l is a multiple of $m = n/\gcd(p, n)$ we have

$$\sum_{q=1}^{n} \exp(2\pi i p/n)^{ql} = n.$$

We get zero otherwise. This yields for the remaining terms without the scalar in front

$$\sum_{p=1}^{n-1} [w^k] \exp\left(\sum_{l\ge 1} (-1)^{ml-1} n \frac{w^{ml}}{ml}\right) = \sum_{p=1}^{n-1} [w^k] \exp\left(-\frac{n}{m} \sum_{l\ge 1} \frac{(-w)^{ml}}{l}\right)$$

$$= \sum_{p=1}^{n-1} [w^k] \exp\left(-\frac{n}{m}\log\frac{1}{1-(-w)^m}\right) = \sum_{p=1}^{n-1} [w^k](1-(-w)^{n/\gcd(p,n)})^{\gcd(p,n)}$$

$$= \sum_{p=1}^{n-1} [w^k](1+(-1)^{1+n/\gcd(p,n)} w^{n/\gcd(p,n)})^{\gcd(p,n)}.$$

Using an Iverson bracket this becomes

$$\sum_{p=1}^{n-1} [[n/\gcd(p,n)|k]] \times \binom{\gcd(p,n)}{k\gcd(p,n)/n} (-1)^{k\gcd(p,n)/n+k}.$$

Putting it all together we thus obtain

$$\frac{1}{n} + (-1)^k \binom{n}{k}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} [[n/\gcd(p,n)|k]] \times \binom{\gcd(p,n)}{k\gcd(p,n)/n} (-1)^{k\gcd(p,n)/n}.$$

Note that $n/\gcd(p,n)$ is a divisor of n that is at least two (we would need $p = n$ to get $n/\gcd(p,n) = 1$ but $p < n$). This means when $\gcd(k,n) = 1$ the Iverson bracket fails for all p and only the first term remains, which is what we wanted to prove.

As a sanity check we evaluate the formula when $k = n$. The sum of the one subset is $n(n+1)/2$ so we should get for the probability one when n is odd and zero when it is even. We find

$$\frac{1}{n} + (-1)^n \binom{n}{n}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} \binom{\gcd(p,n)}{\gcd(p,n)} (-1)^{\gcd(p,n)}$$

$$= \frac{1}{n} + (-1)^n \frac{1}{n} \sum_{p=1}^{n-1} (-1)^{\gcd(p,n)} = (-1)^n \frac{1}{n} \sum_{p=1}^{n} (-1)^{\gcd(p,n)}$$

$$= (-1)^n \frac{1}{n} \sum_{d|n} \sum_{\gcd(p,n)=d} (-1)^d = (-1)^n \frac{1}{n} \sum_{d|n} (-1)^d \sum_{\gcd(q,n/d)=1} 1$$

$$= (-1)^n \frac{1}{n} \sum_{d|n} (-1)^d \varphi(n/d).$$

We evaluate this using formal Dirichlet series, starting from $\sum_{d|n} \varphi(d) = n$ which yields

$$\sum_{n\ge1} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

Furthermore we have

$$\sum_{n\ge1} \frac{(-1)^n}{n^s} = -\left(1 - \frac{2}{2^s}\right)\zeta(s).$$

This means that

$$\sum_{n\ge 1} \frac{1}{n^s} \sum_{d|n} (-1)^d \varphi(n/d) = -\left(1-\frac{2}{2^s}\right)\zeta(s-1).$$

Now we have for n even

$$-(-1)^n \frac{1}{n}[n^{-s}] \left(1-\frac{2}{2^s}\right)\zeta(s-1) = -(-1)^n \frac{1}{n}(n - 2[(n/2)^{-s}]\zeta(s-1))$$

$$= -(-1)^n \frac{1}{n}(n - 2\times n/2) = 0.$$

We obtain zero as required. On the other hand with n odd we find

$$-(-1)^n \frac{1}{n}[n^{-s}] \left(1-\frac{2}{2^s}\right)\zeta(s-1) = -(-1)^n \frac{1}{n}[n^{-s}]\zeta(s-1)$$

$$= -(-1)^n \frac{1}{n} \times n = 1,$$

again as required. This concludes the sanity check. and indeed the entire argument.

This was math.stackexchange.com problem 2417918

2 Probability that a subset of n elements of $[kn]$ sums to a multiple of n

We ask about the probability that a set of size n drawn from $[kn]$ has sum divisible by n. The exponential formula tells us that the cycle index $Z(P_n)$ of the unlabeled set operator

$$\textsc{SET}$$

on n slots has OGF

$$Z(P_n) = [w^n] \exp\left(\sum_{l\ge 1} (-1)^{l-1} a_l \frac{w^l}{l}\right).$$

The desired probability is given by

$$\binom{kn}{n}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} Z\left(P_n; \sum_{q=1}^{kn} z^q\right)\Bigg|_{z=\exp(2\pi i p/n)}.$$

This is

$$\binom{kn}{n}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} [w^n] \exp\left(\sum_{l\ge 1} (-1)^{l-1} \left(\sum_{q=1}^{kn} z^{ql} \right) \frac{w^l}{l} \right) \Bigg|_{z=\exp(2\pi i p/n)}.$$

Evaluating the contribution for $p = 0$ first we get

$$\binom{kn}{n}^{-1} \frac{1}{n} [w^n] \exp\left(\sum_{l\ge 1} (-1)^{l-1} kn \frac{w^l}{l} \right)$$
$$= \binom{kn}{n}^{-1} \frac{1}{n} [w^n] \exp\left(kn \log(1+w) \right)$$
$$= \binom{kn}{n}^{-1} \frac{1}{n} [w^n] (1+w)^{kn} = \binom{kn}{n}^{-1} \frac{1}{n} \binom{kn}{n} = \frac{1}{n}.$$

It remains to evaluate the contribution from $1 \le p \le n-1$. Now for these p if l is a multiple of $m = n/\gcd(p,n)$ we have

$$\sum_{q=1}^{kn} \exp(2\pi i p/n)^{ql} = kn.$$

We get zero otherwise. This yields for the remaining terms without the scalar in front

$$\sum_{p=1}^{n-1} [w^n] \exp\left(\sum_{l\ge 1} (-1)^{ml-1} kn \frac{w^{ml}}{ml} \right) = \sum_{p=1}^{n-1} [w^n] \exp\left(-\frac{kn}{m} \sum_{l\ge 1} \frac{(-w)^{ml}}{l} \right)$$
$$= \sum_{p=1}^{n-1} [w^n] \exp\left(-\frac{kn}{m} \log \frac{1}{1-(-w)^m} \right) = \sum_{p=1}^{n-1} [w^n] (1-(-w)^{n/\gcd(p,n)})^{k \gcd(p,n)}$$
$$= \sum_{p=1}^{n-1} [w^n] (1 + (-1)^{1+n/\gcd(p,n)} w^{n/\gcd(p,n)})^{k \gcd(p,n)}.$$

This is

$$\sum_{p=1}^{n-1} \binom{k \gcd(p,n)}{\gcd(p,n)} (-1)^{(1+n/\gcd(p,n)) \gcd(p,n)}.$$

Putting it all together we thus obtain

$$\frac{1}{n} + (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} \binom{k \gcd(p,n)}{\gcd(p,n)} (-1)^{\gcd(p,n)}.$$

While this formula will produce results it may perhaps be simplified. Write

$$\frac{1}{n} = (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \binom{k\gcd(n,n)}{\gcd(n,n)} (-1)^{\gcd(n,n)}$$

to get

$$(-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{p=1}^{n} \binom{k\gcd(p,n)}{\gcd(p,n)} (-1)^{\gcd(p,n)}$$

$$= (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{d|n} \sum_{\gcd(p,n)=d} \binom{kd}{d} (-1)^d$$

$$= (-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{d|n} \binom{kd}{d} (-1)^d \sum_{\gcd(q,n/d)=1} 1.$$

We find the alternate closed form

$$(-1)^n \binom{kn}{n}^{-1} \frac{1}{n} \sum_{d|n} \binom{kd}{d} (-1)^d \varphi(n/d).$$

These two formulae were verified by simple enumeration for $1 \le n \le 7$ and $1 \le k \le 6$, see below.

This was math.stackexchange.com problem 2894653.

3 Probability that a multiset of k elements of $[n]$ sums to a multiple of n

We ask about the probability that a multiset of size k drawn from $[n]$ has sum divisible by n.

The exponential formula tells us that the cycle index $Z(S_k)$ of the unlabeled multiset operator MSET on k slots has OGF

$$Z(S_k) = [w^k] \exp\left(\sum_{l\ge 1} a_l \frac{w^l}{l} \right).$$

The number of these multisets is thus given by

$$Z(S_k)\left(\sum_{q=1}^{n} z^q \right)\Bigg|_{z=1} = [w^k] \exp\left(\sum_{l\ge 1} \left(\sum_{q=1}^{n} z^{ql} \right) \frac{w^l}{l} \right)\Bigg|_{z=1}$$

$$= [w^k] \exp\left(n \sum_{l\ge 1} \frac{w^l}{l} \right) = [w^k] \frac{1}{(1-z)^n} = \binom{n+k-1}{k}.$$

The desired probability is then given by

$$\binom{n+k-1}{k}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} Z(S_k) \left(\sum_{q=1}^{n} z^q \right) \Bigg|_{z=\exp(2\pi i p/n)}.$$

This is

$$\binom{n+k-1}{k}^{-1} \frac{1}{n} \sum_{p=0}^{n-1} [w^k] \exp \left(\sum_{l\ge 1} \left(\sum_{q=1}^{n} z^{ql} \right) \frac{w^l}{l} \right) \Bigg|_{z=\exp(2\pi i p/n)}.$$

Evaluating the contribution for $p=0$ first we get

$$\binom{n+k-1}{k}^{-1} \frac{1}{n} [w^k] \exp \left(\sum_{l\ge 1} n \frac{w^l}{l} \right) = \frac{1}{n}.$$

This was the same as when we counted. It remains to evaluate the contribution from $1 \le p \le n-1$. Now for these p if l is a multiple of $m = n/\gcd(p,n)$ we have

$$\sum_{q=1}^{n} \exp(2\pi i p/n)^{ql} = n.$$

We get zero otherwise. This yields for the remaining terms without the scalar in front

$$\sum_{p=1}^{n-1} [w^k] \exp \left(\sum_{l\ge 1} n \frac{w^{ml}}{ml} \right) = \sum_{p=1}^{n-1} [w^k] \exp \left(\frac{n}{m} \sum_{l\ge 1} \frac{w^{ml}}{l} \right)$$

$$= \sum_{p=1}^{n-1} [w^k] \exp \left(\frac{n}{m} \log \frac{1}{1-w^m} \right) = \sum_{p=1}^{n-1} [w^k] \frac{1}{(1-w^{n/\gcd(p,n)})^{\gcd(p,n)}}.$$

Using an Iverson bracket this becomes

$$\sum_{p=1}^{n-1} [[n/\gcd(p,n) | k]] \times \binom{k\gcd(p,n)/n + \gcd(p,n) - 1}{k\gcd(p,n)/n}.$$

Putting it all together we thus obtain

$$\frac{1}{n} + \binom{n+k-1}{k}^{-1} \frac{1}{n} \sum_{p=1}^{n-1} [[n/\gcd(p,n) | k]] \times \binom{k\gcd(p,n)/n + \gcd(p,n) - 1}{k\gcd(p,n)/n}.$$

Note that $n/\gcd(p,n)$ is a divisor of n that is at least two (we would need $p=n$ to get $n/\gcd(p,n)=1$ but $p<n$). This means when $\gcd(k,n)=1$ the Iverson bracket fails for all p and only the first term remains.

We also have a Maple routine to compute and verify these data including two routines for enumeration, the second one optimized.

This was math.stackexchange.com problem 2533814

4 Generic algorithm for counting subsets of n elements of $[q]$ whose sum is divisible by some k

This is a straightforward application of the Polya Enumeration Theorem. We treat the problem of subsets with n elements of the set $\{1, 2, \ldots, q\}$ whose sum is divisible by k. Suppose $Z(P_n)$ is the cycle index of the set operator $\mathrm{SET}_{=n}$ given by the recurrence by Lovasz which is

$$Z(P_n) = \frac{1}{n} \sum_{l=1}^{n} (-1)^{l+1} a_l Z(P_{n-l}) \quad \text{where} \quad Z(P_0) = 1.$$

We obtain by PET the following formula for the OGF of ordinary sets

$$Z(P_n)\left(w + w^2 + \cdots + w^q\right) = Z(P_n)\left(\sum_{m=1}^{q} w^m\right).$$

With ρ a root of unity namely

$$\rho = \exp(2\pi i/k)$$

we get for the desired count the value

$$\frac{1}{k} \sum_{p=0}^{k-1} Z(P_n)\left(\sum_{m=1}^{q} w^m\right)\Bigg|_{w=\rho^p}.$$

We will compute the value for $p = 0$ separately and to do this recall the OGF of the set operator $\mathfrak{P}_{=n}$ which is

$$Z(P_n) = [z^n] \exp\left(a_1 z - a_2 \frac{z^2}{2} + a_3 \frac{z^3}{3} - a_4 \frac{z^4}{4} + \cdots\right).$$

or

$$Z(P_n) = [z^n] \exp\left(\sum_{r\ge 1} (-1)^{r+1} a_r \frac{z^r}{r}\right).$$

On substituting this into our formula we get

$$\frac{1}{k} \sum_{p=0}^{k-1} [z^n] \exp\left(\sum_{r\ge 1} (-1)^{r+1} \frac{z^r}{r} \sum_{m=1}^{q} w^{rm}\right)\Bigg|_{w=\rho^p}.$$

When $p = 0$ we obtain

$$\frac{1}{k}[z^n] \exp\left(q \sum_{r\ge 1} (-1)^{r+1} \frac{z^r}{r}\right) = \frac{1}{k}[z^n] \exp\left(q \log(1+z)\right)$$

$$= \frac{1}{k}[z^n](1+z)^q = \frac{1}{k}\binom{q}{n}.$$

We switch to *algorithmics* for the remainder of this discussion.

In treating the case $p \ge 1$ we make the following observation. When substituting into the terms of the cycle index those a_r from the product where pr is a multiple of k produce the value q while the remaining a_r create a sequence of period k that depends only on the remainder b when q is divided by k where we take $1 \le b \le k$.

This yields an algorithm where we iterate over the cycle index, extract eventual powers of q from the terms and interpolate the rest in terms of b. The algorithm can be used to compute formulae for fixed combinations of n and k like the ones at this MSE link automatically.

We obtain for $(n, k) = (3, 3)$

$$1/18\, q^3 + 1/3\, b^2 - 1/6\, q^2 - \frac{11\, b}{9} + q/3 + 2/3$$

and sure enough comparing it to the link these are the right values.

Supposing now that we are interested in divisibility by five of three-element subsets i.e. the pair $(n, k) = (3, 5)$ we find

$$1/12\, b^4 - \frac{13\, b^3}{15} + 1/30\, q^3 + \frac{181\, b^2}{60} - 1/10\, q^2 - \frac{127\, b}{30} + q/15 + 2$$

which gives the sequence (starting at $q = 3$)

$$0, 0, 2, 4, 7, 11, 16, 24, 33, 44, 57, 72, 91, \ldots$$

For the pair $(11, 5)$ we obtain

$$\frac{q^{11}}{199584000} - \frac{q^{10}}{3628800} + \frac{q^9}{151200} - \frac{11\, q^8}{120960} + \frac{683\, q^7}{864000} + \frac{b^4 q^2}{1200}$$
$$- \frac{781\, q^6}{172800} - \frac{b^4 q}{80} - \frac{b^3 q^2}{120} + \frac{31063\, q^5}{1814400} + 1/24\, b^4 + 1/8\, b^3 q + \frac{7\, b^2 q^2}{240}$$
$$- \frac{1529\, q^4}{36288} - \frac{631\, b^3}{1500} - \frac{859\, b^2 q}{2000} - \frac{137\, b q^2}{3000} + \frac{16103\, q^3}{252000} + \frac{863\, b^2}{600}$$
$$+ \frac{129\, bq}{200} - \frac{419\, q^2}{12600} - \frac{25\, b}{12} - \frac{31\, q}{110} + 1$$

which gives the sequence (starting at $q = 11$)

$$0, 2, 15, 72, 273, 873, 2474, 6363, 15114, 33592, \ldots.$$

Another interesting pair is $(3, 6)$ which gives

$$-\frac{b^5 q}{90} - \frac{b^5}{90} + \frac{7\,b^4 q}{36} + 1/4\,b^4 - \frac{23\,b^3 q}{18} - 2\,b^3 + \frac{35\,b^2 q}{9}$$
$$+1/36\,q^3 + 7\,b^2 - \frac{242\,bq}{45} - 1/12\,q^2 - \frac{313\,b}{30} + \frac{17\,q}{6} + 5$$

and $(4, 7)$ which produces

$$\frac{b^6}{360} - \frac{3\,b^5}{40} + \frac{389\,b^4}{504} + \frac{q^4}{168} - \frac{215\,b^3}{56} - 1/28\,q^3$$
$$+\frac{3041\,b^2}{315} + \frac{11\,q^2}{168} - \frac{403\,b}{35} - q/28 + 5.$$

The Maple code for this including a total enumeration routine for verification and some code to prettify the formulae for k small was as follows.

The reader is invited to contribute a better simplification routine making more effective use of the mathematical givens. The Maple code should be considered betaware.

Remark Sat Jan 23 2016. I present one of the special cases where radical simplification is possible. Start from the formula

$$\frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\sum_{r\ge 1} (-1)^{r+1} \frac{z^r}{r} \sum_{m=1}^{q} w^{rm}\right)\Bigg|_{w=\rho^p}.$$

Now suppose that q is a multiple of k and k is an odd prime. Observe that the sum

$$\sum_{m=1}^{q} w^{rm}$$

is equal to $q/k \times k = q$ if pr is a multiple of k and zero otherwise. But pr can only be a multiple of k if r is a multiple of k. This yields

$$\frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\sum_{r\ge 1} (-1)^{kr+1} \frac{z^{kr}}{kr} \sum_{m=1}^{q} w^{krm}\right)\Bigg|_{w=\rho^p}$$
$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\sum_{r\ge 1} (-1)^{kr+1} \frac{z^{kr}}{kr} \frac{q}{k} \times k\right)\Bigg|_{w=\rho^p}$$
$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1} [z^n] \exp\left(\frac{q}{k} \sum_{r\ge 1} (-1)^{kr+1} \frac{z^{kr}}{r}\right)$$

$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1}[z^n]\exp\left(-\frac{q}{k}\sum_{r\ge 1}\frac{(-z)^{kr}}{r}\right)$$

$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1}[z^n]\exp\left(-\frac{q}{k}\log\frac{1}{1-(-z)^k}\right)$$

$$= \frac{1}{k}\binom{q}{n} + \frac{1}{k}\sum_{p=1}^{k-1}[z^n](1+z^k)^{q/k}.$$

Therefore if n is coprime to k we obtain

$$\frac{1}{k}\binom{q}{n}$$

and if it is a multiple of k

$$\frac{1}{k}\binom{q}{n} + \frac{k-1}{k}\binom{q/k}{n/k}.$$

This was math.stackexchange.com problem 1618420