Egorychev method and the evaluation of binomial coefficient sums

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The Egorychev method is from the book by G.P.Egorychev [Ego84]. We collect several examples, the focus being on computational methods to produce results. These are from posts to math.stackexchange.com and have retained the question answer format from that site.

The crux of the method is the use of integrals from the Cauchy Residue Theorem to represent binomial coefficients, exponentials and Iverson brackets.

We use three types of integrals:

- *First binomial coefficient integral (B_1)*

$$\binom{n}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{k+1}} \; dz.$$

- *Second binomial coefficient integral (B_2)*

$$\binom{n}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{(1-z)^{k+1} z^{n-k+1}} \; dz.$$

- *Exponentiation integral (E)*

$$n^k = \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(nz)}{z^{k+1}} \; dz.$$

- *Iverson bracket (I)*

$$[[0 \le k \le n]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^k}{z^{n+1}} \frac{1}{1-z} \; dz.$$

The residue at infinity is coded R.

Contents

List of identities in this document

section 1 B_1

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{k}{j}=(-1)^n\binom{n}{j}\binom{n+j}{j}.$$

section 2 B_1B_2

$$\sum_{k=0}^{r}\binom{r-k}{m}\binom{s+k}{n}=\binom{s+r+1}{n+m+1}.$$

section 3 B_1

$$\sum_{q=0}^{2m}(-1)^q\binom{p-1+q}{q}\binom{2m+2p+q-1}{2m-q}2^q=(-1)^m\binom{p-1+m}{m}.$$

section 4 B_1

$$\sum_{k=0}^{\lfloor m/2\rfloor}\binom{n}{k}(-1)^k\binom{m-2k+n-1}{n-1}=\binom{n}{m}.$$

section 5 B_2

$$\sum_{k=0}^{n}k\binom{2n}{n+k}=\frac{1}{2}n\binom{2n}{n}.$$

section 6 I

$$\sum_{k=0}^{n}\frac{n!}{k!}(n-k)n^k=n^{n+1}.$$

section 7 B_1

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{n+1}\frac{1}{a+bk+1}\binom{a+bk}{m}\binom{k-n-1}{k}=\binom{n}{m}.$$

section 8 B_1, R

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{m+k}{2n} = \binom{2m}{2n}.$$

section 9 $B_1 E$

$$(-1)^p \sum_{q=r}^{p} \binom{p}{q}\binom{q}{r}(-1)^q q^{p-r} = \frac{p!}{r!}.$$

section 10 B_1

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor} = \binom{2n+1}{n}.$$

section 11 B_1

Verify that $f_1(n,k) = f_2(n,k)$ where

$$f_1(n,k) = \sum_{v=0}^{n} \frac{(2k+2v)!}{(k+v)! \times v! \times (2k+v)! \times (n-v)!} 2^{-v}$$

and

$$f_2(n,k) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{(k+m)! \times m! \times (n-2m)!} 2^{n-4m}.$$

section 12 B_1

If

$$T(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} T(n-k)$$

for $n \geq 2$ then

$$T(n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

section 13 B_1

$$\sum_{k=0}^{n}\sum_{l=0}^{n} (-1)^{k+l} \binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{k}\binom{n}{l} = (-1)^m \binom{2m}{m}.$$

section 14 B_1

$$\sum_{k=0}^{2m+1} \binom{n}{k} 2^k \binom{n-k}{\lfloor (2m+1-k)/2 \rfloor} = \binom{2n+1}{2m+1}.$$

section 15 B_1

$$\sum_{k=m}^{n}(-1)^{n+k}\frac{2k+1}{n+k+1}\binom{n}{k}\binom{n+k}{k}^{-1}\binom{k}{m}\binom{k+m}{m}=\delta_{mn}.$$

section 16 B_1

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{k=\lceil n/2\rceil}^{n}\binom{n}{k}^2\binom{2k}{n}.$$

section 17 B_1

$$\sum_{s}\binom{n+s}{k+l}\binom{k}{s}\binom{l}{s}=\binom{n}{k}\binom{n}{l}.$$

section 18 B_1

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor}(-1)^k\binom{2n}{n+3k}=2\times 3^{n-1}.$$

section 19 B_1B_2

$$\sum_{j=0}^{b}\binom{b}{j}^2\binom{n+j}{2b}=\binom{n}{b}^2.$$

section 20 B_1

$$\sum_{k=0}^{\rho}\binom{2x+1}{2k}\binom{x-k}{\rho-k}=\frac{2x+1}{2\rho+1}\binom{x+\rho}{2\rho}2^{2\rho}.$$

section 21 B_1B_2

$$\sum_{k=0}^{\min(a,b)}\binom{x+y+k}{k}\binom{x}{b-k}\binom{y}{a-k}=\binom{x+a}{b}\binom{y+b}{a}$$

section 22 B_1I

$$\sum_{q=0}^{n}\binom{n}{2q}\binom{n-2q}{p-q}2^{2q}=\binom{2n}{2p}.$$

section 23 B_1R

$$\sum_{k=0}^{n-1}\left(\sum_{q=0}^{k}\binom{n}{q}\right)\left(\sum_{q=k+1}^{n}\binom{n}{q}\right)=\frac{1}{2}n\binom{2n}{n}.$$

section 24 B_1

$$(1-x)^{2k+1}\sum_{n\geq 0}\binom{n+k-1}{k}\binom{n+k}{k}x^n=\sum_{j\geq 0}\binom{k-1}{j-1}\binom{k+1}{j}x^j.$$

section 25 B_1

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k\binom{n+1}{k}\binom{2n-3k}{n}=\sum_{k=\lfloor n/2\rfloor}^{n}\binom{n+1}{k}\binom{k}{n-k}.$$

section 26 B_1

$$\sum_{k=0}^{\lfloor (m+n)/2\rfloor}\binom{n}{k}(-1)^k\binom{m+n-2k}{n-1}=\binom{n}{m+1}.$$

section 27 B_1

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}F_{k+1}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}F_{2k+1}.$$

section 28 B_2R

$$\sum_{p,q\geq 0}\binom{n-p}{q}\binom{n-q}{p}=F_{2n+2}.$$

section 29 B_1I

$$\sum_{r=0}^{n}\binom{r+n-1}{n-1}\binom{3n-r}{n}=\frac{1}{2}\left(\binom{4n}{2n}+\binom{2n}{n}^2\right).$$

section 30 B_1

$$[x^\mu y^\nu]\frac{1}{2}\left(1-x-y-\sqrt{1-2x-2y-2xy+x^2+y^2}\right)=\frac{1}{\mu+\nu-1}\binom{\mu+\nu-1}{\nu}\binom{\mu+\nu-1}{\mu}.$$

section 31 B_1

$$\sum_{r=1}^{n+1}\frac{1}{r+1}\binom{2r}{r}\binom{m+n-2r}{n+1-r}=\binom{m+n}{n}.$$

section 32 B_1

$$\sum_{k\geq 0}\binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}=\binom{n-1}{m-1}.$$

section 33 B_1

$$\sum_{k}\binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{r}{tk+r}=\binom{tn+r+s}{n}$$

section 34 B_1

$$\sum_{q=0}^{n-2}\sum_{k=1}^{n}\binom{k+q}{k}\binom{2n-q-k-1}{n-k+1}=n\times\binom{2n}{n+2}$$

section 35 $B_1 I$

$$\sum_{q=0}^{l}\binom{q+k}{k}\binom{l-q}{k}=\binom{l+k+1}{2k+1},$$

section 36 $B_1 I$

$$\sum_{k=0}^{n}k\binom{m+k}{m+1}=\frac{nm+2n+1}{m+3}\binom{n+m+1}{m+2}.$$

section 37 $B_1 IR$

$$\sum_{k=1}^{n}2^{n-k}\binom{k}{\lfloor k/2\rfloor}=-2^{n+1}+(2n+2+(n \mod 2))\binom{n}{\lfloor n/2\rfloor}.$$

section 38 IR

$$\sum_{q=0}^{m-1}\binom{n-1+q}{q}x^n(1-x)^q+\sum_{q=0}^{n-1}\binom{m-1+q}{q}x^q(1-x)^m=1$$

where $n,m\geq 1$

as well as

$$\sum_{k=0}^{n}\binom{m+k}{k}2^{n-k}+\sum_{k=0}^{m}\binom{n+k}{k}2^{m-k}=2^{n+m+1}.$$

section 39 B_1

$$\sum_{l=0}^{n}\sum_{r=0}^{(n-l)/2}\binom{n}{l}\binom{n-l}{r}\binom{n-l-r}{r}=\binom{2n}{n}$$

section 40 B_1

$$\sum_{k=0}^{n}(-1)^k\binom{1+p+q}{k}\binom{p+n-k}{n-k}\binom{q+n-k}{n-k}=\binom{p}{n}\binom{q}{n}.$$

section 41 B_1B_2

$$\sum_{k\geq 0}\binom{p}{k}\binom{q}{k}\binom{n+k}{p+q}=\binom{n}{p}\binom{n}{q}.$$

section 42 B_1

$$\sum_{k=0}^{n}\binom{n}{k}\binom{pn-n}{k}\binom{pn+k}{k}=\binom{pn}{n}^2.$$

section 43 B_1

$$\sum_{r=0}^{\min\{m,n,p\}}\binom{m}{r}\binom{n}{r}\binom{p+m+n-r}{m+n}=\binom{p+m}{m}\binom{p+n}{n}.$$

section 44 B_1

$$\sum_{p=0}^{l}\sum_{q=0}^{p}(-1)^q\binom{m-p}{m-l}\binom{n}{q}\binom{m-n}{p-q}=2^l\binom{m-n}{l}$$

section 45 E

$$\sum_{q=0}^{n}(n-2q)^k\binom{n}{2q+1}=\sum_{q=0}^{k+1}\binom{n}{q}2^{n-q-1}\times q!\times\left\{{k+1 \atop q+1}\right\}-\frac{1}{2}\times n!\times\left\{{k+1 \atop n+1}\right\}.$$

section 46 B_2

$$\sum_{p+q+r=n}\binom{p+q}{p}\binom{p+r}{r}\binom{q+r}{q}=\sum_{q=0}^{n}\binom{2q}{q}$$

with $p,q,r\geq 0$.

section 47 B_1B_2R

$$\sum_{q=0}^{n}q\binom{2n}{n+q}\binom{m+q-1}{2m-1}=m\times 4^{n-m}\times\binom{n}{m}$$

where $n\geq m$.

section 48 B_1B_2R

$$\sum_{q=0}^{n} q\binom{2n}{n+q}\binom{m+q-1}{2m-1} = m \times 4^{n-m} \times \binom{n}{m}$$

where $n \geq m$.

(different proof).

section 49 B_1EIR

With

$$b_k^n = \sum_{l=1}^{k} (-1)^{k-l} l^n \binom{n+1}{k-l}$$

and we

show that $b_k^n = b_{n+1-k}^n$ where $0 \leq k \leq n+1$.

section 50 B_1B_2

Suppose we have a random variable X where

$$\mathrm{P}[X=k] = \binom{N}{2n+1}^{-1}\binom{N-k}{n}\binom{k-1}{n}$$

for $k = n+1, \ldots, N-n$ and zero otherwise.

We seek to show that these probabilities sum to one and compute the the mean and the variance.

section 51 B_1

Suppose we have the Narayana number

$$N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$$

and let

$$A(n,k,l) = \sum_{\substack{i_0+i_1+\cdots+i_k=n \\ j_0+j_1+\cdots+j_k=l}} \prod_{t=0}^{k} N(i_t, j_t+1)$$

where the compositions for n are regular and the ones for l are weak and we seek to verify that

$$A(n,k,l) = \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1}.$$

section 52 B_1

Same as previous, generalized.

section 53 B_1

$$(-1)^m \frac{(n+m)!}{(n-m)!} \left(\frac{d}{dz}\right)^{n-m} (1-z^2)^n = (1-z^2)^m \left(\frac{d}{dz}\right)^{n+m} (1-z^2)^n$$

section 54 B_1

$$r^k (r+n)! = \sum_{m=0}^{k} (r+n+m)!(-1)^{k+m} \sum_{p=0}^{k-m} \binom{k}{p} \left\{ {k+1-p \atop m+1} \right\} n^p.$$

section 55 B_1

$$\sum_{k=0}^{n} \sum_{q=0}^{k} (-1)^q \binom{k}{q} \binom{n-1-qm}{k-1} = [z^n] \frac{1}{1-w-w^2-\cdots-w^m}.$$

section 56

$$\left\{ {n \atop m} \right\} = \sum_{k=m}^{n} \binom{k}{m} \sum_{q=0}^{k} (-1)^{n+q} \left\{ {n+q-m \atop k} \right\} (-1)^{k+q} \left[{k \atop q} \right] \binom{n}{n+q-m}$$

section 57 B_1

$$\sum_{k=0}^{m} \frac{q}{pk+q} \binom{pk+q}{k} \binom{pm-pk}{m-k} = \binom{mp+q}{m}.$$

section 58 $B_1 B_2$

$$\sum_{k=q}^{n-1} \frac{q}{k} \binom{2n-2k-2}{n-k-1} \binom{2k-q-1}{k-1} = \binom{2n-q-2}{n-1}.$$

section 59

$$\sum_{k=0}^{n} \frac{x}{x+kz} \binom{x+kz}{k} \frac{y}{y+(n-k)z} \binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz} \binom{x+y+nz}{n}$$

section 60

$$P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y)$$

where

$$P_n(x) = x(x+an)^{n-1}$$

is an Abel polynomial.

section 61

$$\sum_{m=0}^{n}(-1)^m\binom{2n+2m}{n+m}\binom{n+m}{n-m}=(-1)^n 2^{2n}.$$

section 62

$$\sum_{\substack{k=0\\k\text{ odd}}}^{m}\binom{2n}{2n-k}\binom{2m-2n}{m-k}=\frac{1}{2}\binom{2m}{m}+(-1)^{m+1}2^{2m-1}\binom{n-1/2}{m}$$

section 63

$$\sum_{j=0}^{n}(-1)^{n+j}\begin{bmatrix}n\\j\end{bmatrix}\begin{Bmatrix}m+j\\k\end{Bmatrix}=\frac{n!}{k!}\sum_{q=0}^{k}\binom{k}{q}\binom{q}{n}(-1)^{k-q}q^m.$$

section 64

$$[z^k]\frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n=\binom{n+2k}{k}$$

section 65

$$\sum_{k=1}^{m-1}\sin^{2q}(k\pi/m)=m\frac{1}{2^{2q}}\binom{2q}{q}+m\frac{1}{2^{2q-1}}\sum_{l=1}^{\lfloor q/m\rfloor}\binom{2q}{q-lm}(-1)^{lm}.$$

section 66 B_1

$$\sum_{j=-\lfloor n/p\rfloor}^{\lfloor n/p\rfloor}(-1)^j\binom{2n}{n-pj}$$
$$=[z^n]\left(\sum_{q=0}^{\lfloor p/2\rfloor}\frac{p}{p-q}\binom{p-q}{q}(-1)^q z^q\right)^{-1}\sum_{q=0}^{\lfloor (p-1)/2\rfloor}\binom{p-1-q}{q}(-1)^q z^q$$

section 67 B_1

$$\sum_{k=0}^{n}\binom{2k+1}{k}\binom{m-(2k+1)}{n-k}=\sum_{k=0}^{n}\binom{m+1}{k}.$$

section 68

$$\sum_{m\geq 0}m^{m+n}\frac{z^m}{m!}=\frac{1}{(1-T(z))^{2n+1}}\sum_{k=0}^{n}\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle T(z)^k$$

section 69.1 B_1, B_2

$$\sum_{l=0}^{m}(-4)^l\binom{m}{l}\binom{2l}{l}^{-1}\sum_{k=0}^{n}\frac{(-4)^k}{2k+1}\binom{n}{k}\binom{2k}{k}^{-1}\binom{k+l}{l}=\frac{1}{2n+1-2m}.$$

section 69.2 B_1

$$\sum_{l=0}^{n}\binom{n}{l}^2(x+y)^{2l}(x-y)^{2n-2l}=\sum_{l=0}^{n}\binom{2l}{l}\binom{2n-2l}{n-l}x^{2l}y^{2n-2l}.$$

section 69.3 B_2

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+c}=\binom{n+c}{c}^{-1}\frac{(-1)^c}{c}\left(1-2^{n+1}\sum_{q=0}^{c-1}\binom{n+q}{q}(-1)^q\right).$$

section 69.4 B_1

$$\sum_{j=0}^{n-k}(-1)^j\binom{2k+2j}{j}\binom{n+k+j+1}{n-k-j}=[[(n-k)\text{ is even}]]=\frac{1+(-1)^{n-k}}{2}$$

section 69.5 B_2

$$\sum_{k=0}^{b-1}\binom{a+k-1}{a-1}p^a(1-p)^k=\sum_{k=a}^{a+b-1}\binom{a+b-1}{k}p^k(1-p)^{a+b-k-1}$$

section 69.6 B_1

$$(-1)^{n+k}\begin{bmatrix}n\\k\end{bmatrix}=\sum_{j=0}^{n-k}(-1)^j\binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j}\begin{Bmatrix}n-k+j\\j\end{Bmatrix}$$

section 69.7 B_1

$$\binom{m+n}{s+1}-\binom{n}{s+1}=\sum_{q=0}^{s}\frac{m}{q+1}\binom{m+1+2q}{q}\binom{n-2-2q}{s-q}$$

section 69.8 B_1

$$\sum_{k=q}^{2n}\binom{2n+k}{2k}\frac{(2k-1)!!}{(k-q)!}(-1)^k$$

is zero when q is odd, and

$$\frac{(-1)^{n+q/2}}{2^{2n}}\frac{(2n+q)!}{(n-q/2)!\times(n+q/2)!}$$

otherwise.

section 69.9

$$\sum_{j=n}^{2n} \sum_{k=j+1-n}^{j} (-1)^j 2^{j-k} \binom{2n}{j} \left\{ {j \atop k} \right\} \left[{k \atop j+1-n} \right] = 0.$$

section 69.10

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{n-j} \binom{n-j}{j} = \binom{n+k+m}{m+1}.$$

section 69.11

With Fibonacci numbers

$$F_{2n+2} = \sum_{p=0}^{n} \sum_{q=0}^{n} \binom{n-p}{q} \binom{n-q}{p}.$$

section 69.12

$$\sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^q b^K + a^K b^q}{(a+b)^{q+K}} = 1$$

section 69.13

$$\binom{r+2n-1}{n-1} - \binom{2n-1}{n-1} = \sum_{k=1}^{n-1} \binom{2k-1}{k} \binom{r+2(n-k)-1}{r+n-k}$$

section 69.14

$$\sum_{k=1}^{n} \left(-\frac{1}{4}\right)^k \binom{2k}{k}^2 \frac{1}{1-2k} \binom{n+k-2}{2k-2}$$

is zero when n is odd and

$$\left[\left(\frac{1}{4}\right)^m \binom{2m}{m} \frac{1}{1-2m} \right]^2$$

when $n = 2m$ is even.

section 69.15

$$\sum_{n=0}^{N} \sum_{k=0}^{N} \frac{(-1)^{n+k}}{n+k+1} \binom{N}{n} \binom{N}{k} \binom{N+n}{n} \binom{N+k}{k} = \frac{1}{2N+1}.$$

section 69.16

$$\sum_{k=3}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{k-2}\binom{j(n+1)+k-3}{n-2} = (-1)^{n-1}\left[\binom{n}{2}-\binom{2n+1}{n-2}\right]$$

section 69.17

$$n\sum_{k=0}^{n}\frac{(-1)^k}{2n-k}\binom{2n-k}{k}x^k y^{2n-2k} = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n}{2k}y^{2k}(y^2-4x)^{n-k}.$$

section 69.18

$$\sum_{k=0}^{n}(-1)^k 4^{n-k}\binom{2n-k}{k} = 2n+1$$

section 69.19

$$\sum_{k=0}^{l}\binom{k}{m}\binom{k}{n} = \sum_{k=0}^{n}(-1)^k\binom{l+1}{m+k+1}\binom{l-k}{n-k}.$$

section 69.20

$$\sum_{j=0}^{k}\binom{2n}{2j}\binom{n-j}{k-j} = \frac{4^k n}{n+k}\binom{n+k}{n-k}.$$

section 69.21

$$\sum_{q=m}^{n-k}(-1)^{q-m}\binom{k-1+q}{k-1}\left\{ q \atop m \right\}\left[n \atop q+k \right] = \binom{n-1}{m}\left[n-m \atop k \right].$$

section 69.22

$$\sum_{q=0}^{N}(-1)^q\binom{2q}{q}\binom{N+q}{N-q}\frac{q^2}{(q+1)^2} = (-1)^N + \frac{1}{N(N+1)}$$

section 69.23

$$\sum_{k=0}^{n}\binom{n}{k}^2\sum_{l=0}^{k}\binom{k}{l}\binom{n}{l}\binom{2n-l}{n} = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$$

section 69.24

$$\sum_{k=1}^{a}(-1)^{a-k}\binom{a}{k}\binom{b+k}{b+1} = \binom{b}{a-1}$$

section 69.25

$$\sum_{q=0}^{k}(-1)^{q-j}\binom{n+q}{q}\binom{n+k-q}{k-q}\binom{2n}{n+j-q}=\binom{2n}{n}$$

where $0 \le j \le k$.

section 69.26

$$S_{n,m}=\sum_{k=m}^{n}\binom{k+m}{2m}\binom{2n+1}{n+k+1}=\binom{n}{m}4^{n-m}.$$

section 69.27

$$\sum_{j=0}^{k}\binom{k}{j}\binom{j/2}{n}(-1)^{n+k-j}=\frac{k}{n}(-1)^{k}2^{k-2n}\binom{2n-k-1}{n-1}$$

section 69.28

$$\sum_{k=1}^{n}(-1)^{n-k}k^{n}\binom{n+1}{n-k}=1$$

section 69.29

$$\sum_{q=a+1}^{n}\binom{q-1}{a}\binom{n-q}{k-a}=\binom{n}{k+1}$$

or alternatively

$$\sum_{q=0}^{n}\binom{q}{a}\binom{n-q}{b}=\binom{n+1}{a+b+1}.$$

section 69.30

$$\sum_{k\ge 0}\frac{(2k+1)^2}{(p+k+1)(q+k+1)}\binom{2p}{p-k}\binom{2q}{q-k}=\frac{1}{p+q+1}\binom{2p+2q}{p+q}$$

section 69.31

With

$$G_{n,j}=\sum_{k=1}^{n}\frac{k^{j}(-1)^{n-k}\binom{n}{k}}{\frac{1}{2}n(n+1)-k}$$

we have

$$G_{n,j} = \frac{(\frac{1}{2}n(n+1))^{j-1}n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)} - [[j>n]]n! \sum_{q=0}^{j-1-n} \left(\frac{1}{2}n(n+1)\right)^q \left\{ {j-1-q \atop n} \right\}$$

section 69.32

$$\sum_{p=q}^{k} (-1)^p \binom{k}{p} (q-p)^k = \sum_{p=q}^{k} \left\langle {k \atop p} \right\rangle .$$

section 69.33

$$\sum_{k=0}^{n} (-1)^k \frac{2^{n-k}\binom{n}{k}}{(m+k+1)\binom{m+k}{k}} = \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k+1}$$

section 69.34

$$\sum_{k\geq 1} \left[\binom{\lfloor \frac{k}{2} \rfloor}{m} + \binom{\lceil \frac{k}{2} \rceil}{m} \right] \binom{n-1}{k-1} = 2^{n-2m} \binom{n-m}{m-1} \frac{n+1}{m}$$

section 69.35

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^{n-k} = (1-x)^n \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! \left(\frac{x}{1-x}\right)^k$$

section 69.36

$$\sum_{k=0}^{r} k^p \binom{m}{k} \binom{n}{r-k} = \sum_{j=0}^{p} m^{\underline{j}} \binom{m+n-j}{m+n-r} \left\{ {p \atop j} \right\}$$

section 69.37

Li-Shanlan idenity:

$$\binom{m+k}{k}^2 = \sum_{q=0}^{m} \binom{k}{m-q}^2 \binom{2k+q}{q}$$

section 69.38

Two alternate representations of second order Eulerian numbers:

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{2n+1}{k-j} \left\{ {n+j \atop j} \right\} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k} (-1)^j \binom{2n+1}{j} \left[{2n-k-j+1 \atop n-k-j+1} \right]$$

section 69.39

Two alternate representations of second order Eulerian numbers:

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}\left\{\!\!\left\{ n+j \atop j \right\}\!\!\right\} = \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k+1}(-1)^{n-k-j+1}\binom{n-j}{k-1}\left[\!\!\left[n+j \atop j \right]\!\!\right]$$

section 69.40

$$\left[n \atop n-k \right] - \left\{ n \atop n-k \right\} = \sum_{j=0}^{k}\left(\binom{n+j-1}{2k} - \binom{n+k-j}{2k}\right)\left\langle\!\!\left\langle k \atop j \right\rangle\!\!\right\rangle$$

section 69.41

$$\sum_{k=0}^{2n}(-1)^k\binom{n+k}{k}^{-1}\binom{2n}{k}\binom{2k}{k} = 1$$

section 69.42

$$\sum_{k=1}^{n}\binom{2n-2k}{n-k}\frac{H_{2k}-2H_k}{2n-2k-1}\binom{2k}{k} = \frac{1}{n}\left[4^n - 3\binom{2n-1}{n}\right]$$

section 69.43

$$n! + \sum_{k=1}^{n}(-1)^k(n-k)!\sum_{q=1}^{k}2^q\binom{k-1}{q-1}\binom{n-k}{q} = [w^n]\sum_{k\geq 0}k!\left[w\frac{1-w}{1+w}\right]^k$$

section 69.44

$$\sum_{q=0}^{n}\binom{n}{q}q^k = \sum_{q=1}^{k}n^{\underline{q}}\left\{ k \atop q \right\}2^{n-q}$$

section 69.45

$$\sum_{r=0}^{n}r^k = (n+1)\sum_{q=1}^{k}n^{\underline{q}}\frac{1}{q+1}\left\{ k \atop q \right\}$$

1 Introductory example for the method (B_1)

Suppose we seek to evaluate

$$S_j(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k}{j}$$

which is claimed to be

$$(-1)^n \binom{n}{j} \binom{n+j}{j}.$$

Introduce

$$\binom{n+k}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+k}}{z^{k+1}} \; dz$$

and

$$\binom{k}{j} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^k}{w^{j+1}} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1+z)^k (1+w)^k}{z^k} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \left(1 - \frac{(1+w)(1+z)}{z}\right)^n \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} (-1-w-wz)^n \; dw \; dz$$

$$= \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} (1+w+wz)^n \; dw \; dz.$$

This is

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{j+1}} \sum_{q=0}^{n} \binom{n}{q} w^q (1+z)^q \; dw \; dz.$$

Extracting the residue at $w = 0$ we get

$$\frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \binom{n}{j} (1+z)^j \; dz$$

$$= \binom{n}{j} \frac{(-1)^n}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+j}}{z^{n+1}} \; dz$$

$$= (-1)^n \binom{n}{j} \binom{n+j}{n}.$$

thus proving the claim.
This is math.stackexchange.com problem 1331507.

2 Introductory example for the method, convergence about zero ($B_1 B_2$)

Suppose we seek to evaluate

$$\sum_{k=0}^{r} \binom{r-k}{m} \binom{s+k}{n}$$

where $n \ge s$ and $m \le r$.

Introduce

$$\binom{r-k}{m} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-k-m+1}} \frac{1}{(1-z)^{m+1}} \; dz.$$

Note that this is zero when $k > r - m$ so we may extend the sum in k to $k = \infty$.

Introduce furthermore

$$\binom{s+k}{n} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{s+k}}{w^{n+1}} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{s}}{w^{n+1}} \sum_{k\ge 0} z^k (1+w)^k \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{s}}{w^{n+1}} \frac{1}{1-(1+w)z} \; dw \; dz.$$

For the geometric series to converge we must have $|z(1+w)| < 1$, which also ensures that the inner pole is not inside the contour. Observe that $|z(1+w)| = \epsilon|1+w| \le \epsilon(1+\gamma)$. So we need to choose $1+\gamma < 1/\epsilon$ with ϵ in a neighborhood of zero. The choice $\epsilon = 1/2$ and $\gamma = 1/2$ will work.

Continuing we find

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-m+1}} \frac{1}{(1-z)^{m+2}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^{s}}{w^{n+1}} \frac{1}{1-wz/(1-z)} \; dw \; dz.$$

Extracting the inner residue we get

$$\sum_{q=0}^{n} \binom{s}{n-q} \frac{z^q}{(1-z)^q}.$$

Now

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{r-m-q+1}} \frac{1}{(1-z)^{m+q+2}} \; dz = \binom{r+1}{m+q+1}$$

which yields for the sum

$$\sum_{q=0}^{n} \binom{s}{n-q} \binom{r+1}{m+q+1}.$$

Continue by re-indexing for

$$\sum_{q=0}^{s} \binom{s}{q} \binom{r+1}{m+n-q+1}$$

where we have lowered the upper limit to s since the first binomial coefficient is zero when $q > s$.

Using

$$\binom{r+1}{m+n-q+1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{r+1}}{z^{m+n-q+2}} dz$$

we thus obtain for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{r+1}}{z^{m+n+2}} \sum_{q=0}^{s} \binom{s}{q} z^q dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{r+s+1}}{z^{m+n+2}} = \binom{s+r+1}{n+m+1}.$$

Remark. This can be done using formal power series only.

We have for the sum

$$\sum_{k=0}^{r} \binom{r-k}{m} \binom{s+k}{n} = \sum_{k=0}^{r} [z^{r-k-m}] \frac{1}{(1-z)^{m+1}} [w^n](1+w)^{s+k}$$

$$= [z^{r-m}] \frac{1}{(1-z)^{m+1}} [w^n](1+w)^s \sum_{k=0}^{r} z^k (1+w)^k.$$

Now we may certainly extend the sum to infinity as there is no contribution to the coefficient extractor when $k > r-m$ (recall that $r \ge m$) getting

$$[z^{r-m}] \frac{1}{(1-z)^{m+1}} [w^n](1+w)^s \sum_{k \ge 0} z^k (1+w)^k$$

$$= [z^{r-m}] \frac{1}{(1-z)^{m+1}} [w^n](1+w)^s \frac{1}{1-z(1+w)}$$

$$= [z^{r-m}] \frac{1}{(1-z)^{m+1}} [w^n](1+w)^s \frac{1}{1-z-wz}$$

$$= [z^{r-m}] \frac{1}{(1-z)^{m+2}} [w^n](1+w)^s \frac{1}{1-wz/(1-z)}.$$

Now with $n \ge s$ we get for the inner coefficient

$$\sum_{q=0}^{s} \binom{s}{q} \frac{z^{n-q}}{(1-z)^{n-q}}.$$

Substitute into the outer coefficient extractor to get

$$[z^{r-m}] \frac{1}{(1-z)^{m+2}} \sum_{q=0}^{s} \binom{s}{q} \frac{z^{n-q}}{(1-z)^{n-q}} = [z^{r-m}] \sum_{q=0}^{s} \binom{s}{q} \frac{z^{n-q}}{(1-z)^{n+m+2-q}}$$

$$= \sum_{q=0}^{s} \binom{s}{q} [z^{r-m-n+q}] \frac{1}{(1-z)^{n+m+2-q}} = \sum_{q=0}^{s} \binom{s}{q} \binom{r+1}{n+m+1-q}$$

$$= \sum_{q=0}^{s} \binom{s}{q} [z^{n+m+1-q}](1+z)^{r+1} = [z^{n+m+1}](1+z)^{r+1} \sum_{q=0}^{s} \binom{s}{q} z^q$$

$$= [z^{n+m+1}](1+z)^{r+1}(1+z)^s = [z^{n+m+1}](1+z)^{r+s+1} = \binom{r+s+1}{n+m+1}.$$

This was math.stackexchange.com problem 928271.

3 Introductory example for the method, an interesting substitution (B_1)

Suppose we seek to verify that

$$\sum_{q=0}^{2m} (-1)^q \binom{p-1+q}{q} \binom{2m+2p+q-1}{2m-q} 2^q = (-1)^m \binom{p-1+m}{m}.$$

Introduce

$$\binom{2m+2p+q-1}{2m-q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-q+1}} (1+z)^{2m+2p+q-1} \; dz.$$

Observe that this controls the range being zero when $q > 2m$ so we may extend q to infinity to obtain for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m+2p-1} \sum_{q\ge 0} \binom{p-1+q}{q} (-1)^q 2^q z^q (1+z)^q \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m+2p-1} \frac{1}{(1+2z(z+1))^p} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m+2p-1} \frac{1}{((1+z)^2+z^2)^p} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m+1}} (1+z)^{2m-1} \frac{1}{(1+z^2/(1+z)^2)^p} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m}} (1+z)^{2m} \frac{1}{z(1+z)} \frac{1}{(1+z^2/(1+z)^2)^p} \; dz.$$

Now put

$$\frac{z}{1+z} = u \quad \text{so that} \quad z = \frac{u}{1-u} \quad \text{and} \quad dz = \frac{1}{(1-u)^2} du$$

to obtain for the integral

$$\frac{1}{2\pi i} \int_{|u|=\epsilon} \frac{1}{u^{2m}} \frac{1}{u/(1-u) \times 1/(1-u)} \frac{1}{(1+u^2)^p} \frac{1}{(1-u)^2} \; du$$

$$= \frac{1}{2\pi i} \int_{|u|=\epsilon} \frac{1}{u^{2m+1}} \frac{1}{(1+u^2)^p} \; du.$$

This is

$$[u^{2m}] \frac{1}{(1+u^2)^p} = [v^m] \frac{1}{(1+v)^p} = (-1)^m \binom{m+p-1}{m},$$

as claimed.
This was math.stackexchange.com problem 557982.

4 Introductory example for the method, another interesting substitution (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k} (-1)^k \binom{m-2k+n-1}{n-1}$$

where $m \leq n$ and introduce

$$\binom{m-2k+n-1}{n-1} = \binom{m-2k+n-1}{m-2k}$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m-2k+1}} (1+z)^{m-2k+n-1} \; dz$$

which has the property that it is zero when $2k > m$ so we may set the upper limit in the sum to n, getting

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} (1+z)^{m+n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{z^{2k}}{(1+z)^{2k}} \; dz$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+1}}(1+z)^{m+n-1}\left(1-\frac{z^2}{(1+z)^2}\right)^n \; dz \\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+1}}(1+z)^{m-n-1}(1+2z)^n \; dz \\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^m}{z^m}\frac{1}{z(1+z)}\frac{(1+2z)^n}{(1+z)^n} \; dz.
\end{aligned}
$$

Now put

$$
\frac{1+2z}{1+z} = u \quad \text{so that} \quad z = -\frac{u-1}{u-2},\ 1+z = -\frac{1}{u-2},\ \frac{1+z}{z} = \frac{1}{u-1},
$$

$$
\frac{1}{z(1+z)} = \frac{(u-2)^2}{u-1} \quad \text{and} \quad dz = \frac{1}{(u-2)^2}\; du
$$

to get for the integral

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{|u-1|=\epsilon} \frac{1}{(u-1)^m}\frac{(u-2)^2}{u-1}u^n\frac{1}{(u-2)^2}\; du \\
&= \frac{1}{2\pi i}\int_{|u-1|=\epsilon} \frac{1}{(u-1)^{m+1}}u^n \; du.
\end{aligned}
$$

This is

$$
[(u-1)^m]u^n = [(u-1)^m]\sum_{q=0}^{n}\binom{n}{q}(u-1)^q = \binom{n}{m}.
$$

This solution is more complicated than the obvious one (which can be found at the stackexchange link) but it serves to illustrate the substitution aspect of the method.

This was math.stackexchange.com problem 1558659.

5 Introductory example for the method, yet another interesting substitution (B_2)

Suppose we seek to evaluate

$$
\sum_{k=1}^{n} k\binom{2n}{n+k}.
$$

Introduce

$$
\binom{2n}{n+k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-k+1}}\frac{1}{(1-z)^{n+k+1}}\; dz.
$$

Observe that this is zero when $k > n$ so we may extend k to infinity to obtain for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{(1-z)^{n+1}}\sum_{k\ge 1}k\frac{z^k}{(1-z)^k}\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{(1-z)^{n+1}}\frac{z/(1-z)}{(1-z/(1-z))^2}\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n}}\frac{1}{(1-z)^{n}}\frac{1}{(1-2z)^2}\;dz.$$

Now put $z(1-z)=w$ so that (observe that with $w=z+\cdots$ the image of $|z|=\epsilon$ with ϵ small is another closed circle-like contour which we may certainly deform to obtain another circle $|w|=\gamma$)

$$z=\frac{1-\sqrt{1-4w}}{2}\quad\text{and}\quad(1-2z)^2=1-4w$$

and furthermore

$$dz=-\frac{1}{2}\times\frac{1}{2}\times(-4)\times(1-4w)^{-1/2}\;dw=(1-4w)^{-1/2}\;dw$$

to get for the integral

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^n}\frac{1}{1-4w}(1-4w)^{-1/2}\;dw=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^n}\frac{1}{(1-4w)^{3/2}}\;dw.$$

This evaluates by inspection to

$$4^{n-1}\binom{n-1+1/2}{n-1}=4^{n-1}\binom{n-1/2}{n-1}=\frac{4^{n-1}}{(n-1)!}\prod_{q=0}^{n-2}(n-1/2-q)$$
$$=\frac{2^{n-1}}{(n-1)!}\prod_{q=0}^{n-2}(2n-2q-1)=\frac{2^{n-1}}{(n-1)!}\frac{(2n-1)!}{2^{n-1}(n-1)!}$$
$$=\frac{n^2}{2n}\binom{2n}{n}=\frac{1}{2}n\binom{2n}{n}.$$

Here the mapping from $z=0$ to $w=0$ determines the choice of square root. This was math.stackexchange.com problem 1585536.

Using formal power series

We may use the change of variables rule 1.8 (5) from the Egorychev text (page 16) on the integral

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{1}{z^n}\frac{1}{(1-z)^n}\frac{1}{(1-2z)^2}\;dz=\mathrm{Res}_{z=0}\frac{1}{z^n}\frac{1}{(1-z)^n}\frac{1}{(1-2z)^2}$$

with $A(z) = \frac{z}{(1-2z)^2}$ and $f(z) = \frac{1}{1-z}$. We get $h(z) = z(1-z)$ and find

$$\mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \left. \left[\frac{A(z)}{f(z)h'(z)} \right] \right|_{z=g(w)}.$$

with g the inverse of h.
This becomes

$$\mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \left. \left[\frac{z/(1-2z)^2}{(1-2z)/(1-z)} \right] \right|_{z=g(w)}$$

or alternatively

$$\mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \left. \left[\frac{z(1-z)}{(1-2z)^3} \right] \right|_{z=g(w)} = \mathrm{Res}_{w=0} \frac{1}{w^{n}} \left. \left[\frac{1}{(1-2z)^3} \right] \right|_{z=g(w)}.$$

Observe that $(1-2z)^2 = 1 - 4z + 4z^2 = 1 - 4z(1-z) = 1 - 4w$ so this is

$$\mathrm{Res}_{w=0} \frac{1}{w^n} \frac{1}{(1-4w)^{3/2}}$$

and the rest of the computation continues as before.
This was math.stackexchange.com problem 4007052.

6 Introductory example for the method, a simple telescoping sum (I)

Suppose we seek to evaluate

$$\sum_{k=0}^{n} \frac{n!}{k!}(n-k)n^k = n! n^n \sum_{k=0}^{n} \frac{n-k}{k!} n^{k-n}.$$

Introduce

$$n^{k-n} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^k}{z^{n+1}} \frac{1}{1-z/n} \; dz.$$

Observe that this integral provides an Iverson bracket, as it vanishes when $k > n$. Therefore we may extend k to infinity.

We get for the sum

$$n! n^n \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1-z/n} \sum_{k\ge 0} \frac{n-k}{k!} z^k \; dz$$

$$= n! n^n \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1-z/n} \left(n \exp(z) - z \sum_{k\ge 1} \frac{1}{(k-1)!} z^{k-1} \right) \; dz$$

$$= n!n^n \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{n}{n-z} (n\exp(z) - z\exp(z))\; dz$$
$$= n!n^{n+1} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \exp(z)\; dz = n!n^{n+1} \frac{1}{n!} = n^{n+1}.$$

This concludes the argument.
This was math.stackexchange.com problem 1805035.

7 Verifying that a certain sum vanishes (B_1)

Suppose we seek to evaluate

$$\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{n+1} \frac{1}{a+bk+1} \binom{a+bk}{m} \binom{k-n-1}{k}.$$

Now we have

$$\binom{a+bk}{m} = \sum_{q=0}^{m} (-1)^{m-q} \binom{a+bk+1}{q}$$
$$= (-1)^m + \sum_{q=1}^{m} (-1)^{m-q} \binom{a+bk+1}{q}$$

and hence

$$\frac{1}{a+bk+1} \binom{a+bk}{m} = \frac{(-1)^m}{a+bk+1} + \sum_{q=1}^{m} \frac{1}{q} (-1)^{m-q} \binom{a+bk}{q-1}.$$

Now from the first component we get in the main sum

$$\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{n+1} \frac{(-1)^m}{a+bk+1} \binom{k-n-1}{k}$$
$$= \sum_{k=0}^{n+1} \frac{1}{a+bk+1} \binom{k-n-1}{k} \sum_{m=0}^{n} \binom{n}{m} (-1)^m = 0.$$

We are thus left with the following sum:

$$\sum_{k=0}^{n+1} \binom{k-n-1}{k} \sum_{m=0}^{n} \binom{n}{m} \sum_{q=1}^{m} \frac{1}{q} (-1)^{m-q} \binom{a+bk}{q-1}.$$

Working with the inner sum we obtain

$$\sum_{m=1}^{n} \binom{n}{m} \sum_{q=1}^{m} \frac{1}{q} (-1)^{m-q} \binom{a+bk}{q-1}$$

$$= \sum_{q=1}^{n} \frac{(-1)^q}{q} \binom{a+bk}{q-1} \sum_{m=q}^{n} \binom{n}{m} (-1)^m$$

$$= \sum_{q=1}^{n} \binom{n-1}{q-1} \frac{1}{q} \binom{a+bk}{q-1}$$

$$= \frac{1}{n} \sum_{q=1}^{n} \binom{n}{q} \binom{a+bk}{q-1}.$$

Now put

$$\binom{a+bk}{q-1} = \binom{a+bk}{a+bk-q+1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{a+bk-q+2}} (1+z)^{a+bk} \; dz$$

to get

$$\frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{a+bk+2}} (1+z)^{a+bk} \sum_{q=1}^{n} \binom{n}{q} z^q \; dz$$

$$= \frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{a+bk+2}} (1+z)^{a+bk} (-1 + (1+z)^n) \; dz$$

The inner constant term does not contribute and we are left with

$$\frac{1}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{a+bk+n}}{z^{a+bk+2}} \; dz = \frac{1}{n} \binom{a+bk+n}{a+bk+1} = \frac{1}{n} \binom{a+bk+n}{n-1}.$$

Returning to the main sum we thus have

$$\frac{1}{n} \sum_{k=0}^{n+1} \binom{k-n-1}{k} \binom{a+bk+n}{n-1}$$

$$= \frac{1}{n} \sum_{k=0}^{n+1} \binom{-k}{n+1-k} \binom{a+b(n+1)+n-bk}{n-1}.$$

Note that

$$\binom{-k}{n+1-k} = \frac{1}{(n+1-k)!} \prod_{q=0}^{n-k} (-k-q) = \frac{(-1)^{n-k+1}}{(n+1-k)!} \prod_{q=0}^{n-k} (k+q)$$

$$= \frac{(-1)^{n-k+1}}{(n+1-k)!} \frac{n!}{(k-1)!} = (-1)^{n-k+1} \binom{n}{k-1}.$$

This means for the main sum

$$\frac{(-1)^{n+1}}{n} \sum_{k=1}^{n+1} \binom{n}{k-1} (-1)^k \binom{a+b(n+1)+n-bk}{n-1}$$
$$= \frac{(-1)^n}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{a+bn+n-bk}{n-1}.$$

Introduce

$$\binom{a+bn+n-bk}{n-1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} (1+z)^{a+bn+n-bk} \; dz$$

We get for the sum

$$\frac{(-1)^n}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} (1+z)^{a+bn+n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(1+z)^{bk}} \; dz$$
$$= \frac{(-1)^n}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} (1+z)^{a+bn+n} \left(1 - \frac{1}{(1+z)^b}\right)^n \; dz$$
$$= \frac{(-1)^n}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} (1+z)^{a+bn+n} \frac{((1+z)^b-1)^n}{(1+z)^{bn}} \; dz$$
$$= \frac{(-1)^n}{n} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} (1+z)^{a+n} ((1+z)^b-1)^n \; dz.$$

This is

$$\frac{(-1)^n}{n} [z^{n-1}] (1+z)^{a+n} ((1+z)^b-1)^n.$$

Note however that

$$((1+z)^b-1)^n = \left(\binom{b}{1} z + \binom{b}{2} z^2 + \cdots\right)^n = b^n z^n + \cdots$$

so there is no coefficient on $[z^{n-1}]$ because the powered term starts at z^n. Therefore the end result of the whole calculation is

$$0.$$

Remark. We have made several uses of

$$\binom{n}{m} = \sum_{q=0}^{m} (-1)^{m-q} \binom{n+1}{q}.$$

If this is not considered obvious we can prove it with the integral

$$\binom{n+1}{q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} (1+z)^{n+1} \; dz$$

to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}(1+z)^{n+1}\sum_{q=0}^{m}(-1)^{m-q}\frac{1}{z^q}\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(-1)^m}{z}(1+z)^{n+1}\frac{1-(-1/z)^{m+1}}{1+1/z}$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} (-1)^m(1+z)^{n+1}\frac{1-(-1/z)^{m+1}}{1+z}\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} (-1)^m(1+z)^{n}(1-(-1/z)^{m+1})\;dz$$

$$= -(-1)^m \times (-1)^{m+1}\binom{n}{m} = \binom{n}{m}.$$

This was math.stackexchange.com problem 1789981.

8 A case of radical cancellation (B_1, R)

Suppose we seek to show that

$$\binom{2m}{2n} = \sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{m+k}{2n}.$$

where $m \ge n$. We introduce

$$\binom{2n+1}{2k+1} = \binom{2n+1}{2n-2k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n-2k+1}}(1+z)^{2n+1}\;dz.$$

Observe that this vanishes when $k > n$ so that we may use it to control the range and extend k to infinity. We also use

$$\binom{m+k}{2n} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{2n+1}}(1+w)^{m+k}\;dw.$$

We thus obtain

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+1}}{z^{2n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^m}{w^{2n+1}}\sum_{k\ge 0} z^{2k}(1+w)^k\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+1}}{z^{2n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^m}{w^{2n+1}}\frac{1}{1-(1+w)z^2}\;dw\;dz.$$

Evalute the inner integral using the negative of the residue at the pole at

$$w = \frac{1-z^2}{z^2}$$

(residues sum to zero) as in

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+1}}{z^{2n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^m}{w^{2n+1}}\frac{1}{1-z^2-wz^2}\,dw\,dz$$

$$=-\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+1}}{z^{2n+3}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^m}{w^{2n+1}}\frac{1}{w-(1-z^2)/z^2}\,dw\,dz.$$

The negative of the residue is

$$\frac{1}{z^{2m}}\frac{z^{4n+2}}{(1-z^2)^{2n+1}}=\frac{1}{z^{2m-4n-2}}\frac{1}{(1-z^2)^{2n+1}}$$

and we obtain from the outer integral

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+1}}{z^{2n+3}}\frac{1}{z^{2m-4n-2}}\frac{1}{(1-z^2)^{2n+1}}\,dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-2n+1}}\frac{1}{(1-z)^{2n+1}}\,dz$$

$$=\binom{2m-2n+2n}{2n}=\binom{2m}{2n}.$$

This is the claim.

Remark. We also need to show that the contribution from the residue at infinity of the inner integral is zero. We get

$$\mathrm{Res}_{w=\infty}\frac{(1+w)^m}{w^{2n+1}}\frac{1}{1-(1+w)z^2}$$

$$=-\mathrm{Res}_{w=0}\frac{1}{w^2}(1+1/w)^m w^{2n+1}\frac{1}{1-z^2-z^2/w}$$

$$=-\mathrm{Res}_{w=0}(1+w)^m w^{2n-m}\frac{1}{w(1-z^2)-z^2}.$$

No contribution when $2n\ge m$. Otherwise,

$$\frac{1}{z^2}\mathrm{Res}_{w=0}(1+w)^m\frac{1}{w^{m-2n}}\frac{1}{1-w(1-z^2)/z^2}$$

$$=\frac{1}{z^2}\sum_{q=0}^{m-2n-1}\binom{m}{m-2n-1-q}\frac{(1-z^2)^q}{z^{2q}}$$

$$=\frac{1}{z^2}\sum_{q=0}^{m-2n-1}\binom{m}{2n+1+q}\left(\frac{1}{z^2}-1\right)^q$$

Combining this with the integral in z yields

$$\sum_{q=0}^{m-2n-1} \binom{m}{2n+1+q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2n+1}} \frac{1}{z^2} \sum_{p=0}^{q} \binom{q}{p} (-1)^{q-p} \frac{1}{z^{2p}} \; dz.$$

The contribution from the residue is

$$[z^{2n+2+2p}](1+z)^{2n+1} = 0.$$

We can express this verbally by saying that the term from the integral is $[z^{2n}](1+z)^{2n+1} = 0$ and the sum only contributes negative powers of z with exponent starting at two.

Remark, II. From the convergence we require that $|z^2(1+w)| < 1$ in the double integral and must choose our contours appropriately. We must also verify that $(1-z^2)/z^2$ is outside the contour $|w| = \gamma$. This is $1/z^2 - 1$ i.e. a circle of radius $1/\epsilon^2$ shifted by one to the left. Therefore when $\epsilon < 1/\sqrt{2}$ the pole is outside the contour.

This was math.stackexchange.com problem 1900578.

9 Basic usage of exponentiation integral (B_1E)

Suppose we seek to verify that

$$(-1)^p \sum_{q=r}^{p} \binom{p}{q} \binom{q}{r} (-1)^q q^{p-r} = \frac{p!}{r!}.$$

We use the integral representation

$$\binom{q}{r} = \binom{q}{q-r} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^q}{z^{q-r+1}} \; dz$$

which is zero when $q < r$ (pole vanishes) so we may extend q back to zero. We also use the integral

$$q^{p-r} = \frac{(p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{\exp(qw)}{w^{p-r+1}} \; dw.$$

We thus obtain for the sum

$$\frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} z^{r-1} \sum_{q=0}^{p} \binom{p}{q} (-1)^q \frac{(1+z)^q}{z^q} \exp(qw) \; dz \; dw$$
$$= \frac{(-1)^p (p-r)!}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{p-r+1}}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon} z^{r-1}\left(1-\frac{1+z}{z}\exp(w)\right)^p \; dz\; dw$$

$$= \frac{(-1)^p(p-r)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{p-r+1}}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p-r+1}}(-\exp(w)+z(1-\exp(w)))^p \; dz\; dw$$

$$= \frac{(p-r)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{p-r+1}}$$

$$\times \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p-r+1}}(\exp(w)+z(\exp(w)-1))^p \; dz\; dw.$$

We extract the residue on the inner integral to obtain

$$\frac{(p-r)!}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{p-r+1}}\binom{p}{p-r}\exp(rw)(\exp(w)-1)^{p-r}\; dw$$
$$= \frac{p!}{r!}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{p-r+1}}\exp(rw)(\exp(w)-1)^{p-r}\; dw.$$

It remains to compute

$$[w^{p-r}]\exp(rw)(\exp(w)-1)^{p-r}.$$

Observe that $\exp(w)-1$ starts at w so $(\exp(w)-1)^{p-r}$ starts at w^{p-r} and hence only the constant coefficient from $\exp(rw)$ contributes, the value being one, which finally yields

$$\frac{p!}{r!}.$$

This was math.stackexchange.com problem 1731648.

10 Introductory example for the method, eliminating odd-even dependence (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\binom{k}{\lfloor k/2\rfloor} = \binom{2n+1}{n}.$$

This is

$$\sum_{q=0}^{n}\binom{n}{2q}2^{n-2q}\binom{2q}{q} + \sum_{q=0}^{n}\binom{n}{2q+1}2^{n-2q-1}\binom{2q+1}{q}.$$

We treat these in turn.

First sum. Observe that

$$\binom{n}{2q}\binom{2q}{q} = \binom{n}{q}\binom{n-q}{q}.$$

This yields for the sum

$$2^n \sum_{q=0}^{n} \binom{n}{q}\binom{n-q}{q} 2^{-2q}.$$

Introduce

$$\binom{n-q}{q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-q}}{z^{q+1}}\, dz$$

which yields for the sum

$$\frac{2^n}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z} \sum_{q=0}^{n} \binom{n}{q} 2^{-2q} \frac{1}{z^q(1+z)^q}\, dz$$

$$= \frac{2^n}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z} \left(1+\frac{1}{4z(1+z)}\right)^n dz$$

$$= \frac{2^{-n}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+2z)^{2n}}{z^{n+1}}\, dz = 2^{-n}\binom{2n}{n}2^n = \binom{2n}{n}.$$

Second sum. Observe that

$$\binom{n}{2q+1}\binom{2q+1}{q} = \binom{n}{q}\binom{n-q}{q+1}.$$

This yields for the sum

$$2^{n-1} \sum_{q=0}^{n} \binom{n}{q}\binom{n-q}{q+1} 2^{-2q}.$$

This time introduce

$$\binom{n-q}{q+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-q}}{z^{q+2}}\, dz$$

which yields for the sum

$$\frac{2^{n-1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^2} \sum_{q=0}^{n} \binom{n}{q} 2^{-2q} \frac{1}{z^q(1+z)^q}\, dz$$

$$= \frac{2^{n-1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^2} \left(1+\frac{1}{4z(1+z)}\right)^n dz$$

$$= \frac{2^{-n-1}}{2\pi i}\int_{|z|=\epsilon} \frac{(1+2z)^{2n}}{z^{n+2}}\, dz = 2^{-n-1}\binom{2n}{n+1}2^{n+1} = \binom{2n}{n+1}.$$

Conclusion.
Collecting the two contributions we obtain

$$\binom{2n}{n} + \binom{2n}{n+1} = \binom{2n+1}{n}$$

as claimed.
This was math.stackexchange.com problem 1442436.

11 Introductory example for the method, proving equality of two double hypergeometrics (B_1)

Suppose we seek to verify that $f_1(n,k) = f_2(n,k)$ where

$$f_1(n,k) = \sum_{v=0}^{n} \frac{(2k+2v)!}{(k+v)! \times v! \times (2k+v)! \times (n-v)!} 2^{-v}$$

and

$$f_2(n,k) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{(k+m)! \times m! \times (n-2m)!} 2^{n-4m}.$$

Multiplying by $(n+k)!$ we obtain

$$g_1(n,k) = \sum_{v=0}^{n} \binom{n+k}{n-v} \binom{2k+2v}{v} 2^{-v}$$

and

$$g_2(n,k) = 2^n \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+k}{m} \binom{n+k-m}{n-2m} 2^{-4m}.$$

We will work with the latter two. Re-write the first sum as follows:

$$2^{-n} \sum_{v=0}^{n} \binom{n+k}{v} \binom{2k+2n-2v}{n-v} 2^{v}$$

Introduce

$$\binom{2k+2n-2v}{n-v} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-v+1}} (1+z)^{2k+2n-2v} \; dz.$$

This integral is zero when $v > n$ so we may extend v to infinity.
We get for $g_1(n,k)$

$$2^{-n}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2k+2n}\sum_{v\ge 0}\binom{n+k}{v}\frac{z^v}{(1+z)^{2v}}2^v\;dz$$

$$= 2^{-n}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2k+2n}\left(1+2\frac{z}{(1+z)^2}\right)^{n+k}\;dz$$

$$= 2^{-n}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\left(1+4z+z^2\right)^{n+k}\;dz.$$

For the second sum introduce

$$\binom{n+k-m}{n-2m} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-2m+1}}(1+z)^{n+k-m}\;dz.$$

This is zero when $2m > n$ so we may extend m to infinity.
We get for $g_2(n,k)$

$$2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{n+k}\sum_{m\ge 0}\binom{n+k}{m}\frac{z^{2m}}{(1+z)^m}2^{-4m}\;dz$$

$$= 2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{n+k}\left(1+\frac{1}{16}\frac{z^2}{1+z}\right)^{n+k}\;dz$$

$$= 2^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\left(1+z+\frac{1}{16}z^2\right)^{n+k}\;dz.$$

Finally put $z = 4w$ in this integral to get

$$2^n\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{4^{n+1}w^{n+1}}\left(1+4w+w^2\right)^{n+k}\;4dw$$

$$= 2^{-n}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+1}}\left(1+4w+w^2\right)^{n+k}\;dw.$$

This concludes the argument.
This was math.stackexchange.com problem 924966.

12 A remarkable case of factorization (B_1)

We clear up some ambiguities in the post and prove it by strong induction. We let $T(0) = 0$ and $T(1) = 1$ and prove that when

$$T(n) = \sum_{k=1}^{\lfloor n/2\rfloor}(-1)^{k+1}\binom{n-k}{k}T(n-k)$$

for $n \ge 2$ then

$$T(n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} = \binom{2n-2}{n-1} - \binom{2n-2}{n}.$$

In fact the case of a zero argument to T is not reached as for $n \geq 2$ we also have $n - \lfloor n/2 \rfloor \geq 1$. Applying the induction hypothesis on the RHS we get two pieces, the first is

$$A = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k-1}$$
$$= \binom{2n-2}{n-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k-1}$$

and the second

$$B = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k}$$
$$= \binom{2n-2}{n} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{k+1} \binom{n-k}{k} \binom{2n-2k-2}{n-k}.$$

As we subtract B from A we see that we only need to show that the contribution from the two sum terms is zero.

For these two pieces we introduce the integral representation

$$\binom{n-k}{k} = \binom{n-k}{n-2k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-2k+1}} (1+z)^{n-k} \, dz.$$

This has the nice property that it vanishes when $k > \lfloor n/2 \rfloor$ so we may extend the upper limit of the sum to infinity. We also introduce for the first sum

$$\binom{2n-2k-2}{n-k-1} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n-k}} (1+w)^{2n-2k-2} \, dw.$$

We thus obtain

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} (1+w)^{2n-2}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^n \sum_{k \geq 0} (-1)^{k+1} \frac{z^{2k} w^k}{(1+z)^k (1+w)^{2k}} \, dz \, dw$$
$$= -\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} (1+w)^{2n-2}$$
$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^n \frac{1}{1 + z^2 w/(1+z)/(1+w)^2} \, dz \, dw$$
$$= -\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^n} (1+w)^{2n}$$

$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+1} \frac{1}{(1+z)(1+w)^2 + z^2 w} \; dz \; dw$$

$$= -\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} (1+w)^{2n}$$

$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+1} \frac{1}{z+1+w} \frac{1}{z+(1+w)/w} \; dz \; dw.$$

We evaluate the inner integral by summing the residues at $z = -(1+w)$ and $z = -(1+w)/w$ and flipping the sign. (We will verify that the residue at infinity is zero.)

The residue at $z = -(1+w)$ yields

$$-\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} (1+w)^{2n}$$

$$\times \frac{(-1)^{n+1}}{(1+w)^{n+1}} (-1)^{n+1} w^{n+1} \frac{1}{-(1+w) + (1+w)/w} \; dw$$

$$= -\frac{1}{2\pi i} \int_{|w|=\gamma} (1+w)^{n-1} \frac{w}{1-w^2} \; dw.$$

This is zero as the pole at zero has been canceled. Next for the residue at $z = -(1+w)/w$ we get

$$-\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} (1+w)^{2n}$$

$$\times \frac{(-1)^{n+1} w^{n+1}}{(1+w)^{n+1}} (-1)^{n+1} \frac{1}{w^{n+1}} \frac{1}{-(1+w)/w + 1 + w} \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n+1}} (1+w)^{n-1} \frac{w}{1-w^2} dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{n}} (1+w)^{n-2} \frac{1}{1-w} dw.$$

With $n \ge 2$ we can evaluate this as

$$\sum_{q=0}^{n-1} \binom{n-2}{q} = 2^{n-2}.$$

To wrap up the residue at infinity of the inner integral is

$$\mathrm{Res}_{z=\infty} \frac{1}{z^{n+1}} (1+z)^{n+1} \frac{1}{z+1+w} \frac{1}{z+(1+w)/w}$$

$$= -\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n+1} \frac{(1+z)^{n+1}}{z^{n+1}} \frac{1}{1/z+1+w} \frac{1}{1/z+(1+w)/w}$$

$$= -\mathrm{Res}_{z=0}(1+z)^{n+1}\frac{1}{1+z(1+w)}\frac{1}{1+z(1+w)/w} = 0.$$

Collecting everything and flipping the sign we have shown that

$$A = -2^{n-2}.$$

For piece B we see that it only differs from A in an extra $1/w$ factor on the extractor in w at the front. We thus obtain

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+2}}(1+w)^{2n}$$
$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{z+1+w}\frac{1}{z+(1+w)/w}\;dz\;dw.$$

The residue at $z = -(1+w)$ vanishes the same because there was an extra w to spare on the $w/(1-w^2)$ term:

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{n-1}\frac{1}{1-w^2}\;dw.$$

For the residue at $z = -(1+w)/w$ we are now extracting from

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}(1+w)^{n-2}\frac{1}{1-w}dw.$$

to get

$$\sum_{q=0}^{n}\binom{n-2}{q} = 2^{n-2}$$

as before. The residue at infinity vanished in z and did not reach the front extractor in w, for another contribution of zero. This means that

$$B = -2^{n-2}$$

and we may conclude the proof. The fact that the sum term from the geometric series factored as it did is the remarkable feature of this problem.

This was math.stackexchange.com problem 2113830

13 Evaluating a quadruple hypergeometric(B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{n}\sum_{l=0}^{n}(-1)^{k+l}\binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{k}\binom{n}{l}$$
$$= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{l=0}^{n}(-1)^l\binom{n+k-l}{n}\binom{k+l}{n}\binom{n}{l}.$$

Evaluate the inner sum first and introduce

$$\binom{n+k-l}{n} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+k-l}}{z^{n+1}}\; dz.$$

and

$$\binom{k+l}{n} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{k+l}}{w^{n+1}}\; dw.$$

This yields for the inner sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+k}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{k}}{w^{n+1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{(1+w)^l}{(1+z)^l}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+k}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{k}}{w^{n+1}} \left(1 - \frac{1+w}{1+z}\right)^n\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{k}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{k}}{w^{n+1}} (z-w)^n\; dw\; dz.$$

Extracting the inner coefficient yields

$$\sum_{q=0}^{n} \binom{k}{q} \binom{n}{n-q} (-1)^{n-q} z^q.$$

The outer coefficient becomes

$$\sum_{q=0}^{n} \binom{k}{q} \binom{n}{n-q} (-1)^{n-q} \binom{k}{n-q}$$

$$= \sum_{q=0}^{n} \binom{k}{q} \binom{n}{q} (-1)^{n-q} \binom{k}{n-q}.$$

Call this S. By symmetry we have on re-indexing that

$$2S = \sum_{q=0}^{n} \binom{k}{q} \binom{n}{q} ((-1)^q + (-1)^{n-q}) \binom{k}{n-q}$$

$$= (1+(-1)^n) \sum_{q=0}^{n} \binom{k}{q} \binom{n}{q} (-1)^q \binom{k}{n-q}.$$

This is zero when n is odd so *the entire sum being evaluated vanishes when n is odd* and we may assume that $n = 2m$ and get

$$\sum_{q=0}^{2m} \binom{k}{q} \binom{2m}{q} (-1)^q \binom{k}{2m-q}.$$

Substituting this into the outer sum yields

$$\sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k \binom{k}{q} \binom{k}{2m-q}.$$

We evaluate the inner sum with the integrals

$$\binom{k}{q} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^k}{z^{q+1}} \; dz.$$

and

$$\binom{k}{2m-q} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^k}{w^{2m-q+1}} \; dw$$

to get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k (1+z)^k (1+w)^k \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} (z+w+wz)^{2m} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2m-q+1}} (w(1+z)+z)^{2m} \; dw \; dz.$$

Extracting the coefficient we get for the inner term

$$\binom{2m}{2m-q} (1+z)^{2m-q} z^q$$

and for the outer integral

$$\binom{2m}{2m-q} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z} (1+z)^{2m-q} \; dz = \binom{2m}{2m-q}.$$

We are now ready to conclude and return to the main sum which has been transformed into

$$\sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \binom{2m}{2m-q}$$

which is

$$[v^{2m}](1-v)^{2m}(1+v)^{2m} = [v^{2m}](1-v^2)^{2m} = [v^m](1-v)^{2m}$$

$$= (-1)^m \binom{2m}{m}.$$

This was math.stackexchange.com problem 1577907.

14 An integral representation of a binomial coefficient involving the floor function (B_1)

Suppose we seek to prove that

$$\sum_{k=0}^{2m+1} \binom{n}{k} 2^k \binom{n-k}{\lfloor (2m+1-k)/2 \rfloor} = \binom{2n+1}{2m+1}.$$

Observe that from first principles we have that

$$\binom{n}{\lfloor q/2 \rfloor} = \binom{n}{n - \lfloor q/2 \rfloor} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \left(1 + z + wz^2 + wz^3 + w^2 z^4 + w^2 z^5 + \cdots \right) \; dw \; dz.$$

This simplifies to

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \left(\frac{1}{1-wz^2} + z \frac{1}{1-wz^2} \right) \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{q+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \; dw \; dz.$$

This correctly enforces the range as the reader is invited to verify and we may extend k beyond $2m+1$, getting for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \sum_{k \ge 0} \binom{n}{k} 2^k z^k \frac{w^k}{(1+w)^k} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^n}{w^{n+1}} \frac{1}{1-wz^2} \left(1 + \frac{2wz}{1+w}\right)^n \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1+z}{z^{2m+2}} \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{(1+w+2wz)^n}{w^{n+1}} \frac{1}{1-wz^2} \; dw \; dz.$$

Extracting the inner coefficient now yields

$$\sum_{q=0}^{n} \binom{n}{q} (1+2z)^q z^{2n-2q} = z^{2n} \sum_{q=0}^{n} \binom{n}{q} (1+2z)^q z^{-2q}$$

$$= z^{2n} \left(1 + \frac{1+2z}{z^2}\right)^n = (1+z)^{2n}.$$

We thus get from the outer coefficient

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z^{2m+2}}\; dz$$

which is

$$\binom{2n+1}{2m+1}$$

as claimed. I do believe this is an instructive exercise.
This was math.stackexchange.com problem 2087559

15 Evaluating another quadruple hypergeometric(B_1)

Suppose we seek to verify that

$$\sum_{k=m}^{n} (-1)^{n+k} \frac{2k+1}{n+k+1} \binom{n}{k} \binom{n+k}{k}^{-1} \binom{k}{m} \binom{k+m}{m} = \delta_{mn}.$$

Here we may assume $n \ge m$, the equality holds trivially otherwise.
Now we have

$$\binom{n}{k}\binom{n+k}{k}^{-1} = \frac{n!}{k!(n-k)}\frac{k!n!}{(n+k)!}$$
$$= \frac{n!}{(n-k)}\frac{n!}{(n+k)!} = \binom{2n}{n+k}\binom{2n}{n}^{-1}.$$

We get for the sum

$$\sum_{k=m}^{n} (-1)^{n+k} \frac{2k+1}{n+k+1} \binom{2n}{n+k} \binom{k}{m} \binom{k+m}{m} = \delta_{mn} \times \binom{2n}{n}.$$

which is

$$\sum_{k=m}^{n} (-1)^{n+k} (2k+1) \binom{2n+1}{n+k+1} \binom{k}{m} \binom{k+m}{m}$$
$$= \delta_{mn} \times (2n+1) \times \binom{2n}{n}.$$

Introduce

$$\binom{2n+1}{n+k+1} = \binom{2n+1}{n-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-k+1}} (1+z)^{2n+1}\; dz.$$

Observe that this vanishes when $k > n$ so we may extend k upward to infinity.

Furthermore introduce

$$\binom{k}{m} = \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} (1+w)^k \; dw.$$

Observe once again that the integral vanishes, this time when $0 \le k < m$ so we may extend k back to zero.

We thus get for the sum

$$(-1)^n \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2n+1}$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \sum_{k\ge 0} (-1)^k (2k+1) \binom{k+m}{m} z^k (1+w)^k \; dw \; dz.$$

The inner sum yields two pieces, the first is

$$\sum_{k\ge 0} (-1)^k \binom{k+m}{m} z^k (1+w)^k = \frac{1}{(1+z+wz)^{m+1}}$$
$$= \frac{1}{(1+z)^{m+1}} \frac{1}{(1+wz/(1+z))^{m+1}}.$$

On extracting the residue for the integral in w we obtain

$$(-1)^n \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{2n+1}$$
$$\times \frac{1}{(1+z)^{m+1}} \binom{2m}{m} (-1)^m \frac{z^m}{(1+z)^m} \; dz$$
$$= \binom{2m}{m} (-1)^{n+m} \int_{|z|=\epsilon} \frac{1}{z^{n-m+1}} (1+z)^{2n-2m} \; dz$$
$$= \binom{2m}{m} (-1)^{n+m} \binom{2n-2m}{n-m}.$$

The second piece from the sum is

$$2 \sum_{k\ge 1} (-1)^k k \binom{k+m}{m} z^k (1+w)^k.$$

Write

$$k \binom{k+m}{m} = \frac{(k+m)!}{(k-1)!m!} = (m+1) \frac{(k+m)!}{(k-1)!(m+1)!}$$
$$= (m+1) \binom{k+m}{m+1}$$

to get for the sum

$$2(m+1)z(1+w)\sum_{k\geq 1}(-1)^k\binom{k+m}{m+1}z^{k-1}(1+w)^{k-1}$$
$$= -2(m+1)z(1+w)\frac{1}{(1+z+wz)^{m+2}}$$
$$= -2(m+1)z(1+w)\frac{1}{(1+z)^{m+2}}\frac{1}{(1+wz/(1+z))^{m+2}}.$$

Here we get two pieces, the first is

$$-2(m+1)(-1)^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{z}{z^{n+1}}(1+z)^{2n+1}$$
$$\times\frac{1}{(1+z)^{m+2}}\binom{2m+1}{m}(-1)^m\frac{z^m}{(1+z)^m}\ dz$$
$$= -2(m+1)\binom{2m+1}{m}(-1)^{n+m}\int_{|z|=\epsilon}\frac{1}{z^{n-m}}(1+z)^{2n-2m-1}\ dz$$

We have two cases, we get zero when $n = m$ and when $n > m$ we have

$$-2(m+1)\binom{2m+1}{m}(-1)^{n+m}\binom{2n-2m-1}{n-m-1}.$$

The second piece is

$$-2(m+1)(-1)^n\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{z}{z^{n+1}}(1+z)^{2n+1}$$
$$\times\frac{1}{(1+z)^{m+2}}\binom{2m}{m-1}(-1)^{m-1}\frac{z^{m-1}}{(1+z)^{m-1}}\ dz$$
$$= 2(m+1)\binom{2m}{m-1}(-1)^{n+m}\int_{|z|=\epsilon}\frac{1}{z^{n-m+1}}(1+z)^{2n-2m}\ dz$$
$$= 2(m+1)\binom{2m}{m-1}(-1)^{n+m}\binom{2n-2m}{n-m}.$$

Therefore when $n = m$ we get

$$\binom{2n-2m}{n-m}(-1)^{m+n}\left(2(m+1)\binom{2m}{m-1}+\binom{2m}{m}\right).$$

This simplifies to

$$(-1)^{2m}\left(2(m+1)\binom{2m}{m-1}+\binom{2m}{m}\right)$$
$$= 2m\binom{2m}{m}+\binom{2m}{m} = (2m+1)\binom{2m}{m}.$$

This is precisely the claim we were trying to prove. On the other hand when $n > m$ we obtain

$$\binom{2n-2m}{n-m}(-1)^{m+n}$$
$$\times\left(2(m+1)\binom{2m}{m-1}+\binom{2m}{m}-2(m+1)\binom{2m+1}{m}\frac{n-m}{2n-2m}\right).$$

The factor is

$$(2m+1)\binom{2m}{m}-(m+1)\binom{2m+1}{m}=0.$$

This concludes the argument.

Remark. For $n = m$ we could have evaluated the single term in the initial sum by expanding the four binomial coefficients and assumed $n > m$ thereafter.

This was math.stackexchange.com problem 1817122.

16 An identity by Strehl (B_1)

Suppose we seek to show that

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{k=\lceil n/2\rceil}^{n}\binom{n}{k}^2\binom{2k}{n}.$$

With

$$\binom{n}{k}\binom{2k}{n}=\frac{(2k)!}{k!\times(n-k)!\times(2k-n)!}=\binom{2k}{k}\binom{k}{n-k}$$

we find that the RHS is

$$\sum_{k=\lceil n/2\rceil}^{n}\binom{n}{k}\binom{2k}{k}\binom{k}{n-k}.$$

Introduce

$$\binom{2k}{k}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2k}}{z^{k+1}}\;dz$$

and (this integral is zero when $0 \le k < \lceil n/2\rceil$)

$$\binom{k}{n-k}=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{k}}{w^{n-k+1}}\;dw$$

to get for the RHS

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{w^k(1+w)^k(1+z)^{2k}}{z^k}\;dw\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}\left(1+\frac{w(1+w)(1+z)^2}{z}\right)^n \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(z+w(1+w)(1+z)^2)^n \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}(z+w(z+1))^n(1+w(z+1))^n \; dw\; dz.$$

Extracting first the residue in w in next the residue in z we get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}\sum_{q=0}^n {n\choose q} z^{n-q}(1+z)^q {n\choose n-q}(1+z)^{n-q}\; dz$$

$$= \sum_{q=0}^n {n\choose q}^2 \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{q+1}}\; dz$$

$$= \sum_{q=0}^n {n\choose q}^3$$

QED.

Addendum May 27 2018. We compute this using formal power series as per request in comment. Start from

$${2k\choose k} = [z^k](1+z)^{2k}$$

and

$${k\choose n-k} = [w^{n-k}](1+w)^k.$$

Observe that this coefficient extractor is zero when $n-k>k$ or $k<\lceil n/2\rceil$ where $k\ge 0$. Hence we are justified in lowering k to zero when we substitute these into the sum and we find

$$\sum_{k=0}^n {n\choose k} [z^k](1+z)^{2k}[w^{n-k}](1+w)^k$$

$$= [z^0][w^n]\sum_{k=0}^n {n\choose k}\frac{1}{z^k}(1+z)^{2k} w^k (1+w)^k$$

$$= [z^0][w^n]\left(1+\frac{(1+z)^2 w(1+w)}{z}\right)^n$$

$$= [z^n][w^n](z+(1+z)^2 w(1+w))^n$$

$$= [z^n][w^n](1+w(1+z))^n(z+w(1+z))^n.$$

We extract the coefficient on $[w^n]$ then the one on $[z^n]$ and get

$$[z^n]\sum_{q=0}^n \binom{n}{q}(1+z)^q \binom{n}{n-q}(1+z)^{n-q} z^q$$

$$= \sum_{q=0}^n \binom{n}{q}^2 [z^{n-q}](1+z)^n = \sum_{q=0}^n \binom{n}{q}^2 \binom{n}{n-q} = \sum_{q=0}^n \binom{n}{q}^3.$$

The claim is proved.

This was math.stackexchange.com problem 586138.

17 Shifting the index variable and applying Leibniz' rule (B_1)

We seek to simplify

$$\sum_s \binom{n+s}{k+l}\binom{k}{s}\binom{l}{s}.$$

The substitution $s = t + k + l - n$ yields

$$\sum_t \binom{t+k+l}{k+l}\binom{k}{t+k+l-n}\binom{l}{t+k+l-n}.$$

Working with the assumption that the parameters are positive integers we find that from the first binomial coefficient we get that for it to be non-zero we must have $t \ge 0$ or $t < -(k+l)$. Note however that in the latter case the two remaining coefficients vanish, which leaves $t \ge 0$. Re-writing we find

$$\sum_{t\ge 0} \binom{t+k+l}{k+l}\binom{k}{n-l-t}\binom{l}{n-k-t}.$$

We introduce integral represenations for the two right coefficients that also enforce the fact that $t \le n-l$ and $t \le n-k$ so that we may then let t range to infinity. We use

$$\binom{k}{n-l-t} = \frac{1}{2\pi i}\int_{|z|=\epsilon_1} \frac{1}{z^{n-l-t+1}}(1+z)^k \; dz$$

as well as

$$\binom{l}{n-k-t} = \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k-t+1}}(1+v)^l \; dv.$$

We then get for the sum (no convergence issues here)

$$\frac{1}{2\pi i}\int_{|z|=\epsilon_1} \frac{1}{z^{n-l+1}}(1+z)^k \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k+1}}(1+v)^l \sum_{t\ge 0}\binom{k+l+t}{k+l} v^t z^t \; dv \; dz.$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon_1} \frac{1}{z^{n-l+1}}(1+z)^k \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k+1}}(1+v)^l \frac{1}{(1-vz)^{k+l+1}}\, dv\, dz.$$

We see that this vanishes when $n < k$ or $n < l$, which we label case A. Case B is that $n \geq k, l$. We evaluate the inner integral using the fact that residues sum to zero. With this in mind we write

$$\frac{(-1)^{k+l+1}}{2\pi i}\int_{|z|=\epsilon_1} \frac{1}{z^{n+k+2}}(1+z)^k \frac{1}{2\pi i}\int_{|v|=\epsilon_2} \frac{1}{v^{n-k+1}}(1+v)^l \frac{1}{(v-1/z)^{k+l+1}}\, dv\, dz.$$

We thus require for the pole at $v = 1/z$

$$\frac{1}{(k+l)!}\left(\frac{1}{v^{n-k+1}}(1+v)^l\right)^{(k+l)}$$

which is (apply Leibniz)

$$\frac{1}{(k+l)!}\sum_{q=0}^{k+l}\binom{k+l}{q}(-1)^q\binom{n-k+q}{q}\frac{q!}{v^{n-k+1+q}}$$
$$\times\binom{l}{k+l-q}(k+l-q)!(1+v)^{l-(k+l-q)}$$
$$= \sum_{q=0}^{k+l}(-1)^q\binom{n-k+q}{q}\frac{1}{v^{n-k+1+q}}\binom{l}{k+l-q}(1+v)^{q-k}.$$

Evaluate at $v = 1/z$ to get

$$\sum_{q=0}^{k+l}(-1)^q\binom{n-k+q}{q}z^{n-k+1+q}\binom{l}{k+l-q}\frac{(1+z)^{q-k}}{z^{q-k}}.$$

Substituting this into the integral in z and flipping the sign yields

$$(-1)^{k+l}\sum_{q=0}^{k+l}(-1)^q\binom{n-k+q}{q}\binom{l}{k+l-q}\binom{q}{k}.$$

Now we have

$$\binom{q}{k}\binom{n-k+q}{q} = \frac{(n-k+q)!}{k!\times(q-k)!\times(n-k)!} = \binom{n}{k}\binom{n-k+q}{n}$$

and we obtain

$$(-1)^{k+l}\binom{n}{k}\sum_{q=0}^{k+l}(-1)^q\binom{l}{k+l-q}\binom{n-k+q}{n}$$

$$= \binom{n}{k} \sum_{q=0}^{k+l} (-1)^q \binom{l}{q} \binom{n+l-q}{n}$$

$$= \binom{n}{k} [w^n] \sum_{q=0}^{k+l} (-1)^q \binom{l}{q} (1+w)^{n+l-q}$$

$$= \binom{n}{k} [w^n] (1+w)^{n+l} \sum_{q=0}^{k+l} (-1)^q \binom{l}{q} \frac{1}{(1+w)^q}$$

$$= \binom{n}{k} [w^n] (1+w)^{n+l} \left(1 - \frac{1}{1+w}\right)^l$$

$$= \binom{n}{k} [w^n] w^l (1+w)^n = \binom{n}{k} \binom{n}{n-l} = \binom{n}{k} \binom{n}{l}.$$

This is the claim, which we proved for case B.

Remark. To be perfectly rigorous we also need to show that the contribution from the residue at infinity is zero. We find

$$\mathrm{Res}_{v=\infty} \frac{1}{v^{n-k+1}} (1+v)^l \frac{1}{(1-vz)^{k+l+1}}$$

$$= -\mathrm{Res}_{v=0} \frac{1}{v^2} v^{n-k+1} \frac{(1+v)^l}{v^l} \frac{1}{(1-z/v)^{k+l+1}}$$

$$= -\mathrm{Res}_{v=0} \frac{1}{v^2} v^{n-k-l+1} (1+v)^l \frac{v^{k+l+1}}{(v-z)^{k+l+1}}$$

$$= -\mathrm{Res}_{v=0} v^n (1+v)^l \frac{1}{(v-z)^{k+l+1}} = 0$$

and the check goes through.

This was math.stackexchange.com problem 2381429.

18 Working with negative indices (B_1)

Suppose we seek to prove that

$$\sum_{k=-\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k \binom{2n}{n+3k} = 2 \times 3^{n-1}.$$

We start by introducing the integral

$$\binom{2n}{n+3k} = \binom{2n}{n-3k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-3k+1}} (1+z)^{2n} \; dz.$$

Observe that this vanishes for $3k > n$ (pole canceled) and for $3k < -n$ (upper range of polynomial term exceeded) so we may extend the summation to $[-n, n]$ getting

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2n}\sum_{k=-n}^{n}(-1)^k z^{3k}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2n}(-1)^n z^{-3n}\sum_{k=0}^{2n}(-1)^k z^{3k}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{4n+1}}(1+z)^{2n}(-1)^n\frac{1-(-1)^{2n+1}z^{3(2n+1)}}{1+z^3}\;dz.$$

Only the first piece from the difference due to the geometric series contributes and we get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{4n+1}}(1+z)^{2n}(-1)^n\frac{1}{1+z^3}\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{4n+1}}(1+z)^{2n-1}(-1)^n\frac{1}{1-z+z^2}\;dz.$$

We have two poles other than zero and infinity at ρ and $1/\rho$ where

$$\rho=\frac{1+\sqrt{3}i}{2}$$

and using the fact that residues sum to zero we obtain

$$S+\frac{(-1)^n}{\rho(1+\rho)}\frac{1}{\rho-1/\rho}\left(\frac{(1+\rho)^2}{\rho^4}\right)^n+\frac{(-1)^n}{1/\rho(1+1/\rho)}\frac{1}{1/\rho-\rho}\left(\frac{(1+1/\rho)^2}{1/\rho^4}\right)^n$$

$$+\mathrm{Res}_{z=\infty}\frac{1}{z^{4n+1}}(1+z)^{2n-1}(-1)^n\frac{1}{1-z+z^2}=0.$$

We get for the residue at infinity

$$-\mathrm{Res}_{z=0}\frac{1}{z^2}z^{4n+1}(1+1/z)^{2n-1}(-1)^n\frac{1}{1-1/z+1/z^2}$$

$$=-\mathrm{Res}_{z=0}z^{2n+2}(1+z)^{2n-1}(-1)^n\frac{1}{z^2-z+1}=0.$$

Now if $z^2=z-1$ then $z^4=z^2-2z+1=-z$ and thus

$$\frac{(1+1/\rho)^2}{1/\rho^4}=\frac{(1+\rho)^2}{\rho^4}=\frac{\rho-1+2\rho+1}{-\rho}=-3$$

and furthermore with $z(1+z)(z-1/z)=(1+z)(z^2-1)$ and $(1+z)(z-2)=z^2-z-2=-3$ we finally get

$$S+(-1)^n\times\left(-\frac{1}{3}\right)(-3)^n+(-1)^n\times\left(-\frac{1}{3}\right)(-3)^n=0$$

or

$$S = 2 \times 3^{n-1}.$$

This was math.stackexchange.com problem 2054777.

19 Mixing the two types of binomial integrals ($B_1 B_2$)

Suppose we seek to verify that

$$\sum_{j=0}^{b} \binom{b}{j}^2 \binom{n+j}{2b} = \binom{n}{b}^2.$$

where $0 \le b \le n$.

Introduce

$$\binom{b}{j} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{b-j+1}} \frac{1}{(1-z)^{j+1}} \; dz$$

and

$$\binom{n+j}{2b} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2b+1}} (1+w)^{n+j} \; dw.$$

This yields for the sum

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^n}{w^{2b+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{b+1}} \frac{1}{1-z} \sum_{j=0}^{b} \binom{b}{j} (1+w)^j \frac{z^j}{(1-z)^j} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^n}{w^{2b+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{b+1}} \frac{1}{1-z} \left(1 + \frac{(1+w)z}{1-z}\right)^b \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^n}{w^{2b+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{b+1}} \frac{1}{(1-z)^{b+1}} (1+wz)^b \; dz \; dw.$$

The inner residue is

$$\sum_{q=0}^{b} \binom{b}{q} w^q \binom{b-q+b}{b}.$$

Substitute this into the outer integral to get

$$\sum_{q=0}^{b} \binom{b}{q} \binom{2b-q}{b} \binom{n}{2b-q}.$$

Observe that

$$\binom{2b-q}{b} \binom{n}{2b-q} = \frac{(2b-q)!}{b!(b-q)!} \frac{n!}{(2b-q)!(n-2b+q)!}$$

$$= \frac{(n-b)!}{b!(b-q)!} \frac{n!}{(n-b)!(n-2b+q)!} = \binom{n}{b}\binom{n-b}{b-q}.$$

This yields for the sum

$$\binom{n}{b} \sum_{q=0}^{b} \binom{b}{q}\binom{n-b}{b-q}.$$

which evaluates to

$$\binom{n}{b}^2$$

by inspection.

It can also be done with the integral

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{b-q+1}} (1+z)^{n-b} \; dz$$

which yields

$$\binom{n}{b} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{b+1}} (1+z)^{n-b} \sum_{q=0}^{b} \binom{b}{q} z^q \; dz$$

$$= \binom{n}{b} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{b+1}} (1+z)^{n} \; dz = \binom{n}{b}^2.$$

This was math.stackexchange.com problem 1234156.

A more general version of this identity is at section 41.

20 Two companion identities by Gould (B_1)

Suppose we seek to evaluate

$$Q(x,\rho) = \sum_{k=0}^{\rho} \binom{2x+1}{2k}\binom{x-k}{\rho-k}$$

where $x \geq \rho$.

Introduce

$$\binom{x-k}{\rho-k} = \binom{x-k}{x-\rho} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}} (1+z)^{x-k} \; dz.$$

Note that this controls the range being zero when $\rho < k \leq x$ so we can extend the sum to x supposing that $x > \rho$. And when $x = \rho$ we may also set the upper limit to x.

We get for the sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{x-\rho+1}} (1+z)^{x} \sum_{k=0}^{x} \binom{2x+1}{2k} \frac{1}{(1+z)^k} \; dz.$$

This is

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}(1+z)^x\left(\left(1+\frac{1}{\sqrt{1+z}}\right)^{2x+1}+\left(1-\frac{1}{\sqrt{1+z}}\right)^{2x+1}\right)\,dz$$

$$=\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}\frac{1}{\sqrt{1+z}}\left((1+\sqrt{1+z})^{2x+1}+(1-\sqrt{1+z})^{2x+1}\right)\,dz.$$

Observe that the second term in the parenthesis (i.e. $1-\sqrt{1+z}$) has no constant term and hence starts at z^{2x+1} making for a zero contribution. This leaves

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}\frac{1}{\sqrt{1+z}}(1+\sqrt{1+z})^{2x+1}\,dz.$$

Now put $1+z=w^2$ so that $dz=2w\,dw$ to get

$$\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w^2-1)^{x-\rho+1}}\frac{1}{w}(1+w)^{2x+1}\,w\,dw$$

$$=\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho+1}}\frac{1}{(w+1)^{x-\rho+1}}(1+w)^{2x+1}\,dw$$

$$=\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho+1}}(1+w)^{x+\rho}\,dw$$

$$=\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho+1}}\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q\,dw.$$

This is

$$[(w-1)^{x-\rho}]\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q$$

$$=\binom{x+\rho}{x-\rho}2^{x+\rho-(x-\rho)}=\binom{x+\rho}{x-\rho}2^{2\rho}=\binom{x+\rho}{2\rho}2^{2\rho}.$$

We can also prove the companion identity from above. Suppose we seek to evaluate

$$Q(x,\rho)=\sum_{k=0}^{\rho}\binom{2x+1}{2k+1}\binom{x-k}{\rho-k}$$

where $x\ge\rho$.

Introduce

$$\binom{x-k}{\rho-k}=\binom{x-k}{x-\rho}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}(1+z)^{x-k}\,dz.$$

Note that this controls the range being zero when $\rho<k\le x$ so we can extend the sum to x supposing that $x>\rho$. And when $x=\rho$ we may also set the upper limit to x.

We get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}(1+z)^x\sum_{k=0}^{x}\binom{2x+1}{2k+1}\frac{1}{(1+z)^k}\;dz.$$

This is

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^x}{z^{x-\rho+1}}\sqrt{1+z}\left(\left(1+\frac{1}{\sqrt{1+z}}\right)^{2x+1}-\left(1-\frac{1}{\sqrt{1+z}}\right)^{2x+1}\right)\;dz$$

$$=\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}\left((1+\sqrt{1+z})^{2x+1}-(1-\sqrt{1+z})^{2x+1}\right)\;dz.$$

Observe that the second term in the parenthesis (i.e. $1-\sqrt{1+z}$) has no constant term and hence starts at z^{2x+1} making for a zero contribution. This leaves

$$\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{x-\rho+1}}(1+\sqrt{1+z})^{2x+1}\;dz.$$

Now put $1+z=w^2$ so that $dz=2w\;dw$ to get

$$\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w^2-1)^{x-\rho+1}}(1+w)^{2x+1}\;w\;dw$$

$$=\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho+1}}\frac{1}{(w+1)^{x-\rho+1}}(1+w)^{2x+1}\;w\;dw$$

$$=\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho+1}}(1+w)^{x+\rho}\;w\;dw.$$

Writing $w=(w-1)+1$ this produces two pieces, the first is

$$\frac{1}{2\pi i}\int_{|w-1|=\epsilon}\frac{1}{(w-1)^{x-\rho}}\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q\;dw.$$

This is

$$[(w-1)^{x-\rho-1}]\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q$$

$$=\binom{x+\rho}{x-\rho-1}2^{x+\rho-(x-\rho-1)}=\binom{x+\rho}{x-\rho-1}2^{2\rho+1}=\binom{x+\rho}{2\rho+1}2^{2\rho+1}.$$

The second piece is

$$[(w-1)^{x-\rho}]\sum_{q=0}^{x+\rho}\binom{x+\rho}{q}2^{x+\rho-q}(w-1)^q$$

$$= \binom{x+\rho}{x-\rho} 2^{x+\rho-(x-\rho)} = \binom{x+\rho}{x-\rho} 2^{2\rho} = \binom{x+\rho}{2\rho} 2^{2\rho}.$$

Joining the two pieces we finally obtain

$$\left(2 \times \frac{x-\rho}{2\rho+1} + 1\right) \times \binom{x+\rho}{2\rho} 2^{2\rho}$$

$$= \frac{2x+1}{2\rho+1} \binom{x+\rho}{2\rho} 2^{2\rho}.$$

This was math.stackexchange.com problem 1383343.

21 Exercise 1.3 from Stanley's Enumerative Combinatorics (B_2)

We will do this one using coefficient extractors as in the second half of this document. We seek to verify that

$$\sum_{k=0}^{\min a,b} \binom{x+y+k}{k} \binom{x}{b-k} \binom{y}{a-k} = \binom{x+a}{b} \binom{y+b}{a}.$$

where we take $y \geq a$ and $x \geq b$.

Now introduce

$$\binom{x}{b-k} = \binom{x}{x-b+k} = [z^{b-k}] \frac{1}{(1-z)^{x-b+k+1}}$$

and

$$\binom{y}{a-k} = \binom{y}{y-a+k} = [w^{a-k}] \frac{1}{(1-w)^{y-a+k+1}}.$$

We get for the sum

$$[z^b][w^a] \frac{1}{(1-z)^{x-b+1}} \frac{1}{(1-w)^{y-a+1}} \sum_{k=0}^{\min(a,b)} \binom{x+y+k}{k} \frac{z^k w^k}{(1-z)^k} \frac{1}{(1-w)^k}.$$

The coefficient extractors provide range control and we may continue with

$$[z^b][w^a] \frac{1}{(1-z)^{x-b+1}} \frac{1}{(1-w)^{y-a+1}} \sum_{k \geq 0} \binom{x+y+k}{k} \frac{z^k w^k}{(1-z)^k} \frac{1}{(1-w)^k}$$

$$= [z^b][w^a] \frac{1}{(1-z)^{x-b+1}} \frac{1}{(1-w)^{y-a+1}} \frac{1}{(1-zw/(1-z)/(1-w))^{x+y+1}}$$

$$= [z^b][w^a](1-z)^{y+b}(1-w)^{x+a}\frac{1}{(1-z-w)^{x+y+1}}$$

$$= [z^b]\frac{1}{(1-z)^{x-b+1}}[w^a](1-w)^{x+a}\frac{1}{(1-w/(1-z))^{x+y+1}}$$

$$= [z^b]\frac{1}{(1-z)^{x-b+1}}\sum_{k=0}^{a}\binom{x+a}{k}(-1)^k\binom{a-k+x+y}{x+y}\frac{1}{(1-z)^{a-k}}$$

$$= \sum_{k=0}^{a}\binom{x+a}{k}(-1)^k\binom{a-k+x+y}{x+y}[z^b]\frac{1}{(1-z)^{x+a-b-k+1}}$$

$$= \sum_{k=0}^{a}\binom{x+a}{k}(-1)^k\binom{a-k+x+y}{x+y}\binom{x+a-k}{b}.$$

Now

$$\binom{x+a}{k}\binom{x+a-k}{b} = \frac{(x+a)!}{k!\times b!\times(x+a-b-k)!} = \binom{x+a}{b}\binom{x+a-b}{k}$$

so we obtain

$$\binom{x+a}{b}\sum_{k=0}^{a}\binom{x+a-b}{k}(-1)^k\binom{a-k+x+y}{a-k}$$

$$= \binom{x+a}{b}[z^a](1+z)^{a+x+y}\sum_{k=0}^{a}\binom{x+a-b}{k}(-1)^k z^k\frac{1}{(1+z)^k}.$$

Here the coefficient extractor once more enforces the range and we get

$$\binom{x+a}{b}[z^a](1+z)^{a+x+y}\sum_{k\ge 0}\binom{x+a-b}{k}(-1)^k z^k\frac{1}{(1+z)^k}$$

$$= \binom{x+a}{b}[z^a](1+z)^{a+x+y}\left(1-\frac{z}{1+z}\right)^{x+a-b}$$

$$= \binom{x+a}{b}[z^a](1+z)^{y+b} = \binom{x+a}{b}\binom{y+b}{a}.$$

This is the claim.
This was math.stackexchange.com problem 1426447.

22 Counting m-subsets $(B_1 I)$

Permit me to contribute an algebraic proof.

Suppose we seek to verify that

$$\sum_{q=0}^{n} \binom{n}{2q} \binom{n-2q}{p-q} 2^{2q} = \binom{2n}{2p}.$$

Observe that the sum is

$$\sum_{q=0}^{n} \binom{n}{p-q} \binom{n-p+q}{n-p-q} 4^q.$$

which is

$$\sum_{q=0}^{p} \binom{n}{p-q} \binom{n-p+q}{n-p-q} 4^q = 4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{n+q-2p} 4^{-q}.$$

Introduce the Iverson bracket

$$[[0 \le q \le p]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^q}{z^{p+1}} \frac{1}{1-z} \; dz.$$

This provides range control so we may extend q to n.
Introduce furthermore

$$\binom{n-q}{n+q-2p} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{n-q}}{w^{n+q-2p+1}} \; dw.$$

We thus get for the sum

$$\frac{4^p}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^n}{w^{n-2p+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} \sum_{q=0}^{n} \binom{n}{q} z^q \frac{1}{w^q(1+w)^q} 4^{-q} \; dz \; dw$$

$$= \frac{4^p}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^n}{w^{n-2p+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} \left(1 + z\frac{1}{4w(1+w)}\right)^n \; dz \; dw$$

$$= \frac{4^{p-n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} (4w(1+w)+z)^n \; dz \; dw.$$

We evaluate the inner integral using the negative of the residue of the pole at $z = 1$ which yields

$$\frac{4^{p-n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p+1}} (4w + 4w^2 + 1)^n \; dw$$

$$= \frac{4^{p-n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p+1}} (2w+1)^{2n} \; dw$$

$$= 4^{p-n} \binom{2n}{2n-2p} 2^{2n-2p} = \binom{2n}{2p}.$$

If we want to be rigorous we need to verify that the contribution from the residue at infinity of the last integral in z is zero when $n \ge p$. We get for the residue

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{p+1} \frac{1}{1-1/z} (4w(1+w) + 1/z)^n$$
$$= -\mathrm{Res}_{z=0} z^p \frac{1}{z-1} (4w(1+w) + 1/z)^n$$
$$= -\mathrm{Res}_{z=0} \frac{1}{z^{n-p}} \frac{1}{z-1} (4zw(1+w) + 1)^n.$$

This is clearly zero when $n = p$. For $n > p$ we obtain

$$\sum_{q=0}^{n-p-1} \binom{n}{q} 4^q w^q (1+w)^q.$$

This polynomial has degree $2n-2p-2$ but the integral in w extracts the coefficient on $2n-2p$ for a zero contribution.

Addendum. We can use the same method to prove the companion identity

$$\sum_{q=0}^{n} \binom{n}{2q+1} \binom{n-2q-1}{p-q} 2^{2q+1} = \binom{2n}{2p+1}.$$

The sum is

$$\sum_{q=0}^{n} \binom{n}{p-q} \binom{n-p+q}{n-p-q-1} 2^{2q+1}$$

which is

$$\sum_{q=0}^{p} \binom{n}{p-q} \binom{n-p+q}{n-p-q-1} 2^{2q+1} = 2^{2p+1} \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{n+q-2p-1} 2^{-2q}.$$

Using exactly the same substitution as before we obtain the integral

$$\frac{2^{2p+1-2n}}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-2p}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{p+1}} \frac{1}{1-z} \left(4w(1+w) + z\right)^n \; dz \; dw.$$

This time we get from the residue at the pole $z = 1$

$$2^{2p+1-2n} \binom{2n}{2n-2p-1} 2^{2n-2p-1} = \binom{2n}{2p+1}.$$

For the residue at infinity we are extracting the coefficient on $w^{2n-2p-1}$ but the inner term has degree $2n-2p-2$, again for a contribution of zero.

Addendum II. We can actually eliminate the Iverson bracket starting from

$$4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{n+q-2p} 4^{-q}.$$

and observing that this is

$$4^p \sum_{q=0}^{p} \binom{n}{q} \binom{n-q}{2p-2q} 4^{-q}.$$

Now introduce

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p-2q+1}} (1+z)^{n-q} \; dz$$

This is zero when $q > p$ so it provides the range control, which we have now obtained without the Iverson bracket.

We get for the sum

$$4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} (1+z)^n \sum_{q\ge 0} \binom{n}{q} 4^{-q} \frac{z^{2q}}{(1+z)^q} \; dz$$

$$= 4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} (1+z)^n \left(1 + \frac{1}{4} \frac{z^2}{1+z}\right)^n \; dz$$

$$= 4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2p+1}} \left(1 + z + \frac{1}{4} z^2\right)^n \; dz$$

Now put $z = 2w$ to get

$$4^p \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{2^{2p+1} w^{2p+1}} \left(1 + 2w + w^2\right)^n \; 2dw$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{w^{2p+1}} (1+w)^{2n} \; dw.$$

This is

$$\binom{2n}{2p}$$

as claimed. This was math.stackexchange.com problem 1430202.

23 Method applied to an iterated sum (B_1R)

Suppose we seek to show that

$$\sum_{k=0}^{n-1} \left(\sum_{q=0}^{k} \binom{n}{q} \right) \left(\sum_{q=k+1}^{n} \binom{n}{q} \right) = \frac{1}{2} n \binom{2n}{n}.$$

Using the integral representation

$$\binom{n}{q} = \binom{n}{n-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n-q+1}}\; dz$$

we get for the first factor

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \sum_{q=0}^{k} z^q \; dz = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1-z^{k+1}}{1-z}\; dz$$

$$= 2^n - \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{z^{k+1}}{1-z}\; dz$$

and for the second factor

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{z^{k+1}-z^{n+1}}{1-z}\; dz = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^{n+1}} \frac{z^{k+1}}{1-z}\; dz.$$

These add to 2^n as they obviously should.

Summing from $k = 0$ to $n-1$ we get a positive and a negative piece. The positive piece is

$$2^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^n} \sum_{k=0}^{n-1} \frac{z^k}{1-z}\; dz$$

$$= 2^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^n} \frac{1-z^n}{(1-z)^2}\; dz$$

$$= 2^n \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^n}{z^n} \frac{1}{(1-z)^2}\; dz.$$

The negative piece is

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^n}{z_1^n(1-z_1)} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_2)^n}{z_2^n(1-z_2)} \sum_{k=0}^{n-1} z_1^k z_2^k \; dz_2\; dz_1$$

$$= \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^n}{z_1^n(1-z_1)} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_2)^n}{z_2^n(1-z_2)} \frac{1-z_1^n z_2^n}{1-z_1 z_2}\; dz_2\; dz_1$$

$$= \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^n}{z_1^n(1-z_1)} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_2)^n}{z_2^n(1-z_2)} \frac{1}{1-z_1 z_2}\; dz_2\; dz_1.$$

We evaluate the inner integral by taking the sum of the negatives of the residues of the poles at $z_2 = 1$ and $z_2 = 1/z_1$ instead of computing the residue of the pole at zero by using the fact that the residues sum to zero.

Re-write the integral as follows.

$$\frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_2)^n}{z_2^n(z_2-1)} \frac{1}{z_1 z_2 - 1}\; dz_2$$

$$= \frac{1}{z_1}\frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_2)^n}{z_2^n(z_2-1)}\frac{1}{z_2-1/z_1}\; dz_2.$$

Now the negative of the residue at $z_2 = 1$ is

$$-\frac{1}{z_1}2^n\frac{1}{1-1/z_1} = 2^n\frac{1}{1-z_1}.$$

Substituting this into the outer integral we get

$$2^n\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^n}{z_1^n(1-z_1)^2}\; dz_1.$$

We see that this piece precisely cancels the positive piece that we obtained first.

Continuing the negative of the residue at $z_2 = 1/z_1$ is

$$-\frac{1}{z_1}\frac{(1+1/z_1)^n}{1/z_1^n \times (1/z_1-1)} = -\frac{1}{z_1}\frac{(1+z_1)^n}{(1/z_1-1)} = -\frac{(1+z_1)^n}{(1-z_1)}.$$

We now substitute this into the outer integral flipping the sign because this was the negative piece to get

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^{2n}}{z_1^n(1-z_1)^2}\; dz_1.$$

Extracting the residue at $z_1 = 0$ we get

$$\sum_{q=0}^{n-1}\binom{2n}{n-1-q}(q+1) = \sum_{q=0}^{n-1}\binom{2n}{n+q+1}(q+1)$$

$$= -n\sum_{q=0}^{n-1}\binom{2n}{n+q+1} + \sum_{q=0}^{n-1}\binom{2n}{n+q+1}(n+q+1)$$

$$= -n\left(\frac{1}{2}2^{2n} - \frac{1}{2}\binom{2n}{n}\right) + 2n\sum_{q=0}^{n-1}\binom{2n-1}{n+q}$$

$$= -n\left(\frac{1}{2}2^{2n} - \frac{1}{2}\binom{2n}{n}\right) + 2n\frac{1}{2}2^{2n-1}$$

$$= \frac{1}{2}n\binom{2n}{n}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity of the inner integral is zero. We use the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0}\left[-\frac{1}{z^2}h\left(\frac{1}{z}\right)\right]$$

which in the present case gives for the inner term in z_2

$$-\mathrm{Res}_{z_2=0} \frac{1}{z_2^2} \frac{(1+1/z_2)^n}{1/z_2^n \times (1-1/z_2)} \frac{1}{1-z_1/z_2}$$

$$= -\mathrm{Res}_{z_2=0} \frac{1}{z_2^2} \frac{(1+z_2)^n}{(1-1/z_2)} \frac{1}{1-z_1/z_2}$$

$$= -\mathrm{Res}_{z_2=0} \frac{(1+z_2)^n}{(z_2-1)} \frac{1}{z_2-z_1}$$

which is zero by inspection.

This was math.stackexchange.com problem 889892.

24 A pair of two double hypergeometrics (B_1)

We seek to show that

$$(1-x)^{2k+1} \sum_{n\ge 0} \binom{n+k-1}{k} \binom{n+k}{k} x^n = \sum_{j\ge 0} \binom{k-1}{j-1} \binom{k+1}{j} x^j.$$

Suppose we start by evalutating the two sums in turn, where the parameter $k \ge 1$. For the first we will be using the following integral representation:

$$\binom{n+k}{k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+k}}{z^{k+1}} \; dz.$$

We seek

$$\sum_{n\ge 1} \binom{n-1+k}{k} \binom{n+k}{k} x^n.$$

Using the integral we find

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{n\ge 1} \binom{n-1+k}{k} x^n \frac{(1+z)^{n+k}}{z^{k+1}} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^k}{z^{k+1}} \sum_{n\ge 1} \binom{n-1+k}{k} (1+z)^n x^n \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{x(1+z)^{k+1}}{z^{k+1}} \sum_{n\ge 1} \binom{n-1+k}{k} (1+z)^{n-1} x^{n-1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{x(1+z)^{k+1}}{z^{k+1}} \frac{1}{(1-x(1+z))^{k+1}} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{x(1+z)^{k+1}}{z^{k+1}} \frac{1}{(1-x-xz))^{k+1}} \; dz$$

$$= \frac{1}{(1-x)^{k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{x(1+z)^{k+1}}{z^{k+1}} \frac{1}{(1-xz/(1-x)))^{k+1}} \; dz$$

$$= \frac{x}{(1-x)^{k+1}} \sum_{q=0}^{k} \binom{k+1}{k-q} \binom{q+k}{k} \left(\frac{x}{1-x}\right)^q$$

$$= \frac{x}{(1-x)^{k+1}} \sum_{q=0}^{k} \binom{k+1}{q+1} \binom{q+k}{k} \left(\frac{x}{1-x}\right)^q.$$

Applying the integral representation from the beginning a second time we obtain for this sum

$$\frac{x}{(1-x)^{k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{k} \binom{k+1}{q+1} \frac{(1+z)^{q+k}}{z^{k+1}} \left(\frac{x}{1-x}\right)^q \; dz$$

$$= \frac{x}{(1-x)^{k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^k}{z^{k+1}} \sum_{q=0}^{k} \binom{k+1}{q+1} (1+z)^q \left(\frac{x}{1-x}\right)^q \; dz$$

$$= \frac{1}{(1-x)^{k}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \sum_{q=0}^{k} \binom{k+1}{q+1} (1+z)^{q+1} \left(\frac{x}{1-x}\right)^{q+1} \; dz$$

$$= \frac{1}{(1-x)^{k}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(-1 + \left(1 + (1+z)\frac{x}{1-x}\right)^{k+1}\right) \; dz.$$

We have $k+1-(k-1) = 2$, so the first component inside the parentheses drops out, leaving

$$\frac{1}{(1-x)^{k}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(1 + (1+z)\frac{x}{1-x}\right)^{k+1} \; dz$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(1 - x + x(1+z)\right)^{k+1} \; dz$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} \left(1 + xz\right)^{k+1} \; dz.$$

We need one more simplification on this and put $z = 1/w$, getting

$$\frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+1/w)^{k-1}}{(1/w)^{k+1}} \left(1 + x/w\right)^{k+1} \; \frac{1}{w^2} dw$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} w^2 (w+1)^{k-1} \left(\frac{w+x}{w}\right)^{k+1} \; \frac{1}{w^2} dw$$

$$= \frac{1}{(1-x)^{2k+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(w+1)^{k-1}}{w^{k+1}} (w+x)^{k+1} \; dw.$$

The reson this works is because we are essentially evaluating the residue at infinity and the residues sum to zero. This concludes the evaluation of the first sum. For the second we will be using the following integral representation:

$$\binom{k-1}{j-1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^j} \; dz.$$

We seek

$$\sum_{j\ge 1} \binom{k+1}{j} \binom{k-1}{j-1} x^j.$$

Using the integral we find

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{j\ge 1} \binom{k+1}{j} x^j \frac{(1+z)^{k-1}}{z^j} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} (1+z)^{k-1} \sum_{j\ge 1} \binom{k+1}{j} \frac{x^j}{z^j} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} (1+z)^{k-1} \left(-1 + (1+x/z)^{k+1}\right) \; dz.$$

The entire component drops out, leaving

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} (1+z)^{k-1} (1+x/z)^{k+1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{k-1}}{z^{k+1}} (z+x)^{k+1} \; dz.$$

This however is precisely the integral that we had for the first sum without the factor in front, done.

The only infinite sum appearing here is the first one with convergence when $|(1+z)x| < 1$. Therefore choosing $|x| < 1/Q$ and $|z| < 1/Q$ with $Q \ge 2$ we have $|(Q+1)/Q/Q| = |1/Q^2 + 1/Q| < 1$ and get convergence of the first LHS integral in a neighborhood of zero.

This is math.stackexchange.com problem 869982.

25 A two phase application of the method (B_1)

We seek to show that

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n+1}{k} \binom{2n-3k}{n} = \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n+1}{k} \binom{k}{n-k}.$$

Note that the second binomial coefficient in both sums controls the range of the sum, so we can write our claim like this:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \binom{2n-3k}{n-3k} = \sum_{k=0}^{n+1} \binom{n+1}{k} \binom{k}{n-k}.$$

To evaluate the LHS introduce the integral representation

$$\binom{2n-3k}{n-3k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-3k}}{z^{n-3k+1}} \; dz.$$

We can check that this really is zero when $k > \lfloor n/3 \rfloor$.

This gives for the sum the representation

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \left(\frac{z^3}{(1+z)^3}\right)^k \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+1}} \left(1 - \frac{z^3}{(1+z)^3}\right)^{n+1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1+z)^{n+3}} \left(3z^2+3z+1\right)^{n+1} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1+z)^{n+3}} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q z^q (1+z)^q \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q z^{q-n-1} (1+z)^{q-n-3} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q \frac{1}{z^{n+1-q}} \frac{1}{(1+z)^{n+3-q}} \; dz.$$

Computing the residue we find

$$\sum_{q=0}^{n+1} \binom{n+1}{q} 3^q (-1)^{n-q} \binom{n-q+n+2-q}{n+2-q}$$

$$= \sum_{q=0}^{n+1} \binom{n+1}{q} 3^q (-1)^{n-q} \binom{2n-2q+2}{n-q+2}.$$

Now introduce the integral representation

$$\binom{2n-2q+2}{n-q+2} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{2n-2q+2}}{z^{n-q+3}} \; dz$$

which gives for the sum the integral

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+3}} \sum_{q=0}^{n+1} {n+1\choose q} 3^q (-1)^{n-q} \left(\frac{z}{(1+z)^2}\right)^q \; dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n+2}}{z^{n+3}} \left(\frac{3z}{(1+z)^2}-1\right)^{n+1} \; dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+3}} (-1+z-z^2)^{n+1} \; dz.$$

Put $w=-z$ which just rotates the small circle to get

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{(-w)^{n+3}} (-1-w-w^2)^{n+1} \; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+3}} (1+w+w^2)^{n+1} \; dw.$$

We get for the final answer

$$[w^{n+2}](1+w+w^2)^{n+1}$$

but we have $2n+2-n-2=n$ and thus exploiting the symmetry of $1+w+w^2$ we get

$$[w^n](1+w+w^2)^{n+1}.$$

To evaluate the RHS introduce the integral representation

$${k\choose n-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^k}{z^{n-k+1}} \; dz.$$

This gives for the sum the representation

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \sum_{k=0}^{n+1} {n+1\choose k} ((1+z)z)^k \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z(1+z))^{n+1} \; dz.$$

The answer is

$$[z^n](1+z+z^2)^{n+1},$$

the same as the LHS, and we are done.

This was math.stackexchange.com problem 664823.

26 An identity from Mathematical Reflections (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{\lfloor (m+n)/2 \rfloor} \binom{n}{k} (-1)^k \binom{m+n-2k}{n-1}.$$

Observe that in the second binomial coefficient we must have $m+n-2k \geq n-1$ in order to avoid hitting the zero value in the product in the numerator of the binomial coefficient, so the upper limit for the sum is in fact $m+1 \geq 2k$ with the sum being

$$\sum_{k=0}^{\lfloor (m+1)/2 \rfloor} \binom{n}{k} (-1)^k \binom{m+n-2k}{n-1}.$$

Introduce

$$\binom{m+n-2k}{n-1} = \binom{m+n-2k}{m+1-2k} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n-2k}}{z^{m+2-2k}} \; dz.$$

This integral correctly encodes the range for k being zero when k is larger than $\lfloor (m+1)/2 \rfloor$. Therefore we may let k go to infinity in the sum and obtain for $n > m$

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n}}{z^{m+2}} \sum_{k \geq 0} \binom{n}{k} (-1)^k \frac{z^{2k}}{(1+z)^{2k}} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{m+n}}{z^{m+2}} \left(1 - \frac{z^2}{(1+z)^2}\right)^n \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{(1+z)^{n-m} z^{m+2}} (1+2z)^n \; dz.$$

This produces the closed form

$$\sum_{q=0}^{m+1} \binom{n}{q} 2^q (-1)^{m+1-q} \binom{m+1-q+n-m-1}{n-m-1}$$

$$= (-1)^{m+1} \sum_{q=0}^{m+1} \binom{n}{q} (-1)^q 2^q \binom{n-q}{n-m-1}.$$

This is

$$(-1)^{m+1} \sum_{q=0}^{m+1} \binom{n}{q} (-1)^q 2^q \binom{n-q}{m+1-q}.$$

Introduce

$$\binom{n-q}{m+1-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-q}}{z^{m+2-q}}\;dz$$

which once more correctly encodes the range with the pole at $z=0$ disappearing when $q>m+1$. Therefore we may extend the range to n to get

$$\begin{aligned}
&\frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{m+2}}\sum_{q=0}^{n}\binom{n}{q}(-1)^q 2^q \frac{z^q}{(1+z)^q}\;dz\\
&= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{m+2}}\left(1-2\frac{z}{1+z}\right)^n\;dz\\
&= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{m+2}}\frac{(1-z)^n}{(1+z)^n}\;dz\\
&= \frac{(-1)^{m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{(1-z)^n}{z^{m+2}}\;dz\\
&= (-1)^{m+1}\binom{n}{m+1}(-1)^{m+1} = \binom{n}{m+1}.
\end{aligned}$$

This was math.stackexchange.com problem 390321.

27 A triple Fibonacci-binomial coefficient convolution (B_1)

Here is a proof using complex variables. We seek to show that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}F_{k+1} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}F_{2k+1}.$$

Start from

$$\binom{n+k}{k} = \frac{1}{2\pi i}\int_{|z|=1}\frac{1}{z^{k+1}}(1+z)^{n+k}\;dz.$$

This yields the following expression for the sum on the LHS

$$\frac{1}{2\pi i}\int_{|z|=1}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{z^{k+1}}(1+z)^{n+k}\frac{\varphi^{k+1}-(-1/\varphi)^{k+1}}{\sqrt{5}}\;dz$$

This simplifies to

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z}\sum_{k=0}^{n}\binom{n}{k}\left(\varphi\left(\varphi\frac{1+z}{z}\right)^k+\frac{1}{\varphi}\left(-\frac{1}{\varphi}\frac{1+z}{z}\right)^k\right)\;dz$$

This finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z}\left(\varphi\left(1+\varphi\frac{1+z}{z}\right)^n+\frac{1}{\varphi}\left(1-\frac{1}{\varphi}\frac{1+z}{z}\right)^n\right)\,dz$$

or

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{n+1}}\left(\varphi\left(z+\varphi(1+z)\right)^n+\frac{1}{\varphi}\left(z-\frac{1}{\varphi}(1+z)\right)^n\right)\,dz$$

Continuing we have the following expression for the sum on the RHS

$$\frac{1}{2\pi i}\int_{|z|=1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{1}{z^{k+1}}(1+z)^{n+k}\frac{\varphi^{2k+1}-(-1/\varphi)^{2k+1}}{\sqrt{5}}\,dz$$

This simplifies to

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\varphi\left(\varphi^2\frac{1+z}{z}\right)^k+\frac{1}{\varphi}\left(\frac{1}{\varphi^2}\frac{1+z}{z}\right)^k\right)\,dz$$

This finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z}\left(\varphi\left(-1+\varphi^2\frac{1+z}{z}\right)^n+\frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}\frac{1+z}{z}\right)^n\right)\,dz$$

or

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{n+1}}\left(\varphi\left(-z+\varphi^2(1+z)\right)^n+\frac{1}{\varphi}\left(-z+\frac{1}{\varphi^2}(1+z)\right)^n\right)\,dz$$

Apply the substitution $z = 1/w$ to this integral to obtain (the sign to correct the reverse orientation of the circle is canceled by the minus on the derivative)

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\left(1+\frac{1}{w}\right)^n w^{n+1}$$
$$\times\left(\varphi\left(-\frac{1}{w}+\varphi^2(1+\frac{1}{w})\right)^n+\frac{1}{\varphi}\left(-\frac{1}{w}+\frac{1}{\varphi^2}(1+\frac{1}{w})\right)^n\right)\frac{1}{w^2}\,dw$$

which is

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\left(1+\frac{1}{w}\right)^n\frac{1}{w}$$
$$\times\left(\varphi\left(-1+\varphi^2(w+1)\right)^n+\frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}(w+1)\right)^n\right)\,dw$$

which finally yields

$$\frac{1}{\sqrt{5}}\frac{1}{2\pi i}\int_{|w|=1}\frac{(1+w)^n}{w^{n+1}}$$

$$\times \left(\varphi\left(-1+\varphi^2(w+1)\right)^n + \frac{1}{\varphi}\left(-1+\frac{1}{\varphi^2}(w+1)\right)^n\right)\; dw$$

This shows that the LHS is the same as the RHS because

$$-1+\varphi^2(w+1) = -1+(1+\varphi)(w+1) = w+\varphi(w+1)$$

and

$$-1+\frac{1}{\varphi^2}(w+1) = -1+(1-\frac{1}{\varphi})(w+1)$$

$$= -1+(w+1)-\frac{1}{\varphi}(w+1) = w-\frac{1}{\varphi}(w+1).$$

This is math.stackexchange.com problem 53830.

28 Fibonacci numbers and the residue at infinity (B_2R)

Suppose we seek to evaluate in terms of Fibonacci numbers

$$\sum_{p,q\ge 0} {n-p \choose q}{n-q \choose p}.$$

We use the integrals

$${n-p \choose q} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)^{q+1} z^{n-p-q+1}}\; dz$$

and

$${n-q \choose p} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{(1-w)^{p+1} w^{n-p-q+1}}\; dw.$$

These correctly control the range so we may let p and q go to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{(1-w)w^{n+1}} \sum_{p,q\ge 0} \frac{z^{p+q}w^{p+q}}{(1-w)^p(1-z)^q}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{(1-z)z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{(1-w)w^{n+1}}$$

$$\times \frac{1}{1-zw/(1-w)} \frac{1}{1-zw/(1-z)}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-w-zw} \frac{1}{1-z-zw}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+2}(1+z)} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{w-1/(1+z)} \frac{1}{w-(1-z)/z}\; dw\; dz.$$

We evaluate the inner integral using the fact that the residues of the function in w sum to zero. We have two simple poles. We get for the first pole at $w = (1-z)/z$

$$\frac{z^{n+1}}{(1-z)^{n+1}} \frac{1}{(1-z)/z - 1/(1+z)} = \frac{z^{n+1}}{(1-z)^{n+1}} \frac{z(1+z)}{(1-z)(1+z) - z}$$
$$= \frac{z^{n+2}}{(1-z)^{n+1}} \frac{1+z}{(1-z)(1+z) - z}.$$

Substituting this expression into the outer integral we see that the pole at $z = 0$ is canceled making for a contribution of zero.

For the second pole at $w = 1/(1+z)$ we get

$$(1+z)^{n+1} \frac{1}{1/(1+z) - (1-z)/z} = (1+z)^{n+1} \frac{z(1+z)}{z - (1-z)(1+z)}.$$

This yields the contribution (taking into account the sign flip from the sum of residues)

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+2}(1+z)} (1+z)^{n+1} \frac{z(1+z)}{1-z-z^2} \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (1+z)^{n+1} \frac{1}{1-z-z^2} \; dz.$$

We evaluate this using again the fact that the residues sum to zero. There are simple poles at $z = -\varphi$ and $z = 1/\varphi$.

These yield

$$\left(\frac{1-\varphi}{-\varphi}\right)^{n+1} \frac{1}{-1+2\varphi} + \left(\frac{1+1/\varphi}{1/\varphi}\right)^{n+1} \frac{1}{-1-2/\varphi}$$
$$= \frac{1}{\sqrt{5}} \frac{1}{\varphi^{2n+2}} - \frac{1}{\sqrt{5}} \varphi^{2n+2}.$$

Taking into account the sign flip this is obviously Binet / de Moivre for

$$F_{2n+2}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity in both cases is zero. For example in the first application we use the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z}\right) \right]$$

which in the present case gives for the inner term in w

$$-\mathrm{Res}_{w=0} \frac{1}{w^2} w^{n+1} \frac{1}{1/w - 1/(1+z)} \frac{1}{1/w - (1-z)/z}$$

$$= -\mathrm{Res}_{w=0} w^{n+1} \frac{1}{1-w/(1+z)} \frac{1}{1-w(1-z)/z}$$

which is zero by inspection.

This was math.stackexchange.com problem 801730.

29 Permutations containing a given subsequence ($B_1 I$)

The WZ machinery is very powerful but it is also an incentive to evaluate these sums manually e.g. by using the Egorychev method which I hope will make for a rewarding read.

Suppose as before that we are trying to evaluate

$$S = \sum_{r=0}^{n} \binom{r+n-1}{n-1} \binom{3n-r}{n}$$

which is

$$S_2 - S_1 = \sum_{r=0}^{2n} \binom{r+n-1}{n-1} \binom{3n-r}{n} - \sum_{r=n+1}^{2n} \binom{r+n-1}{n-1} \binom{3n-r}{n}.$$

Start by evaluating S_2.

Put

$$\binom{3n-r}{n} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n-r+1}} \frac{1}{(1-w)^{n+1}} \; dw.$$

and use the following Iverson bracket

$$[[0 \le r \le 2n]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^r}{z^{2n+1}} \frac{1}{1-z} \; dz.$$

This second integral controls the range so that we may extend the sum to infinity to get

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n+1}} \frac{1}{(1-w)^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n+1}} \frac{1}{1-z} \sum_{r=0}^{\infty} \binom{r+n-1}{n-1} z^r w^r \; dz \; dw.$$

This simplifies to

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{2n+1}} \frac{1}{(1-w)^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n+1}} \frac{1}{1-z} \frac{1}{(1-wz)^n} \; dz \; dw.$$

We evaluate the inner integral using the fact that the residues at the three poles sum to zero. The residue at $z=0$ is the sum S_2 which we are trying to compute. The residue at $z=1$ yields

$$-\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{2n+1}}\frac{1}{(1-w)^{2n+1}}\;dw = -\binom{2n+2n}{2n} = -\binom{4n}{2n}.$$

For the residue at $z = 1/w$ re-write the integral as follows:

$$\frac{(-1)^n}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{3n+1}}\frac{1}{(1-w)^{n+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n+1}}\frac{1}{1-z}\frac{1}{(z-1/w)^n}\;dz\;dw.$$

We require a derivative which we compute using Leibniz' rule:

$$\frac{1}{(n-1)!}\left(\frac{1}{z^{2n+1}}\frac{1}{1-z}\right)^{(n-1)}$$

$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^q\frac{(2n+q)!}{(2n)!\times z^{2n+1+q}}\frac{(n-1-q)!}{(1-z)^{1+n-1-q}}$$

$$= \sum_{q=0}^{n-1}\binom{2n+q}{2n}(-1)^q\frac{1}{z^{2n+1+q}}\frac{1}{(1-z)^{n-q}}.$$

Evaluate at $z = 1/w$ to get

$$\sum_{q=0}^{n-1}\binom{2n+q}{2n}(-1)^q w^{2n+1+q}\frac{w^{n-q}}{(w-1)^{n-q}}.$$

Substitute this back into the integral in w to obtain

$$\sum_{q=0}^{n-1}\binom{2n+q}{2n}(-1)^q\frac{(-1)^n}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{3n+1}}\frac{1}{(1-w)^{n+1}}\frac{w^{3n+1}}{(w-1)^{n-q}}\;dw$$

$$= \sum_{q=0}^{n-1}\binom{2n+q}{2n}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{(1-w)^{2n-q+1}}\;dw = 0.$$

We have shown that

$$S_2 = \binom{4n}{2n}.$$

Continuing with S_1 we see that

$$S_1 = \sum_{r=0}^{n-1}\binom{r+2n}{n-1}\binom{2n-1-r}{n} = \sum_{r=0}^{n-1}\binom{r+2n}{n-1}\binom{2n-1-r}{n-1-r}.$$

For this sum no Iverson bracket is needed as the second binomial coefficient controls the range via the following integral:

$$\binom{2n-1-r}{n-1-r} = \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{2n-1-r}}{w^{n-r}}\;dw.$$

Furthermore introduce

$$\binom{r+2n}{n-1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{r+2n}}{z^n}\; dz.$$

This gives the integral

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n-1}}{w^n} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^n} \sum_{r\ge 0} \frac{w^r(1+z)^r}{(1+w)^r}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n-1}}{w^n} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^n} \frac{1}{1-w(1+z)/(1+w)}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n}}{w^n} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^n} \frac{1}{1+w-w(1+z)}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n}}{w^n} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^n} \frac{1}{1-wz}\; dz\; dw.$$

Extracting the inner residue we obtain

$$\sum_{q=0}^{n-1} \binom{2n}{n-1-q} w^q$$

which yields

$$\sum_{q=0}^{n-1} \binom{2n}{n-1-q} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2n}}{w^{n-q}}\; dw$$

$$= \sum_{q=0}^{n-1} \binom{2n}{n-1-q}\binom{2n}{n-1-q}.$$

This is

$$\sum_{q=0}^{n-1} \binom{2n}{q}^2$$

which may be evaluated by inspection as in the first version and we are done.

Remark. To be fully rigorous we must also show that the residue at infinity of

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{2n+1}} \frac{1}{1-z} \frac{1}{(1-wz)^n}\; dz$$

is zero. Recall the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0}\left[-\frac{1}{z^2} h\left(\frac{1}{z}\right)\right]$$

which in this case yields

$$-\mathrm{Res}_{z=0}\frac{1}{z^2}z^{2n+1}\frac{1}{1-1/z}\frac{1}{(1-w/z)^n}$$
$$= -\mathrm{Res}_{z=0}z^{2n}\frac{1}{z-1}\frac{1}{(1-w/z)^n}$$
$$= -\mathrm{Res}_{z=0}z^{3n}\frac{1}{z-1}\frac{1}{(z-w)^n}$$

which is zero by inspection.

This is math.stackexchange.com problem 1255356.

30 Catalan numbers and Lagrange inversion (B_1)

Suppose we seek the series for

$$\frac{1}{2}\left(1-x-y-\sqrt{1-2x-2y-2xy+x^2+y^2}\right).$$

Introduce $u = xy$ and $v = x + y$ to get

$$\frac{1}{2}\left(1-v-\sqrt{1-2v+v^2-4u}\right) = \frac{1}{2}\left(1-v-\sqrt{(1-v)^2-4u}\right).$$

Lagrange inversion now asks us to compute the integral

$$\frac{1}{2\pi i}\int_{|u|=\epsilon}\frac{1}{u^{n+1}}\frac{1}{2}\left(1-v-\sqrt{(1-v)^2-4u}\right)\;du.$$

Put $w^2 = (1-v)^2 - 4u$ so that $4u = (1-v)^2 - w^2$ and $du = -\frac{1}{2}w\;dw$. This yields

$$-\frac{1}{4}\frac{1}{2\pi i}\int_{|w-(1-v)|=\epsilon}\frac{4^{n+1}}{((1-v)^2-w^2)^{n+1}}(1-v-w)w\;dw$$
$$= -\frac{4^n}{2\pi i}\int_{|w-(1-v)|=\epsilon}\frac{1}{((1-v)-w)^n}\frac{1}{((1-v)+w)^{n+1}}w\;dw.$$

This has two pieces, piece B_1 is

$$-\frac{4^n}{2\pi i}\int_{|w-(1-v)|=\epsilon}\frac{1}{((1-v)-w)^n}\frac{1}{((1-v)+w)^n}\;dw$$

and piece B_2 is

$$(1-v)\frac{4^n}{2\pi i}\int_{|w-(1-v)|=\epsilon}\frac{1}{((1-v)-w)^n}\frac{1}{((1-v)+w)^{n+1}}\;dw$$

Observe that

$$\frac{1}{((1-v)+w)^n} = \frac{1}{(w-(1-v)+2(1-v))^n}$$

$$= \frac{1}{2^n(1-v)^n} \frac{1}{((w-(1-v))/2/(1-v)+1)^n}$$

$$= \frac{1}{2^n(1-v)^n} \sum_{q\ge 0} \binom{q+n-1}{n-1} (-1)^q \frac{(w-(1-v))^q}{2^q(1-v)^q}.$$

It follows that B_1 is

$$-4^n(-1)^n \frac{1}{2^n(1-v)^n} \binom{2n-2}{n-1} (-1)^{n-1} \frac{1}{2^{n-1}(1-v)^{n-1}}$$

$$= 2\binom{2n-2}{n-1} \frac{1}{(1-v)^{2n-1}}.$$

Similarly we have for B_2 the series expansion

$$\frac{1}{2^{n+1}(1-v)^{n+1}} \sum_{q\ge 0} \binom{q+n}{n} (-1)^q \frac{(w-(1-v))^q}{2^q(1-v)^q}.$$

It follows that B_2 is

$$(1-v)4^n(-1)^n \frac{1}{2^{n+1}(1-v)^{n+1}} \binom{2n-1}{n} (-1)^{n-1} \frac{1}{2^{n-1}(1-v)^{n-1}}$$

$$= -\binom{2n-1}{n} \frac{1}{(1-v)^{2n-1}}.$$

Collecting the two contributions we finally obtain for the two pieces

$$\left(-\frac{2n-1}{n}+2\right)\binom{2n-2}{n-1}\frac{1}{(1-v)^{2n-1}} = \frac{1}{n}\binom{2n-2}{n-1}\frac{1}{(1-v)^{2n-1}}.$$

We recognize the Catalan numbers here which are OEIS A000108.
We have thus obtained the following series

$$\sum_{n\ge 1} \frac{1}{n}\binom{2n-2}{n-1}\frac{1}{(1-v)^{2n-1}} u^n.$$

Observe that

$$\frac{1}{(1-x-y)^{2n-1}} = \frac{1}{(1-x)^{2n-1}} \frac{1}{(1-y/(1-x))^{2n-1}}$$

$$= \frac{1}{(1-x)^{2n-1}} \sum_{q\ge 0} \binom{q+2n-2}{2n-2} \frac{y^q}{(1-x)^q} = \sum_{q\ge 0} \binom{q+2n-2}{2n-2} \frac{y^q}{(1-x)^{q+2n-1}}.$$

It follows that the coefficient on $[y^\nu]$ is

$$\sum_{n=1}^{\nu} x^n \frac{1}{n}\binom{2n-2}{n-1}\binom{\nu+n-2}{2n-2}\frac{1}{(1-x)^{\nu+n-1}}.$$

This yields for the coefficient on $[x^\mu y^\nu]$

$$\sum_{n=1}^{\min(\mu,\nu)} \frac{1}{n}\binom{2n-2}{n-1}\binom{\nu+n-2}{2n-2}\binom{\mu+\nu-2}{\nu+n-2}.$$

The inner term here is

$$\frac{1}{n}\frac{(2n-2)!}{(n-1)!\times(n-1)!}\frac{(\nu+n-2)!}{(2n-2)!\times(\nu-n)!}\frac{(\mu+\nu-2)!}{(\nu+n-2)!\times(\mu-n)!}$$

which is

$$(\mu+\nu-2)!\frac{1}{n}\frac{1}{(n-1)!\times(n-1)!}\frac{1}{(\nu-n)!}\frac{1}{(\mu-n)!}$$

$$=\frac{1}{\mu+\nu-1}(\mu+\nu-1)!\frac{1}{n}\frac{1}{(n-1)!\times(n-1)!}\frac{1}{(\nu-n)!}\frac{1}{(\mu-n)!}$$

$$=\frac{1}{\mu+\nu-1}\binom{\mu+\nu-1}{n, n-1, \mu-n, \nu-n}.$$

It remains to evaluate the sum in the multinomial coefficient since we already have the correct factor in front. This gives

$$\sum_{n=1}^{\min(\mu,\nu)} \frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{z_1^n}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^{\mu-n+1}}\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^{\nu-n+1}}$$
$$\times(1+z_1+z_2+z_3)^{\mu+\nu-1}\; dz_3\; dz_2\; dz_1.$$

Observe that the poles in z_2 and z_3 disappear when $n>\mu$ or $n>\nu$ so they effectively control the range and we may extend the sum to infinity, obtaining

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^{\mu+1}}\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^{\nu+1}}$$
$$\times(1+z_1+z_2+z_3)^{\mu+\nu-1}\sum_{n\ge 1}\frac{z_2^n z_3^n}{z_1^n}\; dz_3\; dz_2\; dz_1.$$

This is

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^{\mu+1}}\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^{\nu+1}}$$
$$\times(1+z_1+z_2+z_3)^{\mu+\nu-1}\frac{z_2z_3/z_1}{1-z_2z_3/z_1}\; dz_3\; dz_2\; dz_1.$$

or

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^{\mu}}\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^{\nu}}$$

$$\times(1+z_1+z_2+z_3)^{\mu+\nu-1}\frac{1/z_1}{1-z_2z_3/z_1}\;dz_3\;dz_2\;dz_1.$$

or

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^\mu}\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^\nu}$$

$$\times(1+z_1+z_2+z_3)^{\mu+\nu-1}\frac{1}{z_1-z_2z_3}\;dz_3\;dz_2\;dz_1.$$

First treat the pole at $z_1 = z_2z_3$ to get

$$\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^\mu}\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^\nu}(1+z_2z_3+z_2+z_3)^{\mu+\nu-1}\;dz_3\;dz_2$$

$$=\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^\mu}\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^\nu}(1+z_2)^{\mu+\nu-1}(1+z_3)^{\mu+\nu-1}\;dz_3\;dz_2.$$

This factors into

$$\frac{1}{2\pi i}\int_{|z_2|=\epsilon}\frac{1}{z_2^\mu}(1+z_2)^{\mu+\nu-1}\;dz_2\times\frac{1}{2\pi i}\int_{|z_3|=\epsilon}\frac{1}{z_3^\nu}(1+z_3)^{\mu+\nu-1}\;dz_3.$$

Extracting coefficients from this we obtain

$$\binom{\mu+\nu-1}{\mu-1}\binom{\mu+\nu-1}{\nu-1}.$$

This concludes the argument since we have established the value

$$\frac{1}{\mu+\nu-1}\binom{\mu+\nu-1}{\mu-1}\binom{\mu+\nu-1}{\nu-1}$$

which may be rewritten as

$$\frac{1}{\mu+\nu-1}\binom{\mu+\nu-1}{\nu}\binom{\mu+\nu-1}{\mu}.$$

This is math.stackexchange.com problem 1266250.

31 A binomial coefficient - Catalan number convolution (B_1)

Suppose we seek to show that

$$\sum_{r=1}^{n+1}\frac{1}{r+1}\binom{2r}{r}\binom{m+n-2r}{n+1-r}=\binom{m+n}{n}.$$

We will assume familiarity with the generating function of the Catalan numbers (which seems like a reasonable assumption). This generating function is given by

$$\sum_{r\ge 0} \frac{1}{r+1}\binom{2r}{r} z^r = \frac{1-\sqrt{1-4z}}{2z}$$

so that

$$\frac{1}{r+1}\binom{2r}{r} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{r+1}} \frac{1-\sqrt{1-4z}}{2z}\; dz.$$

Note in particular that this generating function is analytic in a neighborhood of the origin $|z| < 1/4$ with the branch cut $[1/4, \infty)$.

Furthermore introduce

$$\binom{m+n-2r}{n+1-r} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n-2r}}{w^{n+2-r}}\; dw.$$

Observe carefully that this last integral is zero when $r > n+1$, so we may extend the range of the sum to infinity.

This yields for the sum

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z}\frac{1-\sqrt{1-4z}}{2z} \sum_{r\ge 1} \frac{w^r}{(1+w)^{2r} z^r}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1-\sqrt{1-4z}}{2z^2} \frac{w/(1+w)^2/z}{1-w/(1+w)^2/z}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1-\sqrt{1-4z}}{2z^2} \frac{1}{z(1+w)^2/w - 1}\; dz\; dw.$$

Observe that with the principal branch of the logarithm

$$1-\sqrt{1-4z} = 2z + 2z^2 + 4z^3 + \cdots$$

and

$$\frac{1}{z(1+w)^2/w-1} = -1 - z\frac{(1+w)^2}{w} - z^2\frac{(1+w)^4}{w^2} - \cdots.$$

so that the contribution from the pole at $z = 0$ is

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1}{2} \times (-2)\; dw = -\binom{m+n}{n+1}.$$

On the other hand the contribution from the simple pole at $z = w/(1+w)^2$ which is inside the contour is

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1-\sqrt{1-4w/(1+w)^2}}{2w^2/(1+w)^4} \frac{w}{(1+w)^2}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n-2}}{w^{n+1}} \frac{(1+w)^4 - (1+w)^3\sqrt{(1+w)^2-4w}}{2w^2}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n-2}}{2w^{n+3}}((1+w)^4 - (1-w)(1+w)^3)\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n-2}}{2w^{n+3}}(1+w)^3 \times (2w)\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n+1}}{w^{n+2}}\; dw.$$

which yields

$$\binom{m+n+1}{n+1}.$$

Collecting the two contributions we obtain

$$\binom{m+n+1}{n+1} - \binom{m+n}{n+1} = \left(\frac{m+n+1}{n+1} - \frac{m}{n+1}\right)\binom{m+n}{n}$$

$$= \binom{m+n}{n}$$

as claimed.

Addendum. In fact the above admits considerable simplification. Write

$$-\binom{m+n}{n+1} + \sum_{r=0}^{n+1} \frac{1}{r+1}\binom{2r}{r}\binom{m+n-2r}{n+1-r}$$

and use the same integral as before for the binomial coefficient to obtain

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \sum_{r\ge 0} \frac{1}{r+1}\binom{2r}{r} \frac{w^r}{(1+w)^{2r}}\; dw$$

which becomes

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1-\sqrt{1-4w/(1+w)^2}}{2w/(1+w)^2}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n}}{w^{n+2}} \frac{1+w-\sqrt{(1+w)^2-4w}}{2w/(1+w)}\; dw$$

$$= \frac{1}{2}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n+1}}{w^{n+3}} \left(1+w-\sqrt{(1-w)^2}\right)\; dw$$

$$= \frac{1}{2}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n+1}}{w^{n+3}}(2w)\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{m+n+1}}{w^{n+2}}\; dw$$

$$= \binom{m+n+1}{n+1}.$$

We may then conclude as before.

Addendum, Feb 2021. For the first version to be complete we need the conditions on ε and γ for the geometric series to converge. This requires $|w/(1+w)^2| < |z|$. Note that if this holds then the pole at $z = w/(1+w)^2$ is guaranteed to be inside the contour in $|z|$ as claimed. We take $\gamma < 1$ positive somewhere close to zero and we then require $\gamma/(1-\gamma)^2 < \varepsilon < 1/4$ where the last term is from the Catalan GF. The values $\gamma = 1/7$ and $\varepsilon = 1/5$ will work. This also ensures convergence of the Catalan GF series in the compact version.

This was math.stackexchange.com problem 563307.

32 A new obstacle from Concrete Mathematics (Catalan numbers) (B_1)

Suppose we seek to evaluate

$$\sum_{k\ge 0} \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

where $m, n \ge 0$. In fact we may assume that $n \ge m$ because if $m > n$ when counting down from the non-negative value $n+k$ with $m+2k$ terms we invariably hit zero and the sum vanishes.

Furthermore observe that when $k = n-m+q$ with $q > 0$ we obtain $\binom{2n-m+q}{2n-m+2q}$ which is zero by the same argument. This gives

$$\sum_{k=0}^{n-m} \binom{n+k}{n-m-k}\binom{2k}{k}\frac{(-1)^k}{k+1}.$$

Introduce

$$\binom{n+k}{n-m-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m-k+1}}(1+z)^{n+k}\; dz.$$

Observe that this is zero when $k > n-m$ so we may extend k to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+1}}(1+z)^n \sum_{k\ge 0}\binom{2k}{k}\frac{(-1)^k}{k+1} z^k (1+z)^k\; dz.$$

Here we recognize the generating function of the Catalan numbers

$$\sum_{k\ge 0}\binom{2k}{k}\frac{1}{k+1}w^k = \frac{1-\sqrt{1-4w}}{2w}$$

where the branch cut of the logarithm is on the negative real axis and hence the branch cut of the square root term is $(1/4, \infty)$ so we certainly have analyticity in a neighborhood of zero. We obtain

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+1}}(1+z)^n \frac{1-\sqrt{1+4z(1+z)}}{2z(1+z)}\; dz$$
$$= -\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+2}}(1+z)^{n-1}\left(1-\sqrt{(1+2z)^2}\right)\; dz.$$

Now with z in a neighborhood of zero the square root produces the positive root so we finally have

$$-\frac{1}{2}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+2}}(1+z)^{n-1}(-2z)\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-m+1}}(1+z)^{n-1}\; dz$$

which evaluates by inspection to $\binom{n-1}{n-m}$ which is

$$\binom{n-1}{m-1}.$$

This problem has not yet appeared at math.stackexchange.com.

33 Abel-Aigner identity from Table 202 of Concrete Mathematics (B_1)

Seeking to prove that

$$\sum_k \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{r}{tk+r} = \binom{tn+r+s}{n}$$

we see that our identity is in fact

$$\sum_{k=0}^n \binom{tk+r}{k}\binom{tn-tk+s}{n-k} - \sum_{k=0}^n \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{tk}{tk+r}$$
$$= \binom{tn+r+s}{n}.$$

With integers $t, r, s \geq 1$ and starting with the first sum we introduce

$$\binom{tk+r}{k} = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{k+1}}(1+w)^{tk+r}\; dw$$

and

$$\binom{tn-tk+s}{n-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n-k+1}}(1+z)^{tn-tk+s}\; dz.$$

This last integral vanishes when $k > n$ so we may extend the sum to infinity, getting

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{tn+s}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^r}{w} \sum_{k\ge 0} z^k (1+z)^{-tk} \frac{1}{w^k}(1+w)^{tk}\; dw\; dz.$$

Now with ϵ and γ small in a neighborhood of the origin we get that for this to converge we must have $\epsilon/(1-\epsilon)^t < \gamma/(1+\gamma)^t$. We shall see that we may solve this with an additional constraint, namely that $\gamma > \epsilon$. Doing the summation we find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{tn+s}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^r}{w} \frac{1}{1-z(1+w)^t/w/(1+z)^t}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{tn+s}}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} (1+w)^r \frac{1}{w-z(1+w)^t/(1+z)^t}\; dw\; dz.$$

The pole at $w = 0$ has been canceled. There is a pole at $w = z$ however and with the chosen parameters it is inside the contour. We get for the residue

$$(1+w)^r \frac{1}{1-tz(1+w)^{t-1}/(1+z)^t}\bigg|_{w=z} = (1+z)^r \frac{1}{1-tz/(1+z)}$$

The derivative would have vanished if the pole had not been simple. Substituting into the outer integral we get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{tn+r+s+1}}{z^{n+1}} \frac{1}{1-(t-1)z}\; dz.$$

Continuing with the second sum we obtain

$$\sum_{k=1}^{n} \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{tk}{tk+r} = t\sum_{k=1}^{n}\binom{tk+r-1}{k-1}\binom{tn-tk+s}{n-k}$$

$$= t\sum_{k=0}^{n-1}\binom{tk+t+r-1}{k}\binom{t(n-1)-tk+s}{(n-1)-k}.$$

We can recycle the earlier computation and find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{t(n-1)+t+r-1+s+1}}{z^{n}} \frac{t}{1-(t-1)z}\; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{tn+r+s}}{z^{n+1}} \frac{tz}{1-(t-1)z} \; dz.$$

Subtracting the two the result is

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{tn+r+s}}{z^{n+1}} \frac{(1+z)-tz}{1-(t-1)z} \; dz = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{tn+r+s}}{z^{n+1}} \; dz.$$

This evaluates to

$$\binom{tn+r+s}{n}$$

by inspection and we have proved the theorem.

To show that the pole at $w = z$ is the only one inside the contour apply Rouche's theorem to

$$h(w) = w(1+z)^t - z(1+w)^t$$

with $f(w) = w(1+z)^t$ and $g(w) = z(1+w)^t$. We need $|g(w)| < |f(w)|$ on $|w| = \gamma$ and since $f(w)$ has only one root there so does $h(w)$, which must be $w = z$. We thus require

$$|g(w)| \le |z|(1+\gamma)^t < \gamma|1+z|^t = |f(w)|.$$

Now $\gamma/(1+\gamma)^t$ starts at zero and is increasing since $(1+\gamma-\gamma t)/(1+\gamma)^{t+1}$ is positive for $\gamma < 1/(t-1)$ with a local maximum there. Since $|z|/|1+z|^t \le \epsilon/(1-\epsilon)^t$ we may choose ϵ for this to take on a value from the range of $\gamma/(1+\gamma)^t$ on $[0, 1/(t-1)]$. Instantiating γ to the right of this point yields a value $> \epsilon$ that fulfils the requirements of the theorem. Here we have used that $\epsilon/(1+\epsilon)^t < \epsilon/(1-\epsilon)^t < \gamma/(1+\gamma)^t$ by construction. No need for Rouche when $t = 1$.

This was math.stackexchange.com problem 2814898.

34 Reducing the form of a double hypergeometric (B_1)

Suppose we seek to evaluate

$$S(n) = \sum_{q=0}^{n-2} \sum_{k=1}^{n} \binom{k+q}{k} \binom{2n-q-k-1}{n-k+1}.$$

which we re-write as

$$-\sum_{q=0}^{n-2} \binom{2n-q-1}{n+1} - \sum_{q=0}^{n-2} \binom{n+1+q}{n+1} + \sum_{q=0}^{n-2} \sum_{k=0}^{n+1} \binom{k+q}{k} \binom{2n-q-k-1}{n-k+1}.$$

Call these pieces up to sign from left to right S_1, S_2 and S_3.

The two pieces in front cancel the quantities introduced by extending k to include the values zero and $n+1$.

Evaluation of S_1.

Introduce

$$\binom{2n-q-1}{n+1} = \binom{2n-q-1}{n-q-2} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-q-1}}{z^{n-q-1}}\; dz.$$

This vanishes when $q > n-2$ so we may extend the sum to infinity to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n-1}} \sum_{q\ge 0} \frac{z^q}{(1+z)^q}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n-1}} \frac{1}{1-z/(1+z)}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n-1}}\; dz$$

$$= \binom{2n}{n-2}.$$

Evaluation of S_2.

Introduce

$$\binom{n+1+q}{n+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1+q}}{z^{n+2}}\; dz.$$

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+2}} \sum_{q=0}^{n-2} (1+z)^q\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+2}} \frac{(1+z)^{n-1}-1}{1+z-1}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n+1}}{z^{n+3}} ((1+z)^{n-1}-1)\; dz$$

$$= \binom{2n}{n+2}.$$

A more efficient evaluation is to notice that when we re-index q as $n-2-q$ in S_2 we obtain

$$\sum_{q=0}^{n-2} \binom{n+1+n-2-q}{n+1} = \sum_{q=0}^{n-2} \binom{2n-q-1}{n+1}$$

which is S_1.

Evaluation of S_3.

Introduce

$$\binom{2n-q-k-1}{n-k+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-q-k-1}}{z^{n-k+2}}\; dz.$$

This effectively controls the range so we can let k go to infinity to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+2}} \sum_{q=0}^{n-2}\sum_{k\ge 0}\binom{k+q}{q}\frac{z^k}{(1+z)^{q+k}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n-1}}{z^{n+2}} \sum_{q=0}^{n-2}\frac{1}{(1+z)^q}\frac{1}{(1-z/(1+z))^{q+1}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+2}} \sum_{q=0}^{n-2}\frac{1}{(1+z)^{q+1}}\frac{1}{(1-z/(1+z))^{q+1}}\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{2n}}{z^{n+2}} \times (n-1)\times\; dz$$

$$= (n-1)\times\binom{2n}{n+1}.$$

Finally collecting the three contributions we obtain

$$(n-1)\times\binom{2n}{n+1} - 2\binom{2n}{n+2} = (n+2)\binom{2n}{n+2} - 2\binom{2n}{n+2}$$

$$= n\times\binom{2n}{n+2}.$$

This is math.stackexchange.com problem 129913.

35 Basic usage of the Iverson bracket ($B_1 I$)

Suppose we seek to evaluate

$$S(k,l) = \sum_{q=0}^{l}\binom{q+k}{k}\binom{l-q}{k}.$$

We start with the Iverson bracket valid for $q \ge 0$

$$[[0\le q\le l]] = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z^q}{z^{l+1}}\frac{1}{1-z}\; dz$$

This gives for the sum

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^l}{w^{k+1}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l+1}}\frac{1}{1-z}\sum_{q\ge 0}\binom{q+k}{q}\frac{z^q}{(1+w)^q}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^l}{w^{k+1}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l+1}} \frac{1}{1-z} \frac{1}{(1-z/(1+w))^{k+1}}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l+1}} \frac{1}{1-z} \frac{1}{(1+w-z)^{k+1}}\; dw\; dz.$$

We evaluate the inner integral by taking the negative of the sum of the residues at $z = 1$ and at $z = 1 + w$ and $z = \infty$. With ϵ and γ small the second pole is not inside the contour.

The negative of the residue at $z = 1$ is

$$\frac{1}{w^{k+1}}$$

which when substituted into the outer integral yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{2k+2}}\; dw = \binom{l+k+1}{2k+1},$$

which is the formula we are trying to establish.

Next we prove that the residue at infinity is zero. This is given by

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{l+1} \frac{1}{1-1/z} \frac{1}{(1+w-1/z)^{k+1}} = -\mathrm{Res}_{z=0} z^l \frac{1}{z-1} \frac{z^{k+1}}{(z(1+w)-1)^{k+1}}$$

$$= -\frac{1}{(1+w)^{k+1}} \mathrm{Res}_{z=0} \frac{1}{z-1} \frac{z^{l+k+1}}{(z-1/(1+w))^{k+1}}.$$

This is zero by inspection, which leaves the residue at $z = 1 + w$. Write

$$\frac{(-1)^{k+1}}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l+1}} \frac{1}{1-z} \frac{1}{(z-(1+w))^{k+1}}\; dw\; dz.$$

We require the derivative

$$\frac{1}{k!}\left(\frac{1}{z^{l+1}} \frac{1}{1-z}\right)^{(k)} = \frac{1}{k!}\sum_{q=0}^{k} \binom{k}{q} (-1)^q \frac{(l+q)!}{l! \times z^{l+1+q}} \frac{(k-q)!}{(1-z)^{1+k-q}}$$

$$= \sum_{q=0}^{k} \binom{l+q}{q} (-1)^q \frac{1}{z^{l+1+q}} \frac{1}{(1-z)^{1+k-q}}.$$

Evaluate this at $z = 1 + w$ to get

$$\sum_{q=0}^{k} \binom{l+q}{q} (-1)^q \frac{1}{(1+w)^{l+1+q}} \frac{1}{(-w)^{1+k-q}}$$

and substitute into the outer integral to obtain

$$\frac{(-1)^{k+1}}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \sum_{q=0}^{k} \binom{l+q}{q} (-1)^q \frac{1}{(1+w)^{l+1+q}} \frac{1}{(-w)^{1+k-q}}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{l+k+1}}{w^{k+1}} \sum_{q=0}^{k} \binom{l+q}{q} \frac{1}{(1+w)^{l+1+q}} \frac{1}{w^{1+k-q}} \; dw$$

$$= \sum_{q=0}^{k} \binom{l+q}{q} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(1+w)^{k-q}}{w^{2k+2-q}} \; dw.$$

The inner term here is

$$[w^{2k+1-q}](1+w)^{k-q}.$$

But we have $2k+1-q \geq k+1$ while $k-q \leq k$ so these terms are zero, thus concluding the proof.

Simplified solution. As observed elsewhere this can be done without the Iverson bracket.

Introduce

$$\binom{l-q}{k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l-q-k+1}} \frac{1}{(1-z)^{k+1}} \; dz.$$

This controls the range becoming zero when $q > l-k$ so we may extend q to infinity.

We obtain for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{k+1}} \sum_{q\geq 0} \binom{q+k}{k} z^q \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{k+1}} \frac{1}{(1-z)^{k+1}} \; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{l-k+1}} \frac{1}{(1-z)^{2k+2}} \; dz.$$

This evaluates by inspection to

$$\binom{l-k+2k+1}{2k+1} = \binom{l+k+1}{2k+1}.$$

This was math.stackexchange.com problem.

36 Basic usage of the Iverson bracket II ($B_1 I$)

Suppose we seek to compute

$$S(n,m) = \sum_{k=0}^{n} k \binom{m+k}{m+1}.$$

Introduce

$$\binom{m+k}{m+1} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m+k}\; dz$$

as well as the Iverson bracket

$$[[0 \le k \le n]] = \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{w^k}{w^{n+1}}\frac{1}{1-w}\; dw.$$

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}\frac{1}{1-w}\sum_{k\ge 0} kw^k(1+z)^k\; dw\; dz.$$

For this to converge we must have $|w(1+z)| < 1$. We get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+1}}\frac{1}{1-w}\frac{w(1+z)}{(1-w(1+z))^2}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m+1}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n}}\frac{1}{1-w}\frac{1}{(1-w(1+z))^2}\; dw\; dz.$$

We evaluate the inner integral using the fact that the residues at the poles sum to zero. The residue at $w=1$ produces

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m+1}\frac{1}{(-z)^2}\; dz = -\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+4}}(1+z)^{m+1}\; dz = 0.$$

For the residue at $w = 1/(1+z)$ we re-write the inner integral to get

$$\frac{1}{(1+z)^2}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n}}\frac{1}{1-w}\frac{1}{(w-1/(1+z))^2}\; dw.$$

We thus require

$$\left.\left(\frac{1}{w^n}\frac{1}{1-w}\right)'\right|_{w=1/(1+z)}$$

$$= \left.\left(\frac{-n}{w^{n+1}}\frac{1}{1-w} + \frac{1}{w^n}\frac{1}{(1-w)^2}\right)\right|_{w=1/(1+z)}$$

$$= -n(1+z)^{n+1}(1+z)/z + (1+z)^n(1+z)^2/z^2.$$

Substituting this into the outer integral and flipping signs we get two pieces which are

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m-1}n(1+z)^{n+2}/z\; dz$$

$$= \frac{n}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+3}}(1+z)^{n+m+1}\; dz = n \times \binom{n+m+1}{m+2}.$$

The second piece is

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+2}}(1+z)^{m-1}(1+z)^{n+2}/z^2\; dz$$

$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{m+4}}(1+z)^{n+m+1}\; dz = -\binom{n+m+1}{m+3}.$$

It follows that our answer is

$$\left(n - \frac{n-1}{m+3}\right)\binom{n+m+1}{m+2} = \frac{nm+2n+1}{m+3}\binom{n+m+1}{m+2}.$$

Remark. Being rigorous we also verify that the residue at infinity in the calculation of the inner integral is zero. We get

$$-\mathrm{Res}_{w=0}\frac{1}{w^2}w^n \frac{1}{1-1/w}\frac{1}{(1-(1+z)/w)^2}$$

$$= -\mathrm{Res}_{w=0} w^{n-2}\frac{w}{w-1}\frac{w^2}{(w-(1+z))^2} = -\mathrm{Res}_{w=0}\frac{w^{n+1}}{w-1}\frac{1}{(w-(1+z))^2}.$$

There is certainly no pole at zero here and the residue is zero as claimed (the term $1+z$ rotates in a circle around the point one on the real axis and with $\epsilon < 1$ it is never zero). This last result could also be obtained by comparing degrees of numerator and denominator.

This was math.stackexchange.com problem 1836190.

37 Use of a double Iverson bracket (B_1IR)

Suppose we seek to evaluate

$$Y(n) = \sum_{k=1}^{n} 2^{n-k}\binom{k}{\lfloor k/2\rfloor},$$

by considering

$$Y_1(n) = \sum_{k=0}^{\lfloor n/2\rfloor} 2^{n-2k}\binom{2k}{k} \quad \text{and} \quad Y_2(n) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} 2^{n-2k-1}\binom{2k+1}{k}.$$

We will use the following Iverson bracket:

$$[[0 \le k \le n]] = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{z^k}{z^{n+1}}\frac{1}{1-z}\; dz.$$

Evaluation of $Y_1(n)$. Introduce

$$\binom{2k}{k} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{k+1}}(1+w)^{2k}\; dw.$$

With the Iverson bracket controlling the range we can extend k to infinity to get for the sum

$$\frac{2^n}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2\rfloor+1}}\frac{1}{1-z}\sum_{k\ge 0} 2^{-2k} z^k \frac{(1+w)^{2k}}{w^k}\; dz\; dw.$$

We can instantiate these contours to get convergence of the series. We thus obtain

$$\frac{2^n}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2\rfloor+1}}\frac{1}{1-z}\frac{1}{1-z(1+w)^2/w/4}\; dz\; dw$$

$$= \frac{2^{n+2}}{2\pi i}\int_{|w|=\epsilon} \frac{1}{(1+w)^2}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{\lfloor n/2\rfloor+1}}\frac{1}{z-1}\frac{1}{z-4w/(1+w)^2}\; dz\; dw.$$

We evaluate the inner piece by computing the negative of the sum of the residues at $z=1$, $z=4w/(1+w)^2$ and $z=\infty$. We get for $z=1$

$$\frac{1}{1-4w/(1+w)^2} = \frac{(1+w)^2}{(1+w)^2-4w} = \frac{(1+w)^2}{(1-w)^2}$$

for a zero contribution.

We get for $z=\infty$

$$-\mathrm{Res}_{z=0}\frac{1}{z^2}\frac{1}{1/z^{\lfloor n/2\rfloor+1}}\frac{1}{1/z-1}\frac{1}{1/z-4w/(1+w)^2}$$

$$= -\mathrm{Res}_{z=0} z^{\lfloor n/2\rfloor+1}\frac{1}{1-z}\frac{1}{1-4wz/(1+w)^2}$$

again for a zero contribution.

Finally for $z=4w/(1+w)^2$ we get

$$-\frac{(1+w)^{2\lfloor n/2\rfloor+2}}{2^{2\lfloor n/2\rfloor+2}\times w^{\lfloor n/2\rfloor+1}}\frac{(1+w)^2}{(1-w)^2}.$$

Substitute into the outer integral to obtain

$$-\frac{2^{n \mod 2}}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{2\lfloor n/2\rfloor+2}}{w^{\lfloor n/2\rfloor+1}}\frac{1}{(1-w)^2}\; dw.$$

Extracting the negative of the residue we get the sum

$$2^{n \mod 2}\sum_{q=0}^{\lfloor n/2\rfloor} \binom{2\lfloor n/2\rfloor+2}{q}(\lfloor n/2\rfloor-q+1).$$

This yields

$$2^{n \mod 2}(\lfloor n/2 \rfloor + 1)\frac{1}{2}\left(2^{2\lfloor n/2\rfloor+2} - \binom{2\lfloor n/2\rfloor+2}{\lfloor n/2\rfloor+1}\right)$$

$$-2^{n \mod 2}(2\lfloor n/2\rfloor+2)\sum_{q=1}^{\lfloor n/2\rfloor}\binom{2\lfloor n/2\rfloor+1}{q-1}$$

$$= 2^{n \mod 2}(\lfloor n/2 \rfloor + 1)\frac{1}{2}\left(2^{2\lfloor n/2\rfloor+2} - \binom{2\lfloor n/2\rfloor+2}{\lfloor n/2\rfloor+1}\right)$$

$$-2^{n \mod 2}(\lfloor n/2\rfloor+1)\left(2^{2\lfloor n/2\rfloor+1} - 2\binom{2\lfloor n/2\rfloor+1}{\lfloor n/2\rfloor}\right)$$

$$= 2^{n \mod 2}(\lfloor n/2\rfloor+1)\left(2 - \frac{1}{2}\frac{2\lfloor n/2\rfloor+2}{\lfloor n/2\rfloor+1}\right)\binom{2\lfloor n/2\rfloor+1}{\lfloor n/2\rfloor}$$

$$= 2^{n \mod 2}(\lfloor n/2\rfloor+1)\binom{2\lfloor n/2\rfloor+1}{\lfloor n/2\rfloor}.$$

Evaluation of $Y_2(n)$. This is obviously very similar to the first case. We get the integral

$$\frac{2^{n+1}}{2\pi i}\int_{|w|=\epsilon}\frac{1}{1+w}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{\lfloor (n-1)/2\rfloor+1}}\frac{1}{z-1}\frac{1}{z-4w/(1+w)^2}\,dz\,dw.$$

There is no contribution from $z=1$ and $z=\infty$ as before which leaves

$$-\frac{2^{(n+1) \mod 2}}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{2\lfloor (n-1)/2\rfloor+3}}{w^{\lfloor (n-1)/2\rfloor+1}}\frac{1}{(1-w)^2}\,dw.$$

Extracting the negative of the residue we obtain

$$2^{(n+1) \mod 2}\sum_{q=0}^{\lfloor (n-1)/2\rfloor}\binom{2\lfloor (n-1)/2\rfloor+3}{q}(\lfloor (n-1)/2\rfloor - q + 1).$$

This yields

$$2^{(n+1) \mod 2}(\lfloor\frac{n-1}{2}\rfloor+1)\times\frac{1}{2}\left(2^{2\lfloor\frac{n-1}{2}\rfloor+3} - 2\binom{2\lfloor\frac{n-1}{2}\rfloor+3}{\lfloor\frac{n-1}{2}\rfloor+1}\right)$$

$$-2^{(n+1) \mod 2}(2\lfloor\frac{n-1}{2}\rfloor+3)\sum_{q=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{2\lfloor\frac{n-1}{2}\rfloor+2}{q-1}$$

$$= 2^{(n+1) \mod 2}(\lfloor\frac{n-1}{2}\rfloor+1)\times\frac{1}{2}\left(2^{2\lfloor\frac{n-1}{2}\rfloor+3} - 2\binom{2\lfloor\frac{n-1}{2}\rfloor+3}{\lfloor\frac{n-1}{2}\rfloor+1}\right)$$

$$-2^{(n+1) \mod 2}(2\lfloor\frac{n-1}{2}\rfloor+3)$$
$$\times\frac{1}{2}\left(2^{2\lfloor\frac{n-1}{2}\rfloor+2}-2\binom{2\lfloor\frac{n-1}{2}\rfloor+2}{\lfloor\frac{n-1}{2}\rfloor}-\binom{2\lfloor\frac{n-1}{2}\rfloor+2}{\lfloor\frac{n-1}{2}\rfloor+1}\right).$$

Evaluation of $Y(n)$. Keeping in mind that $Y(n)$ does not include a term for $k=0$ we get for $n=2p$ the contributions

$$-2^{2p}+(p+1)\binom{2p+1}{p}+p\left(2^{2p+1}-2\binom{2p+1}{p}\right)$$
$$-(2p+1)\left(2^{2p}-2\binom{2p}{p-1}-\binom{2p}{p}\right)$$
$$=-2^{2p+1}+(4p+2)\binom{2p}{p}.$$

On the other hand for $n=2p+1$ we obtain

$$-2^{2p+1}+2(p+1)\binom{2p+1}{p}+\frac{1}{2}(p+1)\left(2^{2p+3}-2\binom{2p+3}{p+1}\right)$$
$$-\frac{1}{2}(2p+3)\left(2^{2p+2}-2\binom{2p+2}{p}-\binom{2p+2}{p+1}\right)$$
$$=-2^{2p+2}+(4p+5)\binom{2p+1}{p}.$$

Joining the two formulae we get the compact closed form

$$-2^{n+1}+(2n+2+(n \mod 2))\binom{n}{\lfloor n/2\rfloor}.$$

I would conjecture that with the closed form being this simple now that it has been computed we can probably find a much simpler proof.

This was math.stackexchange.com problem 1219731.

38 Iverson bracket and an identity by Gosper, generalized (IR)

Suppose we seek to show that

$$\sum_{q=0}^{m-1}\binom{n-1+q}{q}x^n(1-x)^q+\sum_{q=0}^{n-1}\binom{m-1+q}{q}x^q(1-x)^m=1$$

where $n,m\geq 1$.

We will evaluate the second term by a contour integral and show that is equal to one minus the first term which is the desired result.

Introduce the Iverson bracket

$$[[0 \le q \le n-1]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^q}{z^n} \frac{1}{1-z} \; dz.$$

With this bracket we may extend the sum in q to infinity to get

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \sum_{q\ge 0} \binom{m-1+q}{q} z^q x^q (1-x)^m \; dz$$

$$= \frac{(1-x)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \sum_{q\ge 0} \binom{m-1+q}{q} z^q x^q \; dz$$

$$= \frac{(1-x)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \frac{1}{(1-xz)^m} \; dz.$$

Now we have three poles here at $z=0$ and $z=1$ and $z=1/x$ and the residues at these poles sum to zero, so we can evaluate the residue at zero by computing the negative of the residues at $z=1$ and $z=1/x$.

Observe that the residue at infinity is zero as can be seen from the following computation:

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^n \frac{1}{1-1/z} \frac{1}{(1-x/z)^m}$$

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^n \frac{z}{z-1} \frac{z^m}{(z-x)^m}$$

$$-\mathrm{Res}_{z=0} z^{n+m-1} \frac{1}{z-1} \frac{1}{(z-x)^m} = 0.$$

Returning to the main thread the residue at $z=1$ as seen from

$$-\frac{(1-x)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{z-1} \frac{1}{(1-xz)^m} \; dz.$$

is

$$-(1-x)^m \frac{1}{(1-x)^m} = -1.$$

For the residue at $z=1/x$ we consider

$$\frac{(1-x)^m}{x^m \times 2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \frac{1}{(1/x-z)^m} \; dz$$

$$= \frac{(-1)^m (1-x)^m}{x^m \times 2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \frac{1}{1-z} \frac{1}{(z-1/x)^m} \; dz.$$

and use the following derivative:

$$\frac{1}{(m-1)!} \left(\frac{1}{z^n} \frac{1}{1-z} \right)^{(m-1)}$$

$$= \frac{1}{(m-1)!} \sum_{q=0}^{m-1} \binom{m-1}{q} \frac{(-1)^q (n+q-1)!}{(n-1)! z^{n+q}} \frac{(m-1-q)!}{(1-z)^{m-q}}$$

$$= \sum_{q=0}^{m-1} \frac{1}{q!} \frac{(-1)^q (n+q-1)!}{(n-1)! z^{n+q}} \frac{1}{(1-z)^{m-q}}$$

$$= \sum_{q=0}^{m-1} \binom{n+q-1}{q} \frac{(-1)^q}{z^{n+q}} \frac{1}{(1-z)^{m-q}}.$$

Evaluate this at $z = 1/x$ and multiply by the factor in front to get

$$\frac{(-1)^m (1-x)^m}{x^m} \times \sum_{q=0}^{m-1} \binom{n+q-1}{q} (-1)^q x^{n+q} \frac{1}{(1-1/x)^{m-q}}$$

$$= \frac{(-1)^m (1-x)^m}{x^m} \times \sum_{q=0}^{m-1} \binom{n+q-1}{q} (-1)^q x^{n+q} \frac{x^{m-q}}{(x-1)^{m-q}}$$

$$= (-1)^m (1-x)^m \times \sum_{q=0}^{m-1} \binom{n+q-1}{q} (-1)^q x^n (-1)^{m-q} \frac{1}{(1-x)^{m-q}}$$

$$= \sum_{q=0}^{m-1} \binom{n+q-1}{q} x^n (1-x)^q.$$

This yields for the second sum term the value

$$1 - \sum_{q=0}^{m-1} \binom{n+q-1}{q} x^n (1-x)^q$$

showing that when we add the first and the second sum by cancellation the end result is one, as claimed.

This was math.stackexchange.com problem 538309.

Special case by formal power series

Here we show the special case:

$$\sum_{k=0}^{n} \binom{m+k}{k} 2^{n-k} + \sum_{k=0}^{m} \binom{n+k}{k} 2^{m-k} = 2^{n+m+1}.$$

which is obtained from $x = 1/2$. We have by inspection i.e. same as before that

$$\sum_{k=0}^{n} \binom{m+k}{k} 2^{n-k} = 2^n [z^n] \frac{1}{1-z} \frac{1}{(1-z/2)^{m+1}}.$$

This is

$$\begin{aligned}
&2^n \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{1-z} \frac{1}{(1-z/2)^{m+1}} \\
&= -2^n \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{2^{m+1}}{(2-z)^{m+1}} \\
&= 2^{n+m+1}(-1)^m \times \mathrm{Res}_{z=0} \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{1}{(z-2)^{m+1}}.
\end{aligned}$$

With

$$f(z) = 2^{n+m+1}(-1)^m \frac{1}{z^{n+1}} \frac{1}{z-1} \frac{1}{(z-2)^{m+1}}$$

we will be using the fact that residues sum to zero i.e.

$$\mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=1} f(z) + \mathrm{Res}_{z=2} f(z) + \mathrm{Res}_{z=\infty} f(z) = 0.$$

The residue at infinity is zero since $\lim_{R\to\infty} 2\pi R/R^{n+1}/R/R^{m+1} = 0$.
The residue at one is

$$2^{n+m+1}(-1)^m \times (-1)^{m+1} = -2^{n+m+1}.$$

For the residue at two we use the Leibniz rule:

$$\begin{aligned}
&\frac{1}{m!} \left(\frac{1}{z^{n+1}} \frac{1}{z-1} \right)^{(m)} \\
&= \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{(n+k)!}{n!} \frac{1}{z^{n+1+k}} (-1)^{m-k} \frac{(m-k)!}{(z-1)^{m-k+1}} \\
&= (-1)^m \sum_{k=0}^{m} \binom{n+k}{k} \frac{1}{z^{n+1+k}} \frac{1}{(z-1)^{m-k+1}}.
\end{aligned}$$

Restore factor in front and evaluate at $z = 2$:

$$2^{n+m+1}(-1)^m \times (-1)^m \sum_{k=0}^{m} \binom{n+k}{k} \frac{1}{2^{n+1+k}} = \sum_{k=0}^{m} \binom{n+k}{k} 2^{m-k}.$$

Summing the residues we have shown that

$$\sum_{k=0}^{n} \binom{m+k}{k} 2^{n-k} + \sum_{k=0}^{m} \binom{n+k}{k} 2^{m-k} - 2^{n+m+1} = 0$$

which is the claim.
This was math.stackexchange.com problem 3024722.

39 A double hypergeometric sum (B_1)

Suppose we seek to verify that

$$\binom{2n}{n} = \sum_{l=0}^{n} \sum_{r=0}^{(n-l)/2} \binom{n}{l} \binom{n-l}{r} \binom{n-l-r}{r}$$

which is

$$\sum_{l=0}^{n} \binom{n}{l} \sum_{r=0}^{(n-l)/2} \binom{n-l}{r} \binom{n-l-r}{r}.$$

Start by computing the inner sum.

Introduce

$$\binom{n-l}{r} = \binom{n-l}{n-l-r} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-l-r+1}} (1+z)^{n-l} \; dz.$$

and

$$\binom{n-l-r}{r} = \binom{n-l-r}{n-l-2r} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-l-2r+1}} (1+w)^{n-l-r} \; dw.$$

Observe carefully that this second integral is zero when $2r \ge n-l+1$ so it controls the range and we may extend r to infinity.

This yields for the inner sum

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{n-l}}{w^{n-l+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-l}}{z^{n-l+1}} \sum_{r\ge 0} \frac{w^{2r}}{(1+w)^r} z^r \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{n-l}}{w^{n-l+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-l}}{z^{n-l+1}} \frac{1}{1-zw^2/(1+w)} \; dz \; dw.$$

Substitute this into the outer sum to get

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{n}}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n}}{z^{n+1}} \frac{1}{1-zw^2/(1+w)}$$

$$\times \sum_{l=0}^{n} \binom{n}{l} \frac{w^l}{(1+w)^l} \frac{z^l}{(1+z)^l} \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{n}}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n}}{z^{n+1}} \frac{1}{1-zw^2/(1+w)}$$

$$\times \left(1 + \frac{w}{1+w} \frac{z}{1+z}\right)^n \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1-zw^2/(1+w)} (1+w+z+2wz)^n \; dz \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{1 - zw^2/(1+w)} (1+w+z(1+2w))^n \; dz \; dw.$$

Extracting the inner residue we obtain

$$\sum_{q=0}^{n} \frac{w^{2n-2q}}{(1+w)^{n-q}} {n \choose q} (1+w)^{n-q} (1+2w)^q$$

$$= \sum_{q=0}^{n} {n \choose q} (1+2w)^q w^{2n-2q} = (1+2w+w^2)^n = (1+w)^{2n}.$$

We conclude in extracting the residue in w to get

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} (1+w)^{2n} \; dw = [w^n](1+w)^{2n} = {2n \choose n}.$$

This was math.stackexchange.com problem 1445090.

40 Factoring a triple hypergeometric sum (B_1)

Suppose we seek to evaluate

$$\sum_{k=0}^{n} (-1)^k {1+p+q \choose k} {p+n-k \choose n-k} {q+n-k \choose n-k}$$

which is claimed to be

$${p \choose n} {q \choose n}.$$

Introduce

$${p+n-k \choose n-k} = \frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{(1+z_1)^{p+n-k}}{z_1^{n-k+1}} \; dz_1$$

and

$${q+n-k \choose n-k} = \frac{1}{2\pi i} \int_{|z_2|=\epsilon} \frac{(1+z_2)^{q+n-k}}{z_2^{n-k+1}} \; dz_2.$$

Observe that these integrals vanish when $k > n$ and we may extend k to infinity.

We thus obtain for the sum

$$\frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{(1+z_1)^{p+n}}{z_1^{n+1}} \frac{1}{2\pi i} \int_{|z_2|=\epsilon} \frac{(1+z_2)^{q+n}}{z_2^{n+1}}$$

$$\times \sum_{k \ge 0} {1+p+q \choose k} (-1)^k \frac{z_1^k z_2^k}{(1+z_1)^k (1+z_2)^k} \; dz_2 \; dz_1.$$

This is

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^{p+n}}{z_1^{n+1}} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_2)^{q+n}}{z_2^{n+1}}$$
$$\times \left(1 - \frac{z_1 z_2}{(1+z_1)(1+z_2)}\right)^{p+q+1} \; dz_2 \; dz_1$$

or

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^{n-q-1}}{z_1^{n+1}} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_2)^{n-p-1}}{z_2^{n+1}} (1+z_1+z_2)^{p+q+1} \; dz_2 \; dz_1$$

Supposing that $p \ge n$ and $q \ge n$ this may be re-written as

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{n+1}(1+z_1)^{q+1-n}} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{1}{z_2^{n+1}(1+z_2)^{p+1-n}}$$
$$\times (1+z_1+z_2)^{p+q+1} \; dz_2 \; dz_1$$

Put $z_2 = (1+z_1)z_3$ so that $dz_2 = (1+z_1)\; dz_3$ to get

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{n+1}(1+z_1)^{q+1-n}}$$
$$\times \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{1}{(1+z_1)^{n+1} z_3^{n+1}(1+(1+z_1)z_3)^{p+1-n}}$$
$$\times (1+z_1)^{p+q+1}(1+z_3)^{p+q+1} \; (1+z_1) \; dz_3 \; dz_1$$

which is

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^p}{z_1^{n+1}} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{1}{z_3^{n+1}(1+z_3+z_1 z_3)^{p+1-n}}$$
$$\times (1+z_3)^{p+q+1} \; dz_3 \; dz_1$$
$$= \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{(1+z_1)^p}{z_1^{n+1}} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{(1+z_3)^{n+q}}{z_3^{n+1}(1+z_1 z_3/(1+z_3))^{p+1-n}} \; dz_3 \; dz_1$$

Extracting the residue for z_1 first we obtain

$$\sum_{k=0}^{n} \binom{p}{n-k} \frac{(1+z_3)^{n+q}}{z_3^{n+1}} \binom{k+p-n}{k} (-1)^k \frac{z_3^k}{(1+z_3)^k}.$$

The residue for z_3 then yields

$$\sum_{k=0}^{n} (-1)^k \binom{p}{n-k} \binom{k+p-n}{k} \binom{n-k+q}{n-k}.$$

The sum term here is

$$\frac{p! \times (p+k-n)! \times (q+n-k)!}{(n-k)!(p+k-n)! \times k!(p-n)! \times (n-k)! q!}$$

which simplifies to

$$\frac{p! \times n! \times (q+n-k)!}{(n-k)! \times n! \times k!(p-n)! \times (n-k)!q!}$$

which is

$$\binom{n}{k}\binom{p}{n}\binom{q+n-k}{q}$$

so we have for the sum

$$\binom{p}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{q+n-k}{q}.$$

To evaluae the remaining sum we introduce

$$\binom{q+n-k}{q} = \frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{q+n-k}}{v^{q+1}}\; dv$$

getting for the sum

$$\binom{p}{n}\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{q+n}}{v^{q+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(1+v)^k}\; dv$$

$$= \binom{p}{n}\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{q+n}}{v^{q+1}}\left(1-\frac{1}{1+v}\right)^n\; dv$$

$$= \binom{p}{n}\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{q}}{v^{q-n+1}}\; dv = \binom{p}{n}\binom{q}{q-n}$$

which is

$$\binom{p}{n}\binom{q}{n}.$$

This concludes the argument.

This is math.stackexchange.com problem 174054.

41 Factoring a triple hypergeometric sum II (B_1B_2)

Suppose we seek to evaluate

$$\sum_{k\geq 0}\binom{p}{k}\binom{q}{k}\binom{n+k}{p+q}$$

which is claimed to be

$$\binom{n}{p}\binom{n}{q}.$$

We use the integrals

$$\binom{p}{k} = \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{p-k+1}} \frac{1}{(1-z_1)^{k+1}}\; dz_1$$

and

$$\binom{q}{k} = \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{1}{z_2^{q-k+1}} \frac{1}{(1-z_2)^{k+1}}\; dz_2.$$

These two effectively control the range their product being zero when $k > \min(p, q)$ so that we may extend the sum to infinity.

We also use

$$\binom{n+k}{p+q} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p+q+1}}(1+w)^{n+k}\; dw$$

This yields for the sum

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p+q+1}}(1+w)^{n} \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{1+p}} \frac{1}{1-z_1} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{1}{z_2^{1+q}} \frac{1}{1-z_2}$$

$$\times \sum_{k\ge 0} \frac{(1+w)^k z_1^k z_2^k}{(1-z_1)^k (1-z_2)^k}\; dz_2\; dz_1\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p+q+1}}(1+w)^{n} \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{1+p}} \frac{1}{1-z_1} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{1}{z_2^{1+q}} \frac{1}{1-z_2}$$

$$\times \frac{1}{1-(1+w)z_1 z_2/(1-z_1)/(1-z_2)}\; dz_2\; dz_1\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p+q+1}}(1+w)^{n} \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{1+p}} \frac{1}{2\pi i}\int_{|z_2|=\epsilon} \frac{1}{z_2^{1+q}}$$

$$\times \frac{1}{(1-z_1)(1-z_2)-(1+w)z_1 z_2}\; dz_2\; dz_1\; dw.$$

The inner term here is

$$\frac{1}{1-z_1-z_2+z_1 z_2 - z_1 z_2 - w z_1 z_2}$$

$$= \frac{1}{1-z_1-z_2-w z_1 z_2} = \frac{1}{1-z_1} \frac{1}{1-(1+w z_1)z_2/(1-z_1)}.$$

Extracting the residue in z_2 then yields

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p+q+1}}(1+w)^{n} \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{1+p}} \frac{1}{1-z_1} \frac{(1+w z_1)^q}{(1-z_1)^q}\; dz_1\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p+q+1}}(1+w)^{n} \frac{1}{2\pi i}\int_{|z_1|=\epsilon} \frac{1}{z_1^{1+p}} \frac{(1+w z_1)^q}{(1-z_1)^{q+1}}\; dz_1\; dw.$$

By symmetry of the initial sum we may suppose that $p \le q$, getting for the inner integral

$$\sum_{m=0}^{p} \binom{q}{m} w^m \binom{p+q-m}{q}.$$

The outer integral now yields

$$\sum_{m=0}^{p} \binom{q}{m} \binom{p+q-m}{q} \binom{n}{p+q-m}.$$

The sum term here is

$$\frac{n!}{(q-m)! \times m! \times (p-m)! \times (n+m-p-q)!}$$

$$= \binom{n}{p} \frac{p! \times (n-p)!}{(q-m)! \times m! \times (p-m)! \times (n+m-p-q)!}$$

$$= \binom{n}{p} \binom{p}{m} \binom{n-p}{q-m}.$$

It thus remains to show that

$$\sum_{m=0}^{p} \binom{p}{m} \binom{n-p}{q-m} = \binom{n}{q}$$

which may be done combinatorially or by inspecting the integral

$$\frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{q+1}} (1+v)^{n-p} \sum_{m=0}^{p} \binom{p}{m} v^m \; dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{q+1}} (1+v)^{n-p} (1+v)^p \; dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{q+1}} (1+v)^n \; dv = \binom{n}{q}.$$

This was math stackexchange problem 280481.
A simpler version of this identity is at section 19.

42 Factoring a triple hypergeometric sum III (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{pn-n}{k} \binom{pn+k}{k} = \binom{pn}{n}^2.$$

We use the integrals

$$\binom{pn-n}{k} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z^{k+1}}\; dz$$

and

$$\binom{pn+k}{k} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pn+k}}{w^{k+1}}\; dw.$$

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pn}}{w}\sum_{k=0}^{n}\binom{n}{k}\frac{(1+w)^k}{z^k w^k}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pn}}{w}\left(1+\frac{1+w}{zw}\right)^n\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{pn-n}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pn}}{w^{n+1}}(1+w+zw)^n\; dw\; dz.$$

Expanding the binomial in the inner sum we get

$$\sum_{q=0}^{n}\binom{n}{q} w^q (1+z)^q$$

which yields

$$\sum_{q=0}^{n}\binom{n}{q}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{pn-n+q}}{z^{n+1}}\binom{pn}{n-q}\; dz$$

$$= \sum_{q=0}^{n}\binom{n}{q}\binom{pn-n+q}{n}\binom{pn}{n-q}.$$

The inner term is

$$\binom{n}{q}\binom{pn-n+q}{n}\binom{pn}{pn-n+q}$$

$$= \frac{(pn)!}{q!\times(n-q)!\times(pn-2n+q)!\times(n-q)!}$$

$$= \binom{pn}{n}\frac{n!\times(pn-n)!}{q!\times(n-q)!\times(pn-2n+q)!\times(n-q)!}$$

$$= \binom{pn}{n}\binom{n}{q}\binom{pn-n}{n-q}.$$

Thus it remains to show that

$$\sum_{q=0}^{n}\binom{n}{q}\binom{pn-n}{n-q} = \binom{pn}{n}.$$

This can be done combinatorially or using the integral

$$\frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{(1+v)^{pn-n}}{v^{n+1}}\sum_{q=0}^{n}\binom{n}{q}v^q\;dv$$

$$=\frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{(1+v)^{pn-n}}{v^{n+1}}(v+1)^n\;dv$$

$$=\frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{(1+v)^{pn}}{v^{n+1}}=\binom{pn}{n}.$$

This was math.stackexchange.com problem 656116.

43 Factoring a triple hypergeometric sum IV (B_1)

Suppose we seek to verify that

$$\sum_{r=0}^{\min\{m,n,p\}}\binom{m}{r}\binom{n}{r}\binom{p+m+n-r}{m+n}=\binom{p+m}{m}\binom{p+n}{n}.$$

Introduce

$$\binom{n}{r}=\binom{n}{n-r}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-r+1}}(1+z)^n\;dz$$

and

$$\binom{p+m+n-r}{m+n}=\binom{p+m+n-r}{p-r}$$

$$=\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{p-r+1}}(1+w)^{p+m+n-r}\;dw.$$

Observe carefully that the first of these is zero when $r>n$ and the second one when $r>p$ so we may extend the range of r to infinity.

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+m+n}}{w^{p+1}}\sum_{r\geq 0}\binom{m}{r}z^r\frac{w^r}{(1+w)^r}\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+m+n}}{w^{p+1}}\left(1+\frac{zw}{1+w}\right)^m\;dw\;dz$$

$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+n}}{w^{p+1}}(1+w+zw)^m\;dw\;dz.$$

The inner integral is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+n}}{w^{p+1}}\sum_{q=0}^{m}\binom{m}{q}(1+z)^q w^q\; dw$$

with residue

$$\sum_{q=0}^{\min(m,p)}\binom{m}{q}\binom{p+n}{p-q}(1+z)^q$$

which in combination with the outer integral yields

$$\sum_{q=0}^{\min(m,p)}\binom{m}{q}\binom{p+n}{n+q}\binom{n+q}{n}.$$

Now note that

$$\binom{p+n}{n+q}\binom{n+q}{n} = \frac{(p+n)!}{(p-q)!(n+q)!}\frac{(n+q)!}{q!n!}$$

$$= \frac{(p+n)!}{(p-q)!p!}\frac{p!}{q!n!} = \binom{p+n}{n}\binom{p}{q}.$$

Therefore we just need to verify that

$$\sum_{q=0}^{\min(m,p)}\binom{m}{q}\binom{p}{p-q} = \binom{p+m}{m}$$

which follows by inspection.
It can also be done with the integral

$$\binom{p}{p-q} = \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^p}{w^{p-q+1}}\; dw$$

which is zero when $q > p$ so we can extend q to infinity to get for the sum

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^p}{w^{p+1}}\sum_{q\geq 0}\binom{m}{q}w^q\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{p+m}}{w^{p+1}}\; dw$$

$$= \binom{p+m}{m}.$$

This was math.stackexchange.com problem 1460712.

44 A triple hypergeometric sum V (B_1)

Suppose we seek to verify that

$$\sum_{p=0}^{l}\sum_{q=0}^{p}(-1)^q\binom{m-p}{m-l}\binom{n}{q}\binom{m-n}{p-q} = 2^l\binom{m-n}{l}$$

where $m \geq n$ and $m-n \geq l$.

This is

$$\sum_{p=0}^{l}\binom{m-p}{m-l}\sum_{q=0}^{p}(-1)^q\binom{n}{q}\binom{m-n}{p-q}.$$

Now introduce the integral

$$\binom{m-n}{p-q} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p-q+1}}(1+z)^{m-n}\,dz.$$

Note that this vanishes when $q > p$ so we may extend the range of q to infinity, getting for the sum

$$\sum_{p=0}^{l}\binom{m-p}{m-l}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p+1}}(1+z)^{m-n}\sum_{q\geq 0}(-1)^q\binom{n}{q}z^q\,dz$$

$$= \sum_{p=0}^{l}\binom{m-p}{l-p}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{p+1}}(1+z)^{m-n}(1-z)^n\,dz.$$

Introduce furthermore

$$\binom{m-p}{l-p} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l-p+1}}(1+w)^{m-p}\,dw.$$

This too vanishes when $p > l$ so we may extend p to infinity, getting

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l+1}}(1+w)^m$$

$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}(1+z)^{m-n}(1-z)^n\sum_{p\geq 0}\frac{w^p}{z^p}\frac{1}{(1+w)^p}\,dz\,dw.$$

The geometric series converges when $|w/z/(1+w)| < 1$. We get

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{l+1}}(1+w)^m$$

$$\times\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z}(1+z)^{m-n}(1-z)^n\frac{1}{1-w/z/(1+w)}\,dz\,dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{l+1}} (1+w)^m$$

$$\times \frac{1}{2\pi i} \int_{|z|=\epsilon} (1+z)^{m-n} (1-z)^n \frac{1}{z - w/(1+w)} \; dz \; dw.$$

Now from the convergence we have $|w/(1+w)| < |z|$ which means the pole at $z = w/(1+w)$ is inside the contour $|z| = \epsilon$. Extracting the residue yields (the pole at zero has disappeared)

$$\frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{l+1}} (1+w)^m \left(1 + \frac{w}{1+w}\right)^{m-n} \left(1 - \frac{w}{1+w}\right)^n \; dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{l+1}} (1+2w)^{m-n} \; dw$$

$$= 2^l \binom{m-n}{l}.$$

This was math.stackexchange.com problem 1767709.

45 Basic usage of exponentiation integral to obtain Stirling number formulae (E)

Suppose we seek to evaluate

$$\sum_{q=0}^{n} (n-2q)^k \binom{n}{2q+1}.$$

We observe that

$$(n-2q)^k = \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+1}} \exp((n-2q)z) \; dz.$$

This yields for the sum

$$\frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+1}} \sum_{q=0}^{n} \binom{n}{2q+1} \exp((n-2q)z) \; dz$$

$$= \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp((n+1)z)}{z^{k+1}} \sum_{q=0}^{n} \binom{n}{2q+1} \exp((-2q-1)z) \; dz$$

which is

$$\frac{1}{2} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp((n+1)z)}{z^{k+1}}$$

$$\times \left(\sum_{q=0}^{n} \binom{n}{q} \exp(-qz) - \sum_{q=0}^{n} \binom{n}{q} (-1)^q \exp(-qz) \right) \; dz.$$

This yields two pieces, call them A_1 and A_2. Piece A_1 is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp((n+1)z)}{z^{k+1}}(1+\exp(-z))^n\;dz$$

$$=\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}(\exp(z)+1)^n\;dz$$

and piece A_2 is

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp((n+1)z)}{z^{k+1}}(1-\exp(-z))^n\;dz$$

$$=\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}(\exp(z)-1)^n\;dz.$$

Recall the species equation for labelled set partitions:

$$\mathfrak{P}(\mathcal{U}\mathfrak{P}_{\ge 1}(\mathcal{Z}))$$

which yields the bivariate generating function of the Stirling numbers of the second kind

$$\exp(u(\exp(z)-1)).$$

This implies that

$$\sum_{n\ge q}{n \brace q}\frac{z^n}{n!}=\frac{(\exp(z)-1)^q}{q!}$$

and

$$\sum_{n\ge q}{n \brace q}\frac{z^{n-1}}{(n-1)!}=\frac{(\exp(z)-1)^{q-1}}{(q-1)!}\exp(z).$$

Now to evaluate A_1 proceed as follows:

$$\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}(2+\exp(z)-1)^n\;dz$$

$$=\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}\sum_{q=0}^{n}{n \choose q}2^{n-q}(\exp(z)-1)^q\;dz$$

$$=\sum_{q=0}^{n}{n \choose q}2^{n-q}\times q!\times\frac{1}{2}\frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{\exp(z)}{z^{k+1}}\frac{(\exp(z)-1)^q}{q!}\;dz.$$

Recognizing the differentiated Stirling number generating function this becomes

$$\sum_{q=0}^{n}{n \choose q}2^{n-q-1}\times q!\times{k+1 \brace q+1}.$$

Now observe that when $n>k+1$ the Stirling number for $k+1<q\le n$ is zero, so we may replace n by $k+1$. Similarly, when $n<k+1$ the binomial

coefficient for $n < q \le k+1$ is zero so we may again replace n by $k+1$. This gives the following result for A_1 :

$$\sum_{q=0}^{k+1} \binom{n}{q} 2^{n-q-1} \times q! \times \left\{ {k+1 \atop q+1} \right\}.$$

Moving on to A_2 we observe that when $k < n$ the contribution is zero because the series for $\exp(z) - 1$ starts at z. This integral is simple and we have

$$\frac{1}{2} \frac{k! \times n!}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(z)}{z^{k+1}} \frac{(\exp(z)-1)^n}{n!} \; dz.$$

Recognizing the Stirling number this yields

$$\frac{1}{2} \times n! \times \left\{ {k+1 \atop n+1} \right\}.$$

which correctly represents the fact that we have a zero contribution when $k < n$.

This finally yields the closed form formula

$$\sum_{q=0}^{k+1} \binom{n}{q} 2^{n-q-1} \times q! \times \left\{ {k+1 \atop q+1} \right\} - \frac{1}{2} \times n! \times \left\{ {k+1 \atop n+1} \right\}.$$

confirming the previous results.

This was math.stackexchange.com problem 1353963

46 Evaluation of a three-variable hypergeometric sum (B_2)

The sum

$$\sum_{p+q+r=n} \binom{p+q}{p} \binom{p+r}{r} \binom{q+r}{q}$$

with $p, q, r \ge 0$

is claimed to be

$$\sum_{q=0}^{n} \binom{2q}{q},$$

and is equal to

$$\sum_{p=0}^{n} \sum_{q=0}^{n-p} \binom{p+q}{q} \binom{p+n-p-q}{p} \binom{q+n-p-q}{q}$$

which is

$$\sum_{p=0}^{n} \sum_{q=0}^{n-p} \binom{p+q}{q} \binom{n-q}{p} \binom{n-p}{q}.$$

Re-write this as

$$\sum_{\substack{0 \le p,q \\ p+q \le n}} \binom{p+q}{q} \binom{n-q}{p} \binom{n-p}{q}.$$

Introduce the integral representations

$$\binom{n-q}{p} = \frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{1}{(1-z_1)^{p+1} z_1^{n-p-q+1}} \; dz_1$$

and

$$\binom{n-p}{q} = \frac{1}{2\pi i} \int_{|z_2|=\gamma} \frac{1}{(1-z_2)^{q+1} z_2^{n-p-q+1}} \; dz_2.$$

Observe carefully that these integrals are zero when $p+q > n$ so we may extend the summation in p and q to infinity.

We get for the sum

$$\frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{1}{(1-z_1) z_1^{n+1}} \frac{1}{2\pi i} \int_{|z_2|=\gamma} \frac{1}{(1-z_2) z_2^{n+1}}$$

$$\times \sum_{p,q \ge 0} \binom{p+q}{q} \frac{z_1^{p+q} z_2^{p+q}}{(1-z_1)^p (1-z_2)^q} \; dz_2 \; dz_1.$$

The inner term is

$$\sum_{p \ge 0} \frac{z_1^p z_2^p}{(1-z_1)^p} \sum_{q \ge 0} \binom{q+p}{p} \frac{z_1^q z_2^q}{(1-z_2)^q} = \sum_{p \ge 0} \frac{z_1^p z_2^p}{(1-z_1)^p} \frac{1}{(1 - z_1 z_2/(1-z_2))^{p+1}}$$

$$= \frac{1}{1 - z_1 z_2/(1-z_2)} \frac{1}{1 - z_1 z_2/(1-z_1)/(1 - z_1 z_2/(1-z_2))}$$

$$= \frac{1}{1 - z_1 z_2/(1-z_2) - z_1 z_2/(1-z_1)}$$

$$= \frac{(1-z_1)(1-z_2)}{(1-z_1)(1-z_2) - z_1 z_2 (2 - z_1 - z_2)}$$

$$= \frac{(1-z_1)(1-z_2)}{(1-z_1 z_2)/(1-z_1-z_2)}.$$

Substituting this into the integral yields

$$\frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{1}{z_1^{n+1}} \frac{1}{2\pi i} \int_{|z_2|=\gamma} \frac{1}{z_2^{n+1}} \frac{1}{(1-z_1 z_2)/(1-z_1-z_2)} \; dz_2 \; dz_1.$$

This is

$$\frac{1}{2\pi i} \int_{|z_1|=\epsilon} \frac{1}{z_1^{n+1}} \frac{1}{1-z_1} \frac{1}{2\pi i} \int_{|z_2|=\gamma} \frac{1}{z_2^{n+1}} \frac{1}{(1-z_1 z_2)/(1-z_2/(1-z_1))} \; dz_2 \; dz_1.$$

Extracting the inner residue now yields

$$\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{z_1^{n+1}}\frac{1}{1-z_1}\sum_{q=0}^{n} z_1^q \frac{1}{(1-z_1)^{n-q}}\; dz_1$$

$$= \sum_{q=0}^{n}\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{z_1^{n-q+1}}\frac{1}{(1-z_1)^{n+1-q}}\; dz_1$$

$$= \sum_{q=0}^{n}\frac{1}{2\pi i}\int_{|z_1|=\epsilon}\frac{1}{z_1^{q+1}}\frac{1}{(1-z_1)^{q+1}}\; dz_1.$$

This is

$$\sum_{q=0}^{n}\binom{q+q}{q} = \sum_{q=0}^{n}\binom{2q}{q},$$

which was to be shown, QED.

Remark. For the geometric series to converge we may choose $\epsilon = \gamma = 1/3$. This gives the bound $\frac{1}{3}\frac{1}{3}\frac{1}{2/3}$ for the common ratio, which is $\frac{1}{6}$ and it goes through. We get for the common ratio of the second one that it is $\frac{1}{3}\frac{1}{3}\frac{1}{2/3}\frac{1}{5/6} = \frac{1}{5}$ and it goes through as well.

This is math.stackexchange.com problem 177209.

47 Three phase application including Leibniz' rule (B_1B_2R)

Suppose we seek to verify that

$$\sum_{q=0}^{n} q\binom{2n}{n+q}\binom{m+q-1}{2m-1} = m\times 4^{n-m}\times\binom{n}{m}$$

where $n \geq m$.

We use the integrals

$$\binom{2n}{n+q} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-q+1}}\frac{1}{(1-z)^{n+q+1}}\; dz.$$

and

$$\binom{m+q-1}{2m-1} = \frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{m+q-1}}{w^{2m}}\; dw.$$

Observe that the first integral is zero when $q > n$ so we may extend q to infinity.

This yields for the sum

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}\frac{1}{(1-z)^{n+1}}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{m-1}}{w^{2m}}\sum_{q\geq 0} q\frac{z^q(1+w)^q}{(1-z)^q}\; dw\; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \frac{z(1+w)/(1-z)}{(1-z(1+w)/(1-z))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{(1-z)^{n+1}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m-1}}{w^{2m}} \frac{z(1+w)(1-z)}{(1-z-z(1+w))^2} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n}} \frac{1}{(1-z)^{n}}$$

$$\times \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m}}{w^{2m}} \frac{1}{(1-2z-zw)^2} \; dw \; dz.$$

We evaluate the inner integral using the negative of the residue at the pole at $w = (1-2z)/z$, starting from

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+2}} \frac{1}{(1-z)^{n}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{m}}{w^{2m}} \frac{1}{(w-(1-2z)/z)^2} \; dw \; dz.$$

Differentiating we have

$$m\frac{(1+w)^{m-1}}{w^{2m}} - 2m\frac{(1+w)^{m}}{w^{2m+1}} = (w - 2(1+w))\, m \frac{(1+w)^{m-1}}{w^{2m+1}}$$

$$= (-w-2)m\frac{(1+w)^{m-1}}{w^{2m+1}}.$$

The negative of this evaluated at $w = (1-2z)/z$ is

$$\frac{1}{z} \times m \times \frac{(1-z)^{m-1}}{z^{m-1}} \times \frac{z^{2m+1}}{(1-2z)^{2m+1}}$$

which finally yields

$$\frac{m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \frac{1}{(1-2z)^{2m+1}} \; dz.$$

We have that the residues at zero, one and one half sum to zero with the first one being the sum we are trying to compute. Therefore we evaluate these in turn. We will restore the front factor of m at the end.

For the residue at zero we have using the Cauchy product that

$$\sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{n-m-q} \binom{2m+n-m-q}{n-m-q}$$

$$= \sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{n-m-q} \binom{m+n-q}{2m}.$$

For the residue at one we have that

$$\frac{(-1)^{n-m+1}}{(n-m)!} \left(\frac{1}{z^{n-m+1}} \frac{1}{(1-2z)^{2m+1}} \right)^{(n-m)}$$

$$= \frac{(-1)^{n-m+1}}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times 2^{n-m-q} \frac{(2m+n-m-q)!}{(2m)! \times (1-2z)^{2m+1+n-m-q}}$$

$$= \frac{(-1)^{n-m+1} 2^{n-m}}{(n-m)!} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times 2^{-q} \frac{(m+n-q)!}{(2m)! \times (1-2z)^{m+1+n-q}}.$$

Evaluate this at one to get

$$2^{n-m} \sum_{q=0}^{n-m} \binom{n-m+q}{q} 2^{-q} \binom{m+n-q}{2m}.$$

The residue at one evaluates to the sum we seek just like the residue at zero. This leaves the residue at one half, where we find

$$\frac{(-1)^{2m+1}}{(2m)! \times 2^{2m+1}} \left(\frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \right)^{(2m)}$$

$$= \frac{(-1)^{2m+1}}{(2m)! \times 2^{2m+1}} \sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times \frac{(n-m+2m-q)!}{(n-m)! \times (1-z)^{n-m+1+2m-q}}$$

$$= \frac{(-1)^{2m+1}}{(2m)! \times 2^{2m+1}} \sum_{q=0}^{2m} \binom{2m}{q} (-1)^q \frac{(n-m+q)!}{(n-m)! \times z^{n-m+1+q}}$$

$$\times \frac{(n+m-q)!}{(n-m)! \times (1-z)^{n+m+1-q}}.$$

Evaluate this at one half to get

$$-\frac{1}{2^{2m+1}} \sum_{q=0}^{2m} \binom{n-m+q}{q} (-1)^q 2^{n-m+1+q} \binom{n+m-q}{2m-q} 2^{n+m+1-q}$$

$$= -2^{2n-2m+1} \sum_{q=0}^{2m} \binom{n-m+q}{q} (-1)^q \binom{n+m-q}{2m-q}.$$

For this last sum use the integral

$$\binom{n+m-q}{2m-q} = \binom{n+m-q}{n-m} = \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m-q+1}} \frac{1}{(1-v)^{n-m+1}} \, dv.$$

This controls the range so we can let q go to infinity in the sum to get

$$\frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m+1}} \frac{1}{(1-v)^{n-m+1}} \sum_{q \geq 0} \binom{n-m+q}{q} (-1)^q v^q \, dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m+1}} \frac{1}{(1-v)^{n-m+1}} \frac{1}{(1+v)^{n-m+1}} \, dv$$

$$= \frac{1}{2\pi i} \int_{|v|=\epsilon} \frac{1}{v^{2m+1}} \frac{1}{(1-v^2)^{n-m+1}} \, dv = \binom{n-m+m}{m} = \binom{n}{m}.$$

We have shown that

$$2S - m \times 2 \times 2^{2n-2m} \times \binom{n}{m} = 0$$

and hence may conclude that

$$S = m \times 4^{n-m} \times \binom{n}{m}.$$

Remark. If we want to do this properly we also need to verify that the residue at infinity of the integral in w is zero. Recall the formula for the residue at infinity

$$\mathrm{Res}_{z=\infty} h(z) = \mathrm{Res}_{z=0} \left[-\frac{1}{z^2} h\left(\frac{1}{z}\right) \right]$$

In the present case this becomes

$$-\mathrm{Res}_{w=0} \frac{1}{w^2} \frac{(1+1/w)^m}{1/w^{2m}} \frac{1}{(1-2z-z/w)^2}$$

$$= -\mathrm{Res}_{w=0} \frac{(1+1/w)^m}{1/w^{2m}} \frac{1}{(w(1-2z)-z)^2}$$

$$= -\mathrm{Res}_{w=0} (1+w)^m w^m \frac{1}{(w(1-2z)-z)^2}$$

which is zero by inspection.

The same procedure applied to the main integral yields

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n-m+1} \frac{1}{(1-1/z)^{n-m+1}} \frac{1}{(1-2/z)^{2m+1}}$$

$$= -\mathrm{Res}_{z=0} \frac{1}{z^2} z^{n-m+1} \frac{z^{n-m+1}}{(z-1)^{n-m+1}} \frac{z^{2m+1}}{(z-2)^{2m+1}}$$

$$= -\mathrm{Res}_{z=0} z^{2n+1} \frac{1}{(z-1)^{n-m+1}} \frac{1}{(z-2)^{2m+1}}$$

which is zero as well.

This was math.stackexchange.com problem 1247818.

48 Same problem, streamlined proof (B_1B_2R)

Suppose we seek to verify that

$$S = \sum_{q=0}^{n} q \binom{2n}{n+q} \binom{m+q-1}{2m-1} = m \times 4^{n-m} \times \binom{n}{m}$$

where $n \geq m$.

This is

$$\sum_{q=0}^{n} (n-q) \binom{2n}{q} \binom{m+n-q-1}{2m-1}$$

which has two pieces. We use the integral

$$\binom{m+n-q-1}{2m-1} = \binom{m+n-q-1}{n-m-q} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m-q+1}} (1+w)^{m+n-q-1} \; dw.$$

Observe that this integral vanishes when $q > n-m$ and we may extend q to $2n$. We get for the first piece

$$\frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} (1+w)^{m+n-1} \sum_{q=0}^{2n} \binom{2n}{q} \frac{w^q}{(1+w)^q} \; dw$$

$$= \frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n} \; dw.$$

The second piece is the negative of

$$\sum_{q=0}^{n} q \binom{2n}{q} \binom{m+n-q-1}{2m-1} = \sum_{q=1}^{n} q \binom{2n}{q} \binom{m+n-q-1}{2m-1}$$

$$= 2n \sum_{q=1}^{n} \binom{2n-1}{q-1} \binom{m+n-q-1}{2m-1} = 2n \sum_{q=0}^{n-1} \binom{2n-1}{q} \binom{m+n-q-2}{2m-1}$$

$$= 2n \sum_{q=0}^{n-1} \binom{2n-1}{q} \binom{m+n-q-2}{n-m-q-1}.$$

This vanishes through its integral representation when $q > n - m - 1$ and we obtain

$$\frac{2n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \; dw.$$

Joining the two pieces we arrive at the single integral

$$\frac{n}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \; dw.$$

We know the residues at zero, minus one and infinity sum to zero, where the first represents the queried sum. For the residue at minus one it is given by

$$\begin{aligned} &\frac{n}{2\pi i} \int_{|w+1|=\gamma} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \; dw \\ &= \frac{n}{2\pi i} \int_{|v|=\gamma} \frac{1}{(v-1)^{n-m+1}} \frac{1}{v^{n+1-m}} (2v-1)^{2n-1} \; dv \\ &= -\frac{n}{2\pi i} \int_{|v|=\gamma} \frac{1}{(-v-1)^{n-m+1}} \frac{1}{(-v)^{n+1-m}} (-1-2v)^{2n-1} \; dv \\ &= \frac{n}{2\pi i} \int_{|v|=\gamma} \frac{1}{(1+v)^{n-m+1}} \frac{1}{v^{n+1-m}} (1+2v)^{2n-1} \; dv. \end{aligned}$$

We see that this residue also represents the queried sum. This leaves the residue at infinity which is

$$\begin{aligned} &\mathrm{Res}_{w=\infty} \frac{1}{w^{n-m+1}} \frac{1}{(1+w)^{n+1-m}} (1+2w)^{2n-1} \\ &= -\mathrm{Res}_{w=0} \frac{1}{w^2} w^{n-m+1} \frac{1}{(1+1/w)^{n+1-m}} (1+2/w)^{2n-1} \\ &= -\mathrm{Res}_{w=0} w^{n-m-1} \frac{w^{n+1-m}}{(1+w)^{n+1-m}} \frac{(2+w)^{2n-1}}{w^{2n-1}} \\ &= -\mathrm{Res}_{w=0} \frac{1}{w^{2m-1}} \frac{(2+w)^{2n-1}}{(1+w)^{n+1-m}}. \end{aligned}$$

Extracting coefficients we find

$$-n \sum_{q=0}^{2m-2} {2n-1 \choose 2m-2-q} 2^{2n-2m+1+q} (-1)^q {n-m+q \choose q}.$$

Introduce (this vanishes when $q > 2m - 2$)

$$\begin{aligned} &{2n-1 \choose 2m-2-q} = {2n-1 \choose 2n+1-2m+q} \\ &= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2m-1-q}} \frac{1}{(1-z)^{2n-2m+2+q}} \; dz \end{aligned}$$

to get for the sum

$$-\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z)^{2n-2m+2}}\sum_{q\ge 0}\binom{n-m+q}{q}2^q(-1)^q\frac{z^q}{(1-z)^q}\;dz$$

$$= -\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z)^{2n-2m+2}}\frac{1}{(1+2z/(1-z))^{n-m+1}}\;dz$$

$$= -\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z)^{n-m+1}}\frac{1}{(1+z)^{n-m+1}}\;dz$$

$$= -\frac{n2^{2n-2m+1}}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2m-1}}\frac{1}{(1-z^2)^{n-m+1}}\;dz$$

$$= -n2^{2n-2m+1}[z^{2m-2}]\frac{1}{(1-z^2)^{n-m+1}} = -n2^{2n-2m+1}[z^{m-1}]\frac{1}{(1-z)^{n-m+1}}$$

$$= -n2^{2n-2m+1}\binom{n-m+m-1}{m-1}.$$

It follows that

$$2S - n2^{2n-2m+1}\binom{n-1}{m-1} = 0 \quad\text{or}\quad S = n4^{n-m}\frac{m}{n}\binom{n}{m}$$

which yields

$$S = m\times 4^{n-m}\times\binom{n}{m}$$

as claimed.

49 Symmetry of the Euler-Frobenius coefficient (B_1EIR)

Suppose we have the coefficient of the Euler-Frobenius polynomial

$$b_k^n = \sum_{l=1}^{k}(-1)^{k-l}l^n\binom{n+1}{k-l}$$

and we seek to show that $b_k^n = b_{n+1-k}^n$ where $0\le k\le n+1$.

First re-write this as

$$\sum_{l=0}^{k}(-1)^l(k-l)^n\binom{n+1}{l}.$$

Introduce the Iverson bracket

$$[[0 \le l \le k]] = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{z^l}{z^{k+1}} \frac{1}{1-z} \; dz$$

and the exponentiation integral

$$(k-l)^n = \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp((k-l)w) \; dw.$$

to get for the sum (extend the summation to $n+1$ since the Iverson bracket controls the range)

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw) \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+1}} \frac{1}{1-z} \sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^l z^l \exp(-lw) \; dz \; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw) \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+1}} \frac{1}{1-z} (1 - z\exp(-w))^{n+1} \; dz \; dw.$$

Evaluate this using the residues at the poles at $z = 1$ and at infinity. We obtain for $z = 1$

$$-\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw)(1 - \exp(-w))^{n+1} \; dw,$$

note however that $1 - \exp(-w)$ starts at w so the power starts at w^{n+1} making for a zero contribution.

We get for the residue at infinity

$$-\mathrm{Res}_{z=0} \frac{1}{z^2} z^{k+1} \frac{1}{1-1/z} (1 - \exp(-w)/z)^{n+1}$$

$$= -\mathrm{Res}_{z=0} z^k \frac{1}{z-1} (1 - \exp(-w)/z)^{n+1}$$

$$= \mathrm{Res}_{z=0} \frac{z^k}{z^{n+1}} \frac{1}{1-z} (z - \exp(-w))^{n+1}.$$

We need to flip the sign on this one more time since we are exploiting the fact that the residues at the three poles sum to zero. Actually extracting the coefficient we get

$$-\sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n+1-q} \exp(-(n+1-q)w).$$

Substitute this into the integral in w to get

$$-\sum_{q=0}^{n-k} \binom{n+1}{q} \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(kw)(-1)^{n+1-q} \exp(-(n+1-q)w) \; dw$$

$$= -\sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^{n+1-q} (-1)^n (n+1-k-q)^n$$

$$= \sum_{q=0}^{n-k} \binom{n+1}{q} (-1)^q (n+1-k-q)^n.$$

Using the fact that $n+1-k-q$ is zero at $q = n+1-k$ we finally obtain

$$\sum_{q=0}^{n+1-k} \binom{n+1}{q} (-1)^q (n+1-k-q)^n$$

which is precisely b^n_{n+1-k} by definition, QED.

Addendum. An alternate proof (variation on the theme from above) starts from the unmodified definition and introduces

$$\binom{n+1}{k-l} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k-l+1}} (1+z)^{n+1} \; dz.$$

This controls the range so we may extend l to infinity. Introduce furthermore

$$l^n = \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \exp(lw) \; dw.$$

These two yield for the sum

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \sum_{l\ge 0} (-1)^l z^l \exp(lw) \; dz \; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \frac{1}{1+z\exp(w)} \; dz \; dw$$

$$= \frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(-w)}{w^{n+1}} \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(-1)^k}{z^{k+1}} (1+z)^{n+1} \frac{1}{z+\exp(-w)} \; dz \; dw.$$

We evaluate this using the negatives of the residues at $z = -\exp(-w)$ and at infinity. We get for $z = -\exp(-w)$

$$\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(-w)}{w^{n+1}} \frac{(-1)^k}{(-1)^{k+1}\exp(-(k+1)w)} (1-\exp(-w))^{n+1} \; dw$$

$$= -\frac{n!}{2\pi i} \int_{|w|=\epsilon} \frac{\exp(kw)}{w^{n+1}} (1-\exp(-w))^{n+1} \; dw.$$

As before the exponentiated term starts at w^{n+1} so there is no coefficient on w^n for a contribution of zero.

We get for the residue at infinity (starting from the next-to-last version of the integral)

$$
\begin{aligned}
&-\mathrm{Res}_{z=0}\frac{1}{z^2}(-1)^k z^{k+1}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{1}{1+\exp(w)/z}\\
&= -\mathrm{Res}_{z=0}\frac{1}{z^2}(-1)^k z^{k+1}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{z/\exp(w)}{1+z/\exp(w)}\\
&= -\mathrm{Res}_{z=0}(-1)^k z^{k}\frac{(1+z)^{n+1}}{z^{n+1}}\frac{\exp(-w)}{1+z/\exp(w)}.
\end{aligned}
$$

Doing the sign flip and simplifying we obtain

$$\exp(-w)(-1)^k \times \mathrm{Res}_{z=0}\frac{(1+z)^{n+1}}{z^{n-k+1}}\frac{1}{1+z/\exp(w)}.$$

Extract the residue to get

$$\exp(-w)(-1)^k \sum_{q=0}^{n-k}\binom{n+1}{q}(-1)^{n-k-q}\exp(-(n-k-q)w)$$

Substitute into the integral in w to obtain

$$\sum_{q=0}^{n-k}\binom{n+1}{q}\frac{n!}{2\pi i}\int_{|w|=\epsilon}\frac{1}{w^{n+1}}(-1)^{n-q}\exp(-(n+1-k-q)w)\;dw$$

$$
\begin{aligned}
&= \sum_{q=0}^{n-k}\binom{n+1}{q}(-1)^{n-q}(-1)^n(n+1-k-q)^n\\
&= \sum_{q=0}^{n-k}\binom{n+1}{q}(-1)^{q}(n+1-k-q)^n.
\end{aligned}
$$

We have obtained b^n_{n+1-k} as before.

This was math.stackexchange.com problem 1435648.

50 A probability distribution with two parameters (B_1B_2)

A sum of binomial coefficients CLXVII

Suppose we have a random variable X where

$$\mathrm{P}[X=k] = \binom{N}{2n+1}^{-1}\binom{N-k}{n}\binom{k-1}{n}$$

for $k = n+1, \ldots, N-n$ and zero otherwise.

We seek to show that these probabilities sum to one and compute the the mean and the variance.

Sum of probabilities. This is given by

$$\binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} \binom{N-k}{n} \binom{k-1}{n}.$$

Introduce

$$\binom{N-k}{n} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{N-n-k+1}} \frac{1}{(1-z)^{n+1}} \; dz$$

and

$$\binom{k-1}{n} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{k-1}}{w^{n+1}} \; dw.$$

Observe carefully that the first integral is zero when $k > N-n$ and the second one when $1 \le k \le n$ so we may extend the range of the sum to $1 \le k$.

This gives for the sum (without the scalar)

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \sum_{k \ge 1} z^{k-1} (1+w)^{k-1} \; dw \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-z(1+w)} \; dw \; dz.$$

The integral in w is

$$\frac{1}{1-z} \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{1}{w^{n+1}} \frac{1}{1-wz/(1-z)} \; dw$$

which yields for the integral in z

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{N-n}} \frac{1}{(1-z)^{n+1}} \frac{z^n}{(1-z)^{n+1}} \; dz$$

which is

$$\binom{N-2n-1+2n+1}{2n+1} = \binom{N}{2n+1}.$$

This confirms that the probabilities sum to one.

Expectation. This is given by

$$\mathrm{E}[X] = \binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} k \binom{N-k}{n} \binom{k-1}{n}.$$

Introduce

$$k \binom{k-1}{n} = \frac{k!}{n! \times (k-1-n)!} = (n+1) \binom{k}{n+1}$$

$$= (n+1)\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^k}{w^{n+2}} \; dw.$$

The range control from this integral produces zero when $0 \le k \le n$ so we may extend the sum to zero, getting

$$(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}} \frac{1}{(1-z)^{n+1}} \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \sum_{k\ge 0} z^k (1+w)^k \; dw \; dz.$$

The integral in w is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \frac{1}{1-z(1+w)} \; dw$$

$$= \frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+2}} \frac{1}{1-wz/(1-z)} \; dw$$

which yields for the integral in z including the factor in front

$$(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}} \frac{1}{(1-z)^{n+1}} \frac{z^{n+1}}{(1-z)^{n+2}} \; dz$$

which is

$$(n+1)\binom{N-2n-1+2n+2}{2n+2} = (n+1)\binom{N+1}{2n+2}.$$

We will scale this at the end, same as the variance.

Variance. Start by computing

$$\mathrm{E}[(X+1)X] = \binom{N}{2n+1}^{-1} \sum_{k=n+1}^{N-n} (k+1)k\binom{N-k}{n}\binom{k-1}{n}.$$

Introduce

$$(k+1)k\binom{k-1}{n} = \frac{(k+1)!}{n! \times (k-1-n)!}$$

$$= (n+2)(n+1)\binom{k+1}{n+2} = (n+2)(n+1)\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{k+1}}{w^{n+3}} \; dw.$$

The range control from this integral produces zero when $0 \le k \le n$ as before so we may extend the sum to zero, getting

$$(n+2)(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{N-n+1}} \frac{1}{(1-z)^{n+1}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1+w}{w^{n+3}} \sum_{k\ge 0} z^k (1+w)^k \; dw \; dz.$$

The integral in w is

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1+w}{w^{n+3}}\frac{1}{1-z(1+w)}\;dw$$
$$=\frac{1}{1-z}\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{1+w}{w^{n+3}}\frac{1}{1-wz/(1-z)}\;dw$$

which yields for the integral in z including the factor in front

$$(n+2)(n+1)\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{N-n+1}}\frac{1}{(1-z)^{n+1}}\left(\frac{z^{n+2}}{(1-z)^{n+3}}+\frac{z^{n+1}}{(1-z)^{n+2}}\right)\;dz$$

which is

$$(n+2)(n+1)\left(\binom{N-2n-2+2n+3}{2n+3}+\binom{N-2n-1+2n+2}{2n+2}\right)$$
$$=(n+2)(n+1)\left(\binom{N+1}{2n+3}+\binom{N+1}{2n+2}\right).$$

Simplification for ease of interpretation.
We get for the expectation

$$\mathrm{E}[X]=(n+1)\frac{(N+1)!}{(N-2n-1)!(2n+2)!}\frac{(N-2n-1)!(2n+1)!}{N!}$$
$$=\frac{1}{2}(N+1).$$

We obtain furthermore

$$\mathrm{E}[(X+1)X]=(n+2)(n+1)$$
$$\times\left(\frac{(N+1)!}{(N-2n-2)!(2n+3)!}+\frac{(N+1)!}{(N-2n-1)!(2n+2)!}\right)\frac{(N-2n-1)!(2n+1)!}{N!}$$
$$=\frac{1}{2}(N+1)(n+2)\left(\frac{N-2n-1}{2n+3}+1\right)$$
$$=\frac{1}{2}(N+2)(N+1)\frac{n+2}{2n+3}.$$

This yields for the variance

$$\mathrm{Var}[X]=\mathrm{E}[X^2]-\mathrm{E}[X]^2$$
$$=\frac{1}{2}(N+2)(N+1)\frac{n+2}{2n+3}-\frac{1}{2}(N+1)-\frac{1}{4}(N+1)^2.$$

which simplifies to

$$\mathrm{Var}[X]=\frac{1}{4}(N+1)\frac{N-2n-1}{2n+3}.$$

This was math.stackexchange.com problem 1257644.

51 An identity involving Narayana numbers (B_1)

Suppose we have the Narayana number

$$N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$$

and let

$$A(n,k,l) = \sum_{\substack{i_0+i_1+\cdots+i_k=n\\ j_0+j_1+\cdots+j_k=l}} \prod_{t=0}^{k} N(i_t, j_t+1)$$

where the compositions for n are regular and the ones for l are weak and we seek to verify that

$$A(n,k,l) = \frac{k+1}{n}\binom{n}{l}\binom{n}{l+k+1}.$$

Introducing

$$\begin{aligned} G(z,u) &= \sum_{p\ge 1} z^p \sum_{q\ge 0} u^q \frac{1}{p}\binom{p}{q+1}\binom{p}{q} \\ &= \sum_{p\ge 1} \frac{1}{p} z^p \sum_{q\ge 0} u^q \binom{p}{q+1}\binom{p}{q} \end{aligned}$$

we have by inspection that

$$A(n,k,l) = [z^n][u^l]G(z,u)^{k+1}.$$

To evaluate this introduce for the inner sum term

$$\binom{p}{q+1} = \binom{p}{p-q-1} = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{p-q}}(1+w)^p \; dw.$$

We get for the inner sum

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^p}(1+w)^p \sum_{q\ge 0}\binom{p}{q} u^q w^q \; dw$$

$$\begin{aligned} &= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^p}(1+w)^p(1+uw)^p \; dw \\ &= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^p}(1+w(1+u+uw))^p \; dw. \end{aligned}$$

Extracting the coefficient from this we get

$$[w^{p-1}]\sum_{q=0}^{p}\binom{p}{q} w^q (1+u+uw)^q$$

$$= \sum_{q=0}^{p-1} \binom{p}{q} [w^{p-1-q}](1+u+uw)^q$$

$$= \sum_{q=0}^{p-1} \binom{p}{q} \binom{q}{p-1-q} u^{p-1-q}(1+u)^{2q+1-p}.$$

This is

$$\sum_{q=0}^{p-1} \binom{p}{p-1-q} \binom{p-1-q}{q} u^q (1+u)^{p-1-2q}$$

$$= \sum_{q=0}^{p-1} \binom{p}{q+1} \binom{p-1-q}{q} u^q (1+u)^{p-1-2q}.$$

Now observe that

$$\frac{1}{p}\binom{p}{q+1}\binom{p-1-q}{q} = \frac{1}{q+1}\binom{p-1}{q}\binom{p-1-q}{q}$$

$$= \frac{1}{q+1}\binom{p-1}{p-1-q}\binom{p-1-q}{q} = \frac{1}{q+1}\binom{p-1}{2q}\binom{2q}{q}.$$

where

$$C_q = \frac{1}{q+1}\binom{2q}{q}$$

is a Catalan number.

We thus get for the sum

$$\sum_{p\ge 1} z^p \sum_{q=0}^{p-1} \binom{p-1}{2q} C_q u^q (1+u)^{p-1-2q}$$

$$= z \sum_{p\ge 0} z^p \sum_{q=0}^{p} \binom{p}{2q} C_q u^q (1+u)^{p-2q}$$

$$= z \sum_{q\ge 0} C_q u^q (1+u)^{-2q} \sum_{p\ge q} \binom{p}{2q} z^p (1+u)^p$$

$$= z \sum_{q\ge 0} C_q u^q (1+u)^{-2q} \sum_{p\ge 2q} \binom{p}{2q} z^p (1+u)^p$$

$$= z \sum_{q\ge 0} C_q u^q (1+u)^{-2q} (1+u)^{2q} z^{2q} \sum_{p\ge 0} \binom{p+2q}{2q} z^p (1+u)^p$$

$$= z \sum_{q\ge 0} C_q u^q z^{2q} \frac{1}{(1-z(1+u))^{2q+1}}.$$

Using the generating function of the Catalan numbers

$$Q(w) = \sum_{q\ge 0} C_q w^q = \frac{1-\sqrt{1-4w}}{2w}$$

which has functional equation

$$Q(w) = 1 + wQ(w)^2$$

we obtain

$$Q\left(\frac{uz^2}{(1-z(1+u))^2}\right) = 1 + \frac{uz^2}{(1-z(1+u))^2} Q\left(\frac{uz^2}{(1-z(1+u))^2}\right)^2$$

which is

$$G(z,u)\frac{1-z(1+u)}{z} = 1 + uG(z,u)^2.$$

Extract the coefficient in z first. We get from the functional equation

$$z = \frac{G(z,u)}{uG(z,u)^2 + (1+u)G(z,u) + 1}.$$

The coefficient extractor integral is

$$[z^n]G(z,u) = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} G(z,u)^{k+1}\; dz.$$

which becomes with $G(z,u) = v$

$$\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(uv^2+(1+u)v+1)^{n+1}}{v^{n+1}}$$

$$\times v^{k+1}\left(\frac{1}{uv^2+(1+u)v+1} - \frac{v}{(uv^2+(1+u)v+1)^2}(2uv+(1+u))\right)\; dv$$

$$= \frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(uv^2+(1+u)v+1)^{n-1}}{v^{n-k}}(1-uv^2)\; dv.$$

This is

$$\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{n-1}(1+uv)^{n-1}}{v^{n-k}}(1-uv^2)\; dv.$$

Extracting the coefficient on $[u^l]$ we get two pieces which are, first piece A

$$\binom{n-1}{l}\frac{1}{2\pi i}\int_{|v|=\epsilon} \frac{(1+v)^{n-1}v^l}{v^{n-k}}\; dv = \binom{n-1}{l}\binom{n-1}{n-k-l-1}$$

which is

$$\binom{n-1}{l}\binom{n-1}{k+l} = \binom{n-1}{l}\frac{k+l+1}{n}\binom{n}{k+l+1}$$

$$= (n-l)\frac{k+l+1}{n^2}\binom{n}{l}\binom{n}{k+l+1}.$$

and piece B which is

$$-\binom{n-1}{l-1}\frac{1}{2\pi i}\int_{|v|=\epsilon}\frac{(1+v)^{n-1}v^{l-1}}{v^{n-k}}\;v^2 dv = -\binom{n-1}{l-1}\binom{n-1}{n-k-l-2}$$

which is

$$-\binom{n-1}{l-1}\binom{n-1}{k+l+1} = -\binom{n-1}{l-1}\frac{n-k-l-1}{n}\binom{n}{k+l+1}$$
$$= -l\frac{n-k-l-1}{n^2}\binom{n}{l}\binom{n}{k+l+1}.$$

Collecting the two pieces we finally obtain

$$\left((n-l)\frac{k+l+1}{n^2} + l\frac{-n+k+l+1}{n^2}\right)\binom{n}{l}\binom{n}{k+l+1}$$
$$= \left(n\frac{k+l+1}{n^2} + l\frac{-n}{n^2}\right)\binom{n}{l}\binom{n}{k+l+1}$$
$$= \frac{k+1}{n}\binom{n}{l}\binom{n}{k+l+1}$$

as claimed, QED.

Remark. The closed form of $G(z,u)$ can be computed as follows:

$$\frac{z}{1-z(1+u)}\frac{1-\sqrt{1-4uz^2/(1-z(1+u))^2}}{2uz^2/(1-z(1+u))^2}$$
$$= \frac{z}{(1-z(1+u))^2}\frac{1-z(1+u)-\sqrt{1-2z(1+u)+z^2(1+u)^2-4uz^2}}{2uz^2/(1-z(1+u))^2}$$
$$= \frac{1-z(1+u)-\sqrt{1-2z(1+u)+z^2(1+u)^2-4uz^2}}{2uz}.$$

The above material incorporates data from OEIS A055151 and from OEIS A001263 on Narayana numbers.

This was math.stackechange.com problem 1498014.

52 Convolution of Narayana polynomials (B_1)

This is basically a re-write of the previous entry with a more general conclusion.

Suppose we define

$$C_0^{(1)}(t) = 1 \quad \text{and} \quad C_n^{(1)}(t) = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+1}{k+1}\frac{1}{n+1}t^k$$

and let for $m \ge 2$

$$C_n^{(m)}(t) = \sum_{q=0}^{n} C_q^{(m-1)}(t) C_{n-q}^{(1)}(t).$$

This definition is equivalent to introducing

$$G(w) = \sum_{n\ge 1} C_n^{(1)}(t) w^n$$

and letting

$$C_n^{(m)}(t) = [w^n](1+G(w))^m = [w^n] \sum_{p=0}^{m} \binom{m}{p} G(w)^p.$$

We seek to show that

$$C_0^{(m)}(t) = 1 \quad \text{and} \quad C_n^{(m)}(t) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+m}{k+m} \frac{m}{n+m} t^k.$$

The proof will consist in doing a coefficient extraction operation from the powers of $G(w)$ and showing that these match the proposed formula. We require an alternate representation of $C_n^{(1)}(t)$ and observe that

$$\begin{aligned}\binom{n-1}{k}\binom{n+1}{k+1}\frac{1}{n+1} &= \binom{n}{k+1}\frac{k+1}{n}\binom{n}{k}\frac{n+1}{k+1}\frac{1}{n+1} \\ &= \frac{1}{n}\binom{n}{k+1}\binom{n}{k}.\end{aligned}$$

and introduce

$$\binom{n}{k+1} = \binom{n}{n-k-1} = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z^{n-k}}(1+z)^n \; dz$$

which conveniently vanishes for $k \ge n$. We get for $n \ge 1$

$$\begin{aligned}C_n^{(1)}(t) &= \frac{1}{n}\frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z^n}(1+z)^n \sum_{k\ge 0}\binom{n}{k} z^k t^k \; dz \\ &= \frac{1}{n}\frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z^n}(1+z)^n(1+tz)^n \; dz.\end{aligned}$$

We re-write this as

$$\frac{1}{n}\frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z^n}(1+z(1+t+tz))^n \; dz.$$

Extracting the coefficient yields

$$\frac{1}{n}\sum_{q=0}^{n-1}\binom{n}{q}[z^{n-1-q}](1+t+tz)^q$$

$$= \frac{1}{n}\sum_{q=0}^{n-1}\binom{n}{q}\binom{q}{n-1-q}(1+t)^{q-(n-1-q)}t^{n-1-q}$$

$$= \frac{1}{n}\sum_{q=0}^{n-1}\binom{n}{q}\binom{q}{n-1-q}(1+t)^{2q+1-n}t^{n-1-q}$$

$$= \frac{1}{n}\sum_{q=0}^{n-1}\binom{n}{n-1-q}\binom{n-1-q}{q}(1+t)^{n-1-2q}t^{q}.$$

Now we have

$$\frac{1}{n}\binom{n}{n-1-q}\binom{n-1-q}{q} = \frac{1}{n}\frac{n!}{(q+1)!q!(n-1-2q)!}$$
$$= \frac{1}{n}\binom{2q+1}{q}\binom{n}{2q+1} = \frac{1}{2q+1}\binom{2q+1}{q}\binom{n-1}{2q}$$
$$= \frac{1}{2q+1}\binom{2q+1}{q+1}\binom{n-1}{2q} = \frac{1}{q+1}\binom{2q}{q}\binom{n-1}{2q}$$

where

$$C_q = \frac{1}{q+1}\binom{2q}{q}$$

is a Catalan number. We thus obtain for $G(w)$

$$\sum_{n\geq 1} w^n \sum_{q=0}^{n-1} C_q \binom{n-1}{2q}(1+t)^{n-1-2q}t^q$$

$$= w\sum_{n\geq 0} w^n \sum_{q=0}^{n} C_q \binom{n}{2q}(1+t)^{n-2q}t^q$$

$$= w\sum_{q\geq 0} C_q(1+t)^{-2q}t^q \sum_{n\geq q}\binom{n}{2q} w^n(1+t)^n$$

$$= w\sum_{q\geq 0} C_q(1+t)^{-2q}t^q \sum_{n\geq 2q}\binom{n}{2q} w^n(1+t)^n$$

$$= w\sum_{q\geq 0} C_q(1+t)^{-2q}t^q w^{2q}(1+t)^{2q} \sum_{n\geq 0}\binom{n+2q}{2q} w^n(1+t)^n$$

$$= w\sum_{q\geq 0} C_q t^q w^{2q}\frac{1}{(1-w(1+t))^{2q+1}}$$

$$= \frac{w}{(1-w(1+t))} \sum_{q\ge 0} C_q t^q w^{2q} \frac{1}{(1-w(1+t))^{2q}}.$$

Now the classic generating function of the Catalan numbers is

$$Q(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

We have computed the closed form of $G(w)$ which is

$$\begin{aligned} G(w) &= \frac{w}{(1-w(1+t))} \frac{1-\sqrt{1-4tw^2/(1-w(1+t))^2}}{2tw^2/(1-w(1+t))^2} \\ &= w \frac{1-\sqrt{1-4tw^2/(1-w(1+t))^2}}{2tw^2/(1-w(1+t))} \\ &= w \frac{1-w(1+t)-\sqrt{(1-w(1+t))^2-4tw^2}}{2tw^2} \\ &= \frac{1-w(1+t)-\sqrt{(1-w(1+t))^2-4tw^2}}{2tw}. \end{aligned}$$

Recall the functional equation of the Catalan number generating function which is

$$Q(x) = 1 + xQ(x)^2.$$

We thus obtain

$$Q\left(\frac{tw^2}{(1-w(1+t))^2}\right) = 1 + \frac{tw^2}{(1-w(1+t))^2} Q\left(\frac{tw^2}{(1-w(1+t))^2}\right)^2$$

or

$$\frac{1-w(1+t)}{w} G(w) = 1 + tG(w)^2.$$

Solving for w we get

$$G(w) - w(1+t)G(w) = w(1+tG(w)^2)$$

or

$$G(w) = w(1+(1+t)G(w)+tG(w)^2)$$

which is

$$w = \frac{G(w)}{1+(1+t)G(w)+tG(w)^2}.$$

Recall that we seek

$$[t^k] \sum_{p=0}^{m} \binom{m}{p} [w^n] G(w)^p.$$

We establish the coefficient extraction integral (Lagrange inversion)

$$[w^n]G(w)^p = \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{1}{w^{n+1}} G(w)^p \; dw.$$

Setting $v = G(w)$ and observing that $w = 0$ is mapped to $v = 0$ we get

$$\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+(1+t)v+tv^2)^{n+1}}{v^{n+1}}$$
$$\times v^p \times \left(\frac{1}{1+(1+t)v+tv^2} - \frac{v(1+t+2tv)}{(1+(1+t)v+tv^2)^2}\right) \; dv.$$

This is

$$\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+(1+t)v+tv^2)^{n-1}}{v^{n-p+1}} (1+(1+t)v+tv^2 - v(1+t+2tv)) \; dv$$
$$= \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n-1}(1+tv)^{n-1}}{v^{n-p+1}} (1-tv^2) \; dv.$$

Now substituting this into the target formula yields

$$C_n^{(m)}(t) = \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n-1}(1+tv)^{n-1}}{v^{n+1}} (1-tv^2) \sum_{p=0}^{m} \binom{m}{p} v^p \; dv$$
$$= \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n+m-1}(1+tv)^{n-1}}{v^{n+1}} (1-tv^2) \; dv.$$

To conclude the proof we must treat the case of $n = 0$ which is different from the case $n \geq 1$ and extract coefficients on $[t^k]$ in the latter case. With $n = 0$ we get

$$\frac{1}{2\pi i}\int_{|v|=\gamma} (1+v)^{m-1} \frac{1}{v} \frac{1}{1+tv} (1-tv^2) \; dv.$$

This is the constant coefficient and is equal to

$$(1+v)^{m-1} \frac{1}{1+tv} (1-tv^2)\bigg|_{v=0} = 1,$$

as required. For the case of $n \geq 1$ we have two subcases, $k = 0$ and $k \geq 1$. For $k = 0$ the second term in $1 - tv^2$ does not contribute and we have just

$$\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n+m-1}}{v^{n+1}} \; dv$$
$$= \binom{n+m-1}{n} = \binom{n+m-1}{m-1} = \binom{n+m}{m} \frac{m}{n+m}.$$

This is again the required value. Finally when $n \ge 1$ and $k \ge 1$ we get two pieces, namely

$$\binom{n-1}{k} \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{(1+v)^{n+m-1}}{v^{n-k+1}} \; dv = \binom{n-1}{k} \binom{n+m-1}{n-k}$$

and

$$-\binom{n-1}{k-1} \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{(1+v)^{n+m-1}}{v^{n-k}} \; dv = -\binom{n-1}{k-1} \binom{n+m-1}{n-k-1}.$$

Note that when $k = n$ the first of these integrals never appears in the first place because $[t^k](1+tv)^{n-1} = 0$ and the second vanishes due to the residue. When $k > n$ we have $[t^k](1+tv)^{n-1}(1-tv^2) = 0$ and everything vanishes. This is the required behavior. We get for the non-zero cases

$$\binom{n-1}{k} \binom{n+m-1}{k+m-1} - \binom{n-1}{k-1} \binom{n+m-1}{k+m}$$
$$= \binom{n-1}{k} \binom{n+m}{k+m} \frac{k+m}{n+m} - \binom{n-1}{k} \frac{k}{n-k} \binom{n+m}{k+m} \frac{n-k}{n+m}$$
$$= \binom{n-1}{k} \binom{n+m}{k+m} \frac{m}{n+m}.$$

We have the required value as was to be shown and may end the computation. This was math.stackexchange.com problem 1997791.

53 A property of Legendre polynomials (B_1)

Suppose we seek to determine the constant Q in the equality

$$Q_{n,m} \left(\frac{d}{dz}\right)^{n-m} (1-z^2)^n = (1-z^2)^m \left(\frac{d}{dz}\right)^{n+m} (1-z^2)^n$$

where $n \ge m$. We will compute the coefficients on $[z^q]$ on the LHS and the RHS. Writing $1 - z^2 = (1+z)(1-z)$ we get for the LHS

$$\sum_{p=0}^{n-m} \binom{n-m}{p} \binom{n}{p} p!(1+z)^{n-p}$$
$$\times \binom{n}{n-m-p} (n-m-p)!(-1)^{n-m-p}(1-z)^{m+p}$$
$$= (n-m)!(-1)^{n-m} \sum_{p=0}^{n-m} \binom{n}{p} \binom{n}{n-m-p} (1+z)^{n-p}(-1)^p(1-z)^{m+p}.$$

Extracting the coefficient we get

$$(n-m)!(-1)^{n-m}\sum_{p=0}^{n-m}\binom{n}{p}\binom{n}{n-m-p}(-1)^p$$

$$\times\sum_{k=0}^{n-p}\binom{n-p}{k}(-1)^{q-k}\binom{m+p}{q-k}.$$

We use the same procedure on the RHS and merge in the $(1-z^2)^m$ term to get

$$(n+m)!(-1)^{n+m}\sum_{p=0}^{n+m}\binom{n}{p}\binom{n}{n+m-p}(-1)^p$$

$$\times\sum_{k=0}^{n+m-p}\binom{n+m-p}{k}(-1)^{q-k}\binom{p}{q-k}.$$

Working in parallel with LHS and RHS we treat the inner sum of the LHS first, putting

$$\binom{m+p}{q-k}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{q-k+1}}(1+z)^{m+p}\,dz$$

to get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{q+1}}(1+z)^{m+p}\sum_{k=0}^{n-p}\binom{n-p}{k}(-1)^{q-k}z^k\,dz$$

$$=\frac{(-1)^q}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{q+1}}(1+z)^{m+p}(1-z)^{n-p}\,dz.$$

Adapt and repeat to obtain for the inner sum of the RHS

$$\frac{(-1)^q}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{q+1}}(1+z)^{p}(1-z)^{n+m-p}\,dz.$$

Moving on to the two outer sums we introduce

$$\binom{n}{n-m-p}=\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n-m-p+1}}(1+w)^n\,dw$$

to obtain for the LHS

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n-m+1}}(1+w)^n$$

$$\times\frac{(-1)^q}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{q+1}}(1+z)^m(1-z)^n\sum_{p=0}^{n-m}\binom{n}{p}(-1)^p w^p\frac{(1+z)^p}{(1-z)^p}\,dz\,dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n-m+1}}(1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{q+1}}(1+z)^m(1-z)^n\left(1-w\frac{1+z}{1-z}\right)^n\;dz\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n-m+1}}(1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{q+1}}(1+z)^m(1-z-w-wz)^n\;dz\;dw.$$

Repeat for the RHS to get

$$\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{n+m+1}}(1+w)^n$$

$$\times \frac{(-1)^q}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{q+1}}(1-z)^m(1-z-w-wz)^n\;dz\;dw.$$

Extracting coefficients from the first integral (LHS) we write

$$(1-z-w-wz)^n = (2-(1+z)(1+w))^n$$
$$= \sum_{k=0}^{n} \binom{n}{k}(-1)^k(1+z)^k(1+w)^k 2^{n-k}$$

and the inner integral yields

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{m+k}{q}(1+w)^k 2^{n-k}$$

followed by the outer one which gives

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{m+k}{q}\binom{n+k}{n-m}2^{n-k}.$$

For the second integral (RHS) we write

$$(1-z-w-wz)^n = ((1-z)(1+w)-2w)^n$$
$$= \sum_{k=0}^{n} \binom{n}{k}(1-z)^k(1+w)^k(-1)^{n-k}2^{n-k}w^{n-k}$$

and the inner integral yields

$$(-1)^q \sum_{k=0}^{n} \binom{n}{k}\binom{m+k}{q}(-1)^q(1+w)^k(-1)^{n-k}2^{n-k}w^{n-k}$$

followed by the outer one which produces

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{q} \binom{n+k}{k+m} (-1)^{n-k} 2^{n-k}.$$

The two sums are equal up to a sign and the RHS for the coefficient on $[z^q]$ is obtained from the LHS by multiplying by

$$\frac{(n+m)!}{(n-m)!}(-1)^{n-q}.$$

Observe that powers of z that are present in the LHS and the RHS always have the same parity, the coefficients being zero otherwise (either all even powers or all odd). Therefore $(-1)^{n-q}$ is in fact a constant not dependent on q, the question is which. The leading term has degree $2n - (n-m) = n+m = (2n - (n+m)) + 2m$ on both sides and the sign on the LHS is $(-1)^n$ and on the RHS it is $(-1)^{n+m}$. The conclusion is that the queried factor is given by

$$Q_{n,m} = (-1)^m \frac{(n+m)!}{(n-m)!}.$$

This was math.stackexchange.com problem 2066340.

54 A sum of factorials, OGF and EGF of the Stirling numbers of the second kind (B_1)

We are given that

$$r^k (r+n)! = \sum_{m=0}^{k} \lambda_m (r+n+m)!$$

and seek to determine the λ_m independent of r. We claim and prove that

$$\lambda_m = (-1)^{k+m} \sum_{p=0}^{k-m} \binom{k}{p} \left\{ {k+1-p \atop m+1} \right\} n^p.$$

With this in mind we re-write the initial condition as

$$r^k = \sum_{m=0}^{k} \lambda_m m! \binom{r+n+m}{m}.$$

We evaluate the RHS starting with λ_m using the EGF of the Stirling numbers of the second kind which in the present case says that

$$\left\{ {k+1-p \atop m+1} \right\} = \frac{(k+1-p)!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2-p}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \; dz.$$

We obtain for λ_m

$$(-1)^{k+m} \sum_{p=0}^{k-m} n^p \binom{k}{p} \frac{(k+1-p)!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2-p}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \; dz.$$

The inner term vanishes when $p \ge k+2$ but in fact even better it also vanishes when $p > k-m$ which implies $m+1 > k+1-p$ because $(\exp(z)-1)^{m+1}$ starts at $[z^{m+1}]$ and we are extracting the term on $[z^{k+1-p}]$.

Hence we may extend p to infinity without picking up any extra contributions to get

$$(-1)^{k+m} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} \sum_{p\ge 0} (k+1-p) \frac{n^p z^p}{p!} \; dz.$$

This is

$$(-1)^{k+m} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} \frac{(\exp(z)-1)^{m+1}}{(m+1)!} ((k+1) - nz) \exp(nz) \; dz.$$

Substitute this into the outer sum to get

$$(-1)^k \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1) - nz) \exp(nz)$$
$$\times \sum_{m=0}^{k} \binom{r+n+m}{m} (-1)^m \frac{(\exp(z)-1)^{m+1}}{m+1} \; dz.$$

We have

$$\binom{r+n+m}{m} \frac{1}{m+1} = \binom{r+n+m}{m+1} \frac{1}{r+n}$$

and hence obtain

$$\frac{(-1)^k}{r+n} \frac{k!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{k+2}} ((k+1) - nz) \exp(nz)$$
$$\times \sum_{m=0}^{k} \binom{r+n+m}{m+1} (-1)^m (\exp(z)-1)^{m+1} \; dz.$$

We may extend m to $m > k$ in the remaining sum because the term $(\exp(z)-1)^{m+1}$ as before starts at $[z^{m+1}]$ which would then be $> k+1$ but we are extracting the coefficient on $[z^{k+1}]$, which makes for a zero contribution.

Continuing we find

$$-\sum_{m\ge 0} \binom{r+n+m}{r+n-1} (-1)^{m+1} (\exp(z)-1)^{m+1}$$

$$= 1 - \frac{1}{(1-(1-\exp(z)))^{r+n}} = 1 - \exp(-(r+n)z).$$

We get two pieces on substituting this back into the main integral, the first is

$$\frac{(-1)^k}{r+n}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k+2}}((k+1)-nz)\exp(nz)\; dz$$
$$= \frac{(-1)^k}{r+n}(k+1)!\frac{n^{k+1}}{(k+1)!} - \frac{(-1)^k}{r+n}k!n\frac{n^k}{k!} = 0.$$

and the second is

$$\frac{(-1)^{k+1}}{r+n}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k+2}}((k+1)-nz)\exp(nz)\exp(-(r+n)z)\; dz$$
$$= \frac{(-1)^{k+1}}{r+n}\frac{k!}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k+2}}((k+1)-nz)\exp(-rz)\; dz$$
$$= \frac{(-1)^{k+1}}{r+n}(k+1)!\frac{(-r)^{k+1}}{(k+1)!} - \frac{(-1)^{k+1}}{r+n}k!n\frac{(-r)^k}{k!}$$
$$= \frac{1}{r+n}(k+1)!\frac{r^{k+1}}{(k+1)!} + \frac{1}{r+n}k!n\frac{r^k}{k!}$$
$$= \frac{1}{r+n}r^{k+1} + \frac{1}{r+n}nr^k = r^k.$$

This concludes the argument.

Addendum Nov 27 2016. Markus Scheuer proposes the identity

$$\lambda_m = (-1)^{m+k}\sum_{p=m}^{k} \left\{ {p \atop m} \right\} \binom{k}{p} (n+1)^{k-p}.$$

To see that this is the same as what I presented we extract the coefficient on $[n^q]$ to get

$$(-1)^{m+k}\sum_{p=m}^{k} \left\{ {p \atop m} \right\} \binom{k}{p}\binom{k-p}{q}.$$

Now we have

$$\binom{k}{p}\binom{k-p}{q} = \frac{k!}{p!q!(k-p-q)!} = \binom{k}{q}\binom{k-q}{p}.$$

We get

$$(-1)^{m+k}\binom{k}{q}\sum_{p=m}^{k} \left\{ {p \atop m} \right\} \binom{k-q}{p}.$$

We now introduce

$$\binom{k-q}{p} = \binom{k-q}{k-q-p} = \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k-q-p+1}}(1+z)^{k-q}\; dz.$$

This certainly vanishes when $p > k-q$ so we may extend p to infinity, getting for the sum

$$(-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k-q+1}}(1+z)^{k-q}\sum_{p\ge m} \left\{ p \atop m \right\} z^p\; dz.$$

Using the OGF of the Stirling numbers of the second kind this becomes

$$(-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{k-q+1}}(1+z)^{k-q}\prod_{l=1}^m \frac{z}{1-lz}\; dz.$$

Now put $z/(1+z) = w$ to get $z = w/(1-w)$ and $dz = 1/(1-w)^2\; dw$ to get

$$(-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{k-q}}\frac{1-w}{w}\frac{1}{(1-w)^2}\prod_{l=1}^m \frac{w/(1-w)}{1-lw/(1-w)}\; dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{k-q+1}}\frac{1}{1-w}\prod_{l=1}^m \frac{w}{1-w-lw}\; dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{k-q+1}}\frac{1}{1-w}\prod_{l=1}^m \frac{w}{1-(l+1)w}\; dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{k-q+2}}\frac{w}{1-w}\prod_{l=2}^{m+1} \frac{w}{1-lw}\; dw$$

$$= (-1)^{m+k}\binom{k}{q}\frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{k-q+2}}\prod_{l=1}^{m+1} \frac{w}{1-lw}\; dw$$

$$= (-1)^{m+k}\binom{k}{q}\left\{ k-q+1 \atop m+1 \right\}.$$

This is the claim and we are done.

This was math.stackexchange.com problem 2028293.

55 Fibonacci, Tribonacci, Tetranacci (B_1)

Suppose we seek to evaluate the following sum (with a condition on the binomial coefficient)

$$G(n,m) = \sum_{k=0}^n \sum_{q=0}^k (-1)^q \binom{k}{q}\binom{n-1-qm}{k-1}.$$

Now when $n-1-qm < 0$ we usually get a non-zero value for the binomial coefficient but this is not wanted here. Therefore we have

$$G(n,m) = \sum_{k=0}^{n} \sum_{q=0}^{\lfloor (n-k)/m \rfloor} (-1)^q \binom{k}{q} \binom{n-1-qm}{k-1}.$$

If we have lost any values for q above $\lfloor (n-k)/m \rfloor$ these would render the second binomial coefficient zero. If we have added in any values for q above k the first binomial coefficient is zero there.

Now with the integral

$$\binom{n-1-qm}{k-1} = \binom{n-1-qm}{n-k-qm} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1-qm}}{z^{n-k-qm+1}} \; dz$$

we get range control because the pole vanishes when $q > (n-k)/m$ and we may extend q to infinity. We thus obtain for the inner sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n-k+1}} \sum_{q\ge 0} (-1)^q \binom{k}{q} \frac{z^{qm}}{(1+z)^{qm}} \; dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n-k+1}} \left(1 - \frac{z^m}{(1+z)^m}\right)^k \; dz$$

This yields for the outer sum

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n-1}}{z^{n+1}} \left(1 - z\left(1 - \frac{z^m}{(1+z)^m}\right)\right)^{-1}$$
$$\times \left(1 - z^{n+1}\left(1 - \frac{z^m}{(1+z)^m}\right)^{n+1}\right) \; dz$$

which is

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{(1+z)^{n+m-1}}{z^{n+1}} \left((1-z)(1+z)^m + z^{m+1}\right)^{-1}$$
$$\times \left(1 - z^{n+1}\left(1 - \frac{z^m}{(1+z)^m}\right)^{n+1}\right) \; dz$$

Extracting the second component from the difference we get

$$-\frac{1}{2\pi i} \int_{|z|=\epsilon} (1+z)^{n+m-1} \left((1-z)(1+z)^m + z^{m+1}\right)^{-1} \left(1 - \frac{z^m}{(1+z)^m}\right)^{n+1} \; dz$$

The pole at zero has vanished. We now have non-zero poles at $z=-1$ and from the inverted term. These depend on m and we can certainly choose ϵ small enough so that none of them are inside the contour. Therefore this term does not contribute, leaving only

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n+m-1}}{z^{n+1}}\frac{1}{(1-z)(1+z)^m+z^{m+1}}\;dz.$$

The generating function $f(w)$ of these numbers is thus given by

$$f(w)=\sum_{n\geq 0}w^n\sum_{q=0}^{n}\binom{n+m-1}{n-q}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}.$$

This is

$$\sum_{q\geq 0}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}\sum_{n\geq q}w^n\binom{n+m-1}{n-q}$$
$$=\sum_{q\geq 0}w^q[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}\sum_{n\geq 0}w^n\binom{n+m-1+q}{n}$$
$$=\frac{1}{(1-w)^m}\sum_{q\geq 0}\frac{w^q}{(1-w)^q}[z^q]\frac{1}{(1-z)(1+z)^m+z^{m+1}}.$$

What we have here is an *annihilated coefficient extractor* that simplifies to

$$f(w)=\frac{1}{(1-w)^m}\frac{1}{(1-w/(1-w))(1+w/(1-w))^m+(w/(1-w))^{m+1}}$$
$$=\frac{1}{(1-w)^m}\frac{1}{(1-2w)/(1-w)/(1-w)^m+w^{m+1}/(1-w)^{m+1}}$$
$$=\frac{1-w}{1-2w+w^{m+1}}.$$

Now observe that

$$1-2w+w^{m+1}=(1-w)(1-w-w^2-\cdots-w^{m-1}-w^m)$$

so we finally have

$$f(w)=\left(1-\sum_{q=1}^{m}w^q\right)^{-1}=\frac{1}{1-w-w^2-\cdots-w^m}.$$

We see that by the basic theory of linear recurrences what we have here is a Fibonacci, Tribonacci, Tetranacci etc. recurrence. The question is what are the initial values.

Observe however that $[w^0]f(w)=1$ and for $1\leq q\leq m$ we have

$$[w^q]\frac{1-w}{1-2w+w^{m+1}}=[w^q]\frac{1}{1-2w+w^{m+1}}-[w^{q-1}]\frac{1}{1-2w+w^{m+1}}.$$

But

$$\frac{1}{1-2w+w^{m+1}} = \frac{1}{1-2w(1-w^m/2)} = \sum_{n\ge 0} 2^n w^n (1-w^m/2)^n$$

With the condition on q and $n \ge 1$ only the constant term from the term $(1-w^m/2)^n$ contributes because the degree would be more than m otherwise. This produces just one matching term with coefficient 2^q.

This yields for $f(w)$

$$[w^q]f(w) = 2^q - 2^{q-1} = 2^{q-1}.$$

Therefore we get for the intial terms starting at $q = 0$

$$1, 1, 2, 4, 8, 16, \ldots, 2^{m-1} \quad \text{with recurrence} \quad f_n = \sum_{q=1}^{m} f_{n-q}.$$

This recurrence also shows (by subtraction) that the sequence may be produced starting from $m-1$ zero terms followed by one.

The OEIS has the Fibonacci numbers, OEIS A000045

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

and the Tribonacci numbers, OEIS A000073

$$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, \ldots$$

and the Tetranacci numbers, OEIS A000078

$$1, 2, 4, 8, 15, 29, 56, 108, 208, 401, \ldots$$

and more.

This was math.stackexcange.com problem 1626949.

56 Stirling numbers of two kinds, binomial coefficients

Suppose we seek to verify that

$$\left\{ n \atop m \right\} = \sum_{k=m}^{n} \binom{k}{m} \sum_{q=0}^{k} (-1)^{n+q} \left\{ n+q-m \atop k \right\} (-1)^{k+q} \left[k \atop q \right] \binom{n}{n+q-m}$$

which is

$$\left\{ n \atop m \right\} = (-1)^n \sum_{k=m}^{n} \binom{k}{m} (-1)^k \sum_{q=0}^{k} \left\{ n+q-m \atop k \right\} \left[k \atop q \right] \binom{n}{m-q}$$

where presumably $n \geq m$. We need for the second binomial coefficient that $m \geq q$ so this is

$$\left\{ n \atop m \right\} = (-1)^n \sum_{k=m}^{n} \binom{k}{m} (-1)^k \sum_{q=0}^{m} \left\{ n+q-m \atop k \right\} \left[k \atop q \right] \binom{n}{m-q}.$$

Observe that the Stirling number of the second kind vanishes when $k > n$ so we may extend the summation to infinity, getting

$$\left\{ n \atop m \right\} = (-1)^n \sum_{k \geq m} \binom{k}{m} (-1)^k \sum_{q=0}^{m} \left\{ n+q-m \atop k \right\} \left[k \atop q \right] \binom{n}{m-q}.$$

Recall that

$$\left[k \atop q \right] = [w^q] k! \times \binom{w+k-1}{k}.$$

Starting with the inner sum we obtain

$$n! \sum_{q=0}^{m} \frac{1}{(m-q)!} [z^{n+q-m}] (\exp(z)-1)^k [w^q] \binom{w+k-1}{k}$$

$$= n! \sum_{q=0}^{m} \frac{1}{q!} [z^{n-q}] (\exp(z)-1)^k [w^{m-q}] \binom{w+k-1}{k}.$$

Now when $q > m$ the coefficient extractor in w yields zero, hence we may extend the sum in q to infinity:

$$n! \sum_{q \geq 0} \frac{1}{q!} [z^{n-q}] (\exp(z)-1)^k [w^{m-q}] \binom{w+k-1}{k}.$$

We thus obtain

$$\frac{n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^k \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+k-1}{k} \sum_{q \geq 0} \frac{1}{q!} z^q w^q \; dw \; dz$$

$$= \frac{n!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{n+1}} (\exp(z)-1)^k \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \binom{w+k-1}{k} \exp(zw) \; dw \; dz.$$

Preparing the outer sum we obtain

$$\sum_{k \geq m} \binom{k}{m} (-1)^k (\exp(z)-1)^k \binom{w+k-1}{k}$$

$$= \sum_{k \geq m} \binom{k}{m} (-1)^k (\exp(z)-1)^k [v^k] \frac{1}{(1-v)^w}.$$

Note that for a formal power series $Q(v)$ we have

$$\sum_{k\ge m}\binom{k}{m}(-1)^{k-m}u^{k-m}[v^k]Q(v) = \frac{1}{m!}\left.(Q(v))^{(m)}\right|_{v=-u}.$$

We get for the derivative in v

$$\left(\frac{1}{(1-v)^w}\right)^{(m)} = m!\binom{w+m-1}{m}\frac{1}{(1-v)^{w+m}}.$$

Substituting $u=\exp(z)-1$ yields

$$m!\binom{w+m-1}{m}\exp(-(w+m)z).$$

Returning to the double integral we find

$$\frac{(-1)^n\times n!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(\exp(z)-1)^m(-1)^m$$
$$\times\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{m+1}}\exp(zw)\binom{w+m-1}{m}\exp(-(w+m)z)\,dw\,dz$$
$$=\frac{(-1)^n\times n!}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(\exp(z)-1)^m(-1)^m\exp(-mz)$$
$$\times\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{m+1}}\binom{w+m-1}{m}\,dw\,dz$$
$$=\frac{(-1)^n\times n!}{2\pi i\times m!}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(\exp(z)-1)^m(-1)^m\exp(-mz)\,dz$$
$$=\frac{(-1)^n\times n!}{2\pi i\times m!}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1-\exp(-z))^m(-1)^m\,dz$$
$$=\frac{(-1)^n\times n!}{2\pi i\times m!}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(\exp(-z)-1)^m\,dz.$$

Finally put $z=-v$ to get

$$-\frac{(-1)^n\times n!}{2\pi i\times m!}\int_{|v|=\epsilon}\frac{(-1)^{n+1}}{v^{n+1}}(\exp(v)-1)^m\,dv$$
$$=\frac{n!}{2\pi i\times m!}\int_{|v|=\epsilon}\frac{1}{v^{n+1}}(\exp(v)-1)^m\,dv.$$

This is

$$n![v^n]\frac{(\exp(v)-1)^m}{m!}=\left\{ n \atop m \right\}$$

and we have the claim.
This was math.stackexchange.com problem 1926107.

57 An identity involving two binomial coefficients and a fractional term (B_1)

Suppose we seek to verify that

$$\sum_{k=0}^{m} \frac{q}{pk+q} \binom{pk+q}{k} \binom{pm-pk}{m-k} = \binom{mp+q}{m}.$$

Observe that

$$\binom{pk+q}{k} = \frac{pk+q}{k} \binom{pk+q-1}{k-1}$$

so that

$$\binom{pk+q}{k} - p\binom{pk+q-1}{k-1} = \frac{q}{k}\binom{pk+q-1}{k-1} = \frac{q}{pk+q}\binom{pk+q}{k}.$$

This yields two pieces for the sum, call them S_1

$$\sum_{k=0}^{m} \binom{pk+q}{k} \binom{pm-pk}{m-k}$$

and S_2

$$-p\sum_{k=0}^{m} \binom{pk+q-1}{k-1} \binom{pm-pk}{m-k}.$$

For S_1 introduce the integrals

$$\binom{pk+q}{k} = \frac{1}{2\pi i} \int_{|z|=\gamma} \frac{(1+z)^{pk+q}}{z^{k+1}} \, dz$$

and

$$\binom{pm-pk}{m-k} = \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm-pk}}{w^{m-k+1}} \, dw.$$

The second one controls the range of the sum because the pole at zero vanishes when $k > m$ so we may extend k to infinity, getting for the sum

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm}}{w^{m+1}} \frac{1}{2\pi i} \int_{|z|=\gamma} \frac{(1+z)^q}{z} \sum_{k\ge 0} \frac{w^k}{z^k} \frac{(1+z)^{pk}}{(1+w)^{pk}} \, dz \, dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm}}{w^{m+1}} \frac{1}{2\pi i} \int_{|z|=\gamma} \frac{(1+z)^q}{z} \frac{1}{1 - w(1+z)^p/z/(1+w)^p} \, dz \, dw$$

$$= \frac{1}{2\pi i} \int_{|w|=\epsilon} \frac{(1+w)^{pm+p}}{w^{m+1}} \frac{1}{2\pi i} \int_{|z|=\gamma} (1+z)^q \frac{1}{z(1+w)^p - w(1+z)^p} \, dz \, dw.$$

Suppose $|\epsilon| < |\gamma|$ which makes $\left|\frac{w(1+z)^p}{z(1+w)^p}\right| < 1$ so that we have convergence of the geometric series and suppose we can prove that $z = w$ is the only pole inside the contour and it is simple. We have

$$\begin{aligned}((1+w)^p z - w(1+z)^p)' &= (1+w)^p - pw(1+z)^{p-1} \\ &= (1+w)^{p-1}(1+w-wp).\end{aligned}$$

We can choose $|\epsilon|$ small enough such that $|1+w-wp| > 0$ so the pole is order one which yields

$$\begin{aligned}&\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm+p}}{w^{m+1}}(1+w)^q \frac{1}{(1+w)^{p-1}} \frac{1}{1+w-pw}\; dw \\ &= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm+q+1}}{w^{m+1}} \frac{1}{1+w-pw}\; dw.\end{aligned}$$

Following exactly the same procedure we obtain for S_2

$$-p\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm+q}}{w^{m}} \frac{1}{1+w-pw}\; dw.$$

Adding these two pieces now yields

$$\begin{aligned}&\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm+q}}{w^{m}} \left(\frac{1+w}{w} - p\right) \frac{1}{1+w-pw}\; dw \\ &= \frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm+q}}{w^{m+1}}\; dw \\ &= \binom{pm+q}{m}.\end{aligned}$$

Remark Mon Jan 25 2016.

An alternate proof which is completely rigorous and does not depend on assumptions about the poles of a bivariate complex function proceeds from the integral

$$\frac{1}{2\pi i}\int_{|w|=\epsilon} \frac{(1+w)^{pm}}{w^{m+1}} \sum_{k\ge 0} \frac{w^k}{(1+w)^{pk}} \frac{1}{2\pi i}\int_{|z|=\gamma} \frac{(1+z)^q}{z^{k+1}}(1+z)^{pk}\; dz\; dw$$

Now put

$$u = \frac{z}{(1+z)^p} \quad \text{and introduce} \quad g(u) = z.$$

We then have

$$du = \left(\frac{1}{(1+z)^p} - p\frac{z}{(1+z)^{p+1}}\right)\; dz = \left(\frac{u}{g(u)} - \frac{pu}{1+g(u)}\right)\; dz$$

and

$$dz = \frac{1}{u}\frac{g(u)(1+g(u))}{1+g(u)-pg(u)}\;du.$$

This yields

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm}}{w^{m+1}}\sum_{k\ge 0}\frac{w^k}{(1+w)^{pk}}$$
$$\times\frac{1}{2\pi i}\int_{|u|=\gamma}\frac{1}{g(u)u^k}(1+g(u))^q\frac{1}{u}\frac{g(u)(1+g(u))}{1+g(u)-pg(u)}\;du\;dw$$

or

$$\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm}}{w^{m+1}}\;(1+g(u))^q\frac{1+g(u)}{1+g(u)-pg(u)}\bigg|_{u=w/(1+w)^p}\;dw.$$

Now observe that $g(w/(1+w)^p) = w$ by definition so we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm}}{w^{m+1}}(1+w)^q\frac{1+w}{1+w-pw}\;dw\\&=\frac{1}{2\pi i}\int_{|w|=\epsilon}\frac{(1+w)^{pm+q+1}}{w^{m+1}}\frac{1}{1+w-pw}\;dw.\end{aligned}$$

This is exactly the same as before and the rest of the proof continues unchanged.

This was math.stackexchange.com problem 1620083.

58 Double chain of a total of three integrals (B_1B_2)

Suppose we seek to verify that

$$\sum_{k=q}^{n-1}\frac{q}{k}\binom{2n-2k-2}{n-k-1}\binom{2k-q-1}{k-1} = \binom{2n-q-2}{n-1}.$$

This is the same as

$$\sum_{k=q}^{n}\frac{q}{k}\binom{2n-2k}{n-k}\binom{2k-q-1}{k-1} = \binom{2n-q}{n}.$$

which is equivalent to

$$\sum_{k=q}^{n}\frac{q-k}{k}\binom{2n-2k}{n-k}\binom{2k-q-1}{k-1}+\sum_{k=q}^{n}\binom{2n-2k}{n-k}\binom{2k-q-1}{k-1}$$

$$= \binom{2n-q}{n}.$$

Now

$$\frac{q-k}{k}\binom{2k-q-1}{k-1} = \frac{q-k}{k}\frac{(2k-q-1)!}{(k-1)!(k-q)!}$$

$$= -\frac{(2k-q-1)!}{k!(k-q-1)!} = -\binom{2k-q-1}{k}.$$

It follows that what we have is in fact

$$\sum_{k=q}^{n}\binom{2n-2k}{n-k}\left(\binom{2k-q-1}{k-1} - \binom{2k-q-1}{k}\right) = \binom{2n-q}{n}$$

or alternatively

$$\sum_{k=q}^{n}\binom{2n-2k}{n-k}\left(\binom{2k-q-1}{k-q} - \binom{2k-q-1}{k-q-1}\right) = \binom{2n-q}{n}.$$

There are two pieces here, call them A and B. We use the integral representation

$$\binom{2n-2k}{n-k} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n-2k}}{z^{n-k+1}}\;dz$$

which is zero when $k > n$ (pole vanishes) so we may extend k to infinity. We also use the integral

$$\binom{2k-q-1}{k-q} = \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{2k-q-1}}{w^{k-q+1}}\;dw$$

which is zero when $k < q$ so we may extend k back to zero. We obtain for piece A

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q-1}}{(1+w)^{q+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n}}{z^{n+1}}\sum_{k\ge 0}\frac{z^k}{(1+z)^{2k}}\frac{(1+w)^{2k}}{w^k}\;dz\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q-1}}{(1+w)^{q+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n}}{z^{n+1}}\frac{1}{1-z(1+w)^2/w/(1+z)^2}\;dz\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q}}{(1+w)^{q+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+1}}\frac{1}{w(1+z)^2-z(1+w)^2}\;dz\;dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q-1}}{(1+w)^{q+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+1}}\frac{1}{(z-w)(z-1/w)}\;dz\;dw.$$

The derivation for piece B is the same and yields

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^q}{(1+w)^{q+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+1}}\frac{1}{(z-w)(z-1/w)}\;dz\;dw.$$

The difference of these two is

$$\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q-1}}{(1+w)^{q+1}}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+2}}{z^{n+1}}\frac{1-w}{(z-w)(z-1/w)}\;dz\;dw.$$

Using partial fractions by residues we get

$$\frac{1-w}{(z-w)(z-1/w)} = \frac{1-w}{w-1/w}\frac{1}{z-w} + \frac{1-w}{1/w-w}\frac{1}{z-1/w}$$

$$= \frac{w(1-w)}{w^2-1}\frac{1}{z-w} + \frac{w(1-w)}{1-w^2}\frac{1}{z-1/w} = -\frac{w}{1+w}\frac{1}{z-w} + \frac{w}{1+w}\frac{1}{z-1/w}$$

$$= \frac{1}{1+w}\frac{1}{1-z/w} - \frac{w^2}{1+w}\frac{1}{1-wz}.$$

At this point we can see that there will be no contribution from the second term but this needs to be verified. We get for the residue in z

$$-\frac{w^2}{1+w}\sum_{p=0}^{n}\binom{2n+2}{p}w^{n-p}$$

There is no pole at zero in the outer integral for a contribution of zero. Continuing with the first term we get

$$\frac{1}{1+w}\sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{w^{n-p}}$$

which yields

$$\sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{w^{q-1}}{(1+w)^{q+2}}\frac{1}{w^{n-p}}\;dw$$

$$= \sum_{p=0}^{n}\binom{2n+2}{p}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{(1+w)^{q+2}}\frac{1}{w^{n-q-p+1}}\;dw$$

$$= \sum_{p=0}^{n}\binom{2n+2}{p}(-1)^{n-q-p}\binom{n-p+1}{q+1}.$$

This is

$$\sum_{p=0}^{n}\binom{2n+2}{p}(-1)^{n-q-p}\binom{n-p+1}{n-p-q}.$$

The last integral we will be using is

$$\binom{n-p+1}{n-p-q} = \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n-p+1}}{v^{n-p-q+1}}\; dv.$$

Observe that this is zero when $p \ge n$ so we may extend p to infinity, getting

$$\frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n+1}}{v^{n-q+1}} \sum_{p\ge 0} \binom{2n+2}{p} (-1)^{n-q-p} \frac{v^p}{(1+v)^p}\; dv$$

$$= (-1)^{n-q} \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{(1+v)^{n+1}}{v^{n-q+1}} \left(1-\frac{v}{1+v}\right)^{2n+2} \; dv$$

$$= (-1)^{n-q} \frac{1}{2\pi i}\int_{|v|=\gamma} \frac{1}{v^{n-q+1}} \frac{1}{(1+v)^{n+1}}\; dv$$

$$= (-1)^{n-q}(-1)^{n-q}\binom{n-q+n}{n} = \binom{2n-q}{n}.$$

This is the claim. QED.
This was math.stackexchange.com problem 1708435.

59 Rothe-Hagen identity

The claim we set out to prove is the Rothe-Hagen identity

$$\sum_{k=0}^{n} \frac{x}{x+kz}\binom{x+kz}{k} \frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}.$$

We prove it for x, y, z positive integers and since the LHS and the RHS are in fact polynomials in x, y, z (the fractional terms cancel with the corresponding binomial coefficients e.g. $\frac{x}{x+kz}\binom{x+kz}{k} = \frac{x}{k!}(x+kz-1)^{\underline{k-1}}$ as long as $x+kz \neq 0$ (consult problem statement)) we then have it for arbitrary values (we also get polynomials when $k=0$ or $k=n$.)

Consider the generating function $C(v)$ that satisfies the functional equation again with z a positive integer

$$C(v) = 1 + vC(v)^z.$$

We ask about again with x a positive integer

$$[v^k]C(v)^x = \frac{1}{k}[v^{k-1}]xC(v)^{x-1}C'(v).$$

This is by the Cauchy Coefficient Formula

$$\frac{x}{k\times 2\pi i}\int_{|v|=\epsilon} \frac{1}{v^k} C(v)^{x-1}C'(v)\; dv.$$

Now we put $C(v) = w$ and we have from the functional equation

$$v = \frac{w-1}{w^z}$$

which yields

$$\frac{x}{k \times 2\pi i} \int_{|w-1|=\gamma} \frac{w^{zk}}{(w-1)^k} w^{x-1} \; dw$$
$$= \frac{x}{k \times 2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^k} \sum_{p=0}^{kz+x-1} \binom{kz+x-1}{p} (w-1)^p \; dw$$
$$= \frac{x}{k} \binom{kz+x-1}{k-1} = \frac{x}{x+kz} \binom{x+kz}{k}.$$

Note that this yields the correct value including for $k = 0$.

Now starting from the left of the desired identity we find

$$\sum_{k=0}^{n} [v^k] C_z(v)^x [v^{n-k}] C_z(v)^y = [v^n] C_z(v)^x C_z(v)^y = [v^n] C_z(v)^{x+y}.$$

This is the claim.

The same result may be obtained using Lagrange inversion.

For the LIF computation we put $D(v) = C(v) - 1$ so that we get the functional equation

$$D(v) = v(D(v) + 1)^z.$$

Using the notation from Wikipedia on LIF we have $\phi(w) = (w+1)^z$ and $H(v) = (v+1)^x$ and obtain

$$\frac{1}{k}[w^{k-1}](x(w+1)^{x-1}((w+1)^z)^k) = \frac{x}{k}[w^{k-1}](1+w)^{kz+x-1} = \frac{x}{k}\binom{kz+x-1}{k-1}.$$

This matches the first result.

This was math.stackexchange.com problem 3573304.

60 Abel polynomials are of binomial type

We seek to prove that

$$P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y)$$

where

$$P_n(x) = x(x + an)^{n-1}$$

is an Abel polynomial. Introduce $T(z)$ with functional equation

$$T(z) = z \exp(aT(z))$$

Viewing this as an EGF we seek the coefficient

$$n![z^n] \exp(xT(z)) = x(n-1)![z^{n-1}] \exp(xT(z))T'(z).$$

Note that $[z^0] \exp(xT(z)) = 1$. With the Cauchy Coefficient Formula we find for $n \geq 1$

$$\frac{x(n-1)!}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^n} \exp(xT(z))T'(z)\, dz.$$

Now we put $T(z) = w$ to get $z = w/\exp(aw)$ and

$$\begin{aligned}
&\frac{x(n-1)!}{2\pi i} \int_{|w|=\gamma} \frac{\exp(anw)\exp(xw)}{w^n}\, dw \\
&= \frac{x(n-1)!}{2\pi i} \int_{|w|=\gamma} \frac{\exp((x+an)w)}{w^n}\, dw \\
&= x(x+an)^{n-1}.
\end{aligned}$$

This means that

$$\begin{aligned}
\exp(xT(z)) &= 1 + \sum_{n\geq 1} x(x+an)^{n-1} \frac{z^n}{n!} \\
&= \sum_{n\geq 0} x(x+an)^{n-1} \frac{z^n}{n!} = \sum_{n\geq 0} P_n(x) \frac{z^n}{n!}.
\end{aligned}$$

By convolution of EGFs we thus have

$$\begin{aligned}
P_n(x+y) &= n![z^n] \exp((x+y)T(z)) = n![z^n] \exp(xT(z)) \exp(yT(z)) \\
&= n! \sum_{k=0}^{n} [z^k] \exp(xT(z))[z^{n-k}] \exp(yT(z)) \\
&= n! \sum_{k=0}^{n} \frac{P_k(x)}{k!} \frac{P_{n-k}(y)}{(n-k)!} = \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y).
\end{aligned}$$

The CCF can also be done by Lagrange Inversion, which goes as follows. Using the notation from Wikipedia on Lagrange-Buermann we have $\phi(w) = \exp(aw)$ and $H(w) = \exp(xw)$ and we find

$$n![z^n]\exp(xT(z)) = n!\frac{1}{n}[w^{n-1}]x\exp(xw)\exp(anw)$$
$$= (n-1)!x[w^{n-1}]\exp((x+an)w) = x(x+an)^{n-1}.$$

This was math.stackexchange.com problem 3704156.

61 A summation identity with four poles (B_2)

We seek to show that

$$\sum_{m=0}^{n}(-1)^m\binom{2n+2m}{n+m}\binom{n+m}{n-m} = (-1)^n 2^{2n}.$$

The LHS is

$$[z^n](1+z)^n\sum_{m=0}^{n}(-1)^m\binom{2n+2m}{n+m}(1+z)^m z^m.$$

The coefficient extractor enforces the upper limit of the sum and we may continue with

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\sum_{m\ge 0}(-1)^m\binom{2n+2m}{n+m}(1+z)^m z^m\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{(1-w)^{n+1}}$$
$$\times\sum_{m\ge 0}(-1)^m\frac{1}{w^m}\frac{1}{(1-w)^m}(1+z)^m z^m\; dw\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^{n+1}}\frac{1}{(1-w)^{n+1}}\frac{1}{1+z(1+z)/w/(1-w)}\; dw\; dz$$
$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^n}\frac{1}{(1-w)^n}\frac{1}{w(1-w)+z(1+z)}\; dw\; dz$$
$$= -\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1}{w^n}\frac{1}{(1-w)^n}\frac{1}{(w+z)(w-(1+z))}\; dw\; dz.$$

The contribution from the pole at $w=-z$ is

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+1}}\frac{(-1)^n}{z^n}\frac{1}{(1+z)^n}\frac{1}{1+2z}\; dz$$
$$= \frac{(-1)^n}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{2n+1}}\frac{1}{1+2z}\; dz = (-1)^n[z^{2n}]\frac{1}{1+2z} = (-1)^n(-1)^{2n}2^{2n}$$
$$= \boxed{(-1)^n 2^{2n}.}$$

This is the claim. We will document a choice of γ and ϵ so that $w = 0$ and $w = -z$ are the only poles inside the contour (pole at $w = 1$ not included, nor the pole at $w = 1 + z$.)

Now we have for the pole at $w = 0$

$$-\frac{1}{(w+z)(w-(1+z))} = \frac{1}{1+2z}\frac{1}{w+z} - \frac{1}{1+2z}\frac{1}{w-(1+z)}$$
$$= \frac{1}{z}\frac{1}{1+2z}\frac{1}{1+w/z} + \frac{1}{1+z}\frac{1}{1+2z}\frac{1}{1-w/(1+z)}.$$

We get from the first piece

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{n+2}}\frac{1}{1+2z}\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\frac{1}{z^{n-1-q}}\;dz$$
$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{2n+1-q}}\frac{1}{1+2z}\;dz$$
$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-1-q}\sum_{p=0}^{n}\binom{n}{p}(-1)^{2n-q-p}2^{2n-q-p}$$
$$= \sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}\sum_{p=0}^{n}\binom{n}{p}(-1)^{n-p}2^{n-p}$$
$$= (-1)^n\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}.$$

The second piece yields

$$-\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-1}}{z^{n+1}}\frac{1}{1+2z}\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\frac{1}{(1+z)^{n-1-q}}\;dz$$
$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^q}{z^{n+1}}\frac{1}{1+2z}\;dz$$
$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}\sum_{p=0}^{q}\binom{q}{p}(-1)^{n-p}2^{n-p}$$
$$= -\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}(-1)^{n-q}2^{n-q}\sum_{p=0}^{q}\binom{q}{p}(-1)^{q-p}2^{q-p}$$
$$= -(-1)^n\sum_{q=0}^{n-1}\binom{q+n-1}{n-1}2^{n-q}.$$

We see that the two pieces from $w = 0$ cancel so that the contribution is zero. This almost completes the proof, we only need to choose the contour so that $w = 1$ and $w = 1 + z$ are not included. For the initial geometric series to converge we need $|1 + z|\epsilon < |1 - w|\gamma$. With ϵ and γ in a neigborhood of zero we have $|1 + z|\epsilon \le (1 + \epsilon)\epsilon$ and $(1 - \gamma)\gamma \le |1 - w|\gamma$. The series converges if $(1 + \epsilon)\epsilon < (1 - \gamma)\gamma$. Therefore a good choice is $\epsilon = 1/10$ and $\gamma = 1/5$. The contour in γ clearly includes $w = 0$ and $w = -z$ and definitely does not include $w = 1$ and $w = 1 + z$ with leftmost value $9/10$. This concludes the proof.

We are not required to simplify the sum that appears in $w = 0$, but we may do so. We get

$$S_n = \sum_{q=0}^{n-1} \binom{q+n-1}{n-1} 2^{n-q} = 2^n [z^{n-1}] \frac{1}{1-z} \frac{1}{(1-z/2)^n}$$
$$= (-1)^{n+1} 2^{2n} \mathrm{Res}_{z=0} \frac{1}{z^n} \frac{1}{z-1} \frac{1}{(z-2)^n}.$$

Residues sum to zero and the residue at infinity is zero by inspection. The residue at $z = 1$ contributes -2^{2n}. The residue at $z = 2$ requires

$$\frac{1}{(2+(z-2))^n} \frac{1}{1+(z-2)} = \frac{1}{2^n} \frac{1}{(1+(z-2)/2)^n} \frac{1}{1+(z-2)}.$$

and we get the contribution

$$(-1)^{n+1} 2^n \sum_{q=0}^{n-1} \binom{q+n-1}{n-1} (-1)^q 2^{-q} (-1)^{n-1-q} = S_n.$$

This shows that $2S_n - 2^{2n} = 0$ or $S_n = 2^{2n-1}$.

This was math.stackexchange.com problem 3729998.

62 A summation identity over odd indices with a branch cut (B_2)

In trying to evaluate

$$\sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \binom{2n}{2n-k} \binom{2m-2n}{m-k}$$

we require

$$\sum_{k=0}^{m} \binom{2n}{2n-k} \binom{2m-2n}{m-k} \quad \text{and} \quad \sum_{k=0}^{m} \binom{2n}{2n-k} (-1)^k \binom{2m-2n}{m-k}.$$

For the first one we find

$$\sum_{k=0}^{m} \binom{2n}{k} \binom{2m-2n}{m-k} = [z^m](1+z)^{2m-2n} \sum_{k=0}^{m} \binom{2n}{k} z^k.$$

Here the coefficient extractor enforces the range and we get

$$\begin{aligned}
[z^m](1+z)^{2m-2n} \sum_{k\ge 0} \binom{2n}{k} z^k &= [z^m](1+z)^{2m-2n}(1+z)^{2n} \\
&= [z^m](1+z)^{2m} = \binom{2m}{m}.
\end{aligned}$$

This also follows from Chu-Vandermonde.
Continuing with the second piece we obtain

$$\sum_{k=0}^{m} \binom{2n}{k} (-1)^k \binom{2m-2n}{m-k} = (-1)^m \sum_{k=0}^{m} \binom{2n}{m-k} (-1)^k \binom{2m-2n}{k}$$

$$= (-1)^m \sum_{k=0}^{m} (-1)^k \binom{2m-2n}{k} [z^{2n+k-m}] \frac{1}{(1-z)^{m-k+1}}.$$

Now when $k > m$ we have $[z^{2n+k-m}](1-z)^{k-m-1} = 0$ so the coefficient extractor again enforces the range and we find

$$\frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n-m+1}} \frac{1}{(1-z)^{m+1}} \sum_{k\ge 0} (-1)^k \binom{2m-2n}{k} \frac{(1-z)^k}{z^k} \; dz$$

$$= \frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{2n-m+1}} \frac{1}{(1-z)^{m+1}} \left(1 - \frac{1-z}{z}\right)^{2m-2n} \; dz$$

$$= \frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} \frac{(1-2z)^{2m-2n}}{(1-z)^{m+1}} \; dz$$

$$= \frac{(-1)^m}{2\pi i} \int_{|z|=\epsilon} \frac{1}{z^{m+1}} \frac{(1-2z)^{2(m+1)}}{(1-z)^{m+1}} \frac{1}{(1-2z)^{2n+2}} \; dz.$$

Now put $z(1-z)/(1-2z)^2 = w$ so that

$$z = \frac{1}{2} \pm \frac{1}{2} \frac{1}{\sqrt{1+4w}}.$$

We have that $w = z + 3z^2 + 8z^3 + \cdots$ so $z = 0$ should be mapped to $w = 0$ and in fact we work with

$$z = \frac{1}{2} - \frac{1}{2} \frac{1}{\sqrt{1+4w}}.$$

We also see from the series expansion that the small circle around the origin $|z| = \epsilon$ is mapped to a contour that encircles $w = 0$ once and may in turn be deformed to a small circle $|w| = \gamma$. We choose the branch cut on $(-\infty, -1/4]$ so that we get analyticity in a neighborhood of the origin. We also have

$$dz = \frac{1}{(1+4w)^{3/2}} \; dw.$$

At last making the substitution we obtain

$$\frac{(-1)^m}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{1}{(1/\sqrt{1+4w})^{2n+2}} \frac{1}{(1+4w)^{3/2}} \; dw$$
$$= \frac{(-1)^m}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} (1+4w)^{n-1/2} \; dw = (-1)^m 4^m \binom{n-1/2}{m}.$$

Collecting the two pieces we find

$$\boxed{\frac{1}{2}\binom{2m}{m} + (-1)^{m+1} 2^{2m-1} \binom{n-1/2}{m}.}$$

This was math.stackexchange.com problem 3782050.

63 A stirling number identity

We seek to evaluate (note that this is zero by inspection when $k > n + m$):

$$\sum_{j=0}^{n} (-1)^{n+j} \left[n \atop j \right] \left\{ m+j \atop k \right\}$$

where $k \le n$. It is claimed that it is zero for $k < n$ and n^m for $k = n$. Using standard EGFs this becomes

$$n![z^n] \sum_{j=0}^{n} (-1)^{n+j} \frac{1}{j!} \left(\log \frac{1}{1-z} \right)^j (m+j)! [w^{m+j}] \frac{(\exp(w)-1)^k}{k!}$$

$$= (-1)^n n! m! [z^n] \sum_{j=0}^{n} (-1)^j \binom{m+j}{j} \left(\log \frac{1}{1-z} \right)^j$$
$$\times \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+j+1}} \frac{(\exp(w)-1)^k}{k!} \; dw$$
$$= (-1)^n n! m! [z^n] \frac{1}{2\pi i} \int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!}$$
$$\times \sum_{j=0}^{n} (-1)^j \binom{m+j}{j} \left(\log \frac{1}{1-z} \right)^j \frac{1}{w^j} \; dw.$$

Now $\left(\log\frac{1}{1-z}\right)^j = z^j + \cdots$ so the coefficient extractor $[z^n]$ enforces the upper limit of the sum:

$$(-1)^n n! m! [z^n] \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!}$$

$$\times \sum_{j\ge 0} (-1)^j \binom{m+j}{j} \left(\log\frac{1}{1-z}\right)^j \frac{1}{w^j}\; dw$$

$$= (-1)^n n! m! \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!}$$

$$\times \sum_{j\ge 0} (-1)^j \binom{m+j}{j} \left(\log\frac{1}{1-z}\right)^j \frac{1}{w^j}\; dw\; dz$$

$$= (-1)^n n! m! \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w^{m+1}} \frac{(\exp(w)-1)^k}{k!} \frac{1}{(1+\frac{1}{w}\log\frac{1}{1-z})^{m+1}}\; dw\; dz$$

$$= (-1)^n n! m! \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}}$$

$$\times \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{(\exp(w)-1)^k}{k!} \frac{1}{(w+\log\frac{1}{1-z})^{m+1}}\; dw\; dz.$$

Now observe that for the geometric series in j to converge we must have $|\log\frac{1}{1-z}| < |w|$. Note that with $\log\frac{1}{1-z} = z + \cdots$ the image of $|z| = \epsilon$ makes one turn around the origin, a circle of radius ϵ plus additional lower order fluctuations. We therefore choose ϵ to shrink this pseudo-circle to be entirely contained in $|w| = \gamma$. With this choice the pole at $-\log\frac{1}{1-z}$ is inside the contour in w. We thus require

$$\frac{1}{k! \times m!}\left(\sum_{q=0}^{k} \binom{k}{q} (-1)^{k-q} \exp(qw)\right)^{(m)} = \frac{1}{k! \times m!} \sum_{q=0}^{k} \binom{k}{q} (-1)^{k-q} q^m \exp(qw).$$

Evaluating the integral in w we find

$$(-1)^n \frac{n!}{k!} \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{1}{z^{n+1}} \sum_{q=0}^{k} \binom{k}{q} (-1)^{k-q} q^m (1-z)^q\; dz$$

which is

$$\boxed{\frac{n!}{k!} \sum_{q=0}^{k} \binom{k}{q} \binom{q}{n} (-1)^{k-q} q^m.}$$

Now when $k < n$ we have $\binom{q}{n} = 0$ so the entire sum vanishes as claimed. We get just one term when $k = n$ namely

$$\frac{n!}{n!}\binom{n}{n}\binom{n}{n}(-1)^{n-n}n^m = n^m$$

also as claimed. This concludes the argument.

This was math.stackexchange.com problem 3852633.

64 A Catalan-Central Binomial Coefficient Convolution

We seek to show that with

$$Q(z) = \frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n$$

we have

$$[z^k]Q(z) = \binom{n+2k}{k}.$$

Now with the branch cut on $[1/4, \infty)$ for $\sqrt{1-4z}$ we have analyticity of $Q(z)$ in a neighborhood of the origin (note that the exponentiated term does not in fact have a pole at $z = 0$) and the Cauchy Coefficient Formula applies. We obtain

$$[z^k]Q(z) = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{k+1}}\frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n\; dz.$$

We put $\sqrt{1-4z} = w$ so that $\frac{1}{\sqrt{1-4z}}\; dz = -\frac{1}{2}\; dw$ and $z = (1-w^2)/4$. With $w = 1 - 2z - \cdots$ we get as the image of $|z| = \varepsilon$ a contour that winds around $w = 1$ counterclockwise once and may be deformed to a circle, so that we obtain

$$[z^k]Q(z) = -\frac{1}{2}\frac{1}{2\pi i}\int_{|w-1|=\gamma} \frac{4^{k+1}}{(1-w^2)^{k+1}}(1-w)^n\frac{1}{2^n}\frac{4^n}{(1-w^2)^n}\; dw$$

$$= \frac{(-1)^k \times 2^{n+2k+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{(w-1)^n}{(w^2-1)^{n+k+1}}\; dw$$

$$= \frac{(-1)^k \times 2^{n+2k+1}}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{k+1}}\frac{1}{(w+1)^{n+k+1}}\; dw$$

$$= \frac{(-1)^k \times 2^k}{2\pi i}\int_{|w-1|=\gamma} \frac{1}{(w-1)^{k+1}}\frac{1}{(1+(w-1)/2)^{n+k+1}}\; dw$$

Apply the Cauchy Residue Theorem to get

$$(-1)^k \times 2^k \times (-1)^k \frac{1}{2^k} \binom{n+2k}{n+k} = \boxed{\binom{n+2k}{k}}$$

as claimed.
This was math.stackexchange.com problem 4025969.

65 Post Scriptum I: A trigonometric sum

Suppose we seek to evaluate

$$S = \sum_{k=1}^{m-1} \sin^{2q}(k\pi/m) = \sum_{k=0}^{m-1} \sin^{2q}(2\pi k/2/m).$$

Introducing $\zeta_k = \exp(2\pi ik/2/m)$ (root of unity) we get

$$S = \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}.$$

We also have

$$\sum_{k=m}^{2m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k \exp(2\pi im/2/m) - 1/\zeta_k/\exp(2\pi im/2/m))^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (-\zeta_k + 1/\zeta_k)^{2q}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q} = S.$$

We conclude that

$$S = \frac{1}{2} \sum_{k=0}^{2m-1} \frac{1}{(2i)^{2q}} (\zeta_k - 1/\zeta_k)^{2q}.$$

Introducing

$$f(z) = \frac{(-1)^q}{2^{2q+1}} \left(z - \frac{1}{z}\right)^{2q} \frac{2mz^{2m-1}}{z^{2m}-1}$$

$$= \frac{(-1)^q}{2^{2q+1}} \frac{(z^2-1)^{2q}}{z^{2q}} \frac{2mz^{2m-1}}{z^{2m}-1}$$

we then have

$$S = \sum_{k=0}^{2m-1} \mathrm{Res}_{z=\zeta_k} f(z).$$

Observe that the term $(z^2-1)^{2q}$ cancels the poles at ± 1 produced by $z^{2m}-1$ which however is perfectly acceptable as they correspond to $\zeta_0 = 1$ and $\zeta_m = -1$ where $\zeta_k - 1/\zeta_k$ is zero as well.

Residues sum to zero so we obtain

$$S + \mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=\infty} f(z) = 0.$$

Now for the residue at zero we see that when $2q-1 < 2m-1$ or $q < m$ the residue is zero. Otherwise we get

$$\begin{aligned} &\frac{(-1)^q}{2^{2q+1}} [z^{2q-2m}](z^2-1)^{2q} \frac{2m}{z^{2m}-1} \\ &= \frac{(-1)^q}{2^{2q+1}} [z^{2q}](z^2-1)^{2q} \frac{2mz^{2m}}{z^{2m}-1} \\ = -2m &\frac{(-1)^q}{2^{2q+1}} \sum_{p=0}^{q} \binom{2q}{p} (-1)^{2q-p} [z^{2q-2p}] \frac{z^{2m}}{1-z^{2m}}. \end{aligned}$$

We must have $p = q - lm$ where $l \ge 1$. This yields

$$-2m \frac{1}{2^{2q+1}} \sum_{l=1}^{\lfloor q/m \rfloor} \binom{2q}{q-lm} (-1)^{lm}.$$

This is correct even when $q < m$.

Continuing with the residue at infinity we find

$$\begin{aligned} &\mathrm{Res}_{z=\infty} f(z) = -\mathrm{Res}_{z=0} \frac{1}{z^2} f(1/z) \\ &= -\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{(-1)^q}{2^{2q+1}} \frac{(1/z^2-1)^{2q}}{1/z^{2q}} \frac{2m/z^{2m-1}}{1/z^{2m}-1} \\ &= -\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{(-1)^q}{2^{2q+1}} \frac{(1-z^2)^{2q}}{z^{2q}} \frac{2mz}{1-z^{2m}} \\ &= -\mathrm{Res}_{z=0} \frac{(-1)^q}{2^{2q+1}} \frac{(z^2-1)^{2q}}{z^{2q+1}} \frac{2m}{1-z^{2m}}. \end{aligned}$$

This is the same as the first residue at zero except now l starts at $l = 0$ and we obtain

$$-2m \frac{1}{2^{2q+1}} \sum_{l=0}^{\lfloor q/m \rfloor} \binom{2q}{q-lm} (-1)^{lm}.$$

Joining the two pieces we finally have

$$m\frac{1}{2^{2q}}\binom{2q}{q} + m\frac{1}{2^{2q-1}}\sum_{l=1}^{\lfloor q/m\rfloor}\binom{2q}{q-lm}(-1)^{lm}.$$

This was math.stackexchange.com problem 2051454.

66 Post Scriptum II: A class of polynomials similar to Fibonacci and Lucas Polynomials (B_1)

Suppose we seek to collect information concerning

$$\sum_{j=-\lfloor n/p\rfloor}^{\lfloor n/p\rfloor}(-1)^j\binom{2n}{n-pj}.$$

We will construct a generating function in n with $p \ge 1$ fixed. We introduce

$$\binom{2n}{n-pj} = \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n-pj+1}}(1+z)^{2n}\;dz.$$

Now as we examine this integral we see immediately that it vanishes if $j > \lfloor n/p\rfloor$ (pole at zero disappears). Moreover when $j < -\lfloor n/p\rfloor$ we have that $[z^{n-pj}](1+z)^{2n} = 0$ so this vanishes as well. Hence with this integral in place we may let j range from $-n$ to infinity and get

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2n}\sum_{j=-n}^{\infty}(-1)^j z^{pj}\;dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{n+1}}(1+z)^{2n}\sum_{j=0}^{\infty}(-1)^{j-n} z^{pj-pn}\;dz$$

$$= \frac{(-1)^n}{2\pi i}\int_{|z|=\epsilon}\frac{1}{z^{(p+1)n+1}}(1+z)^{2n}\frac{1}{1+z^p}\;dz.$$

We get zero for the residue at infinity, as can be seen from

$$\mathrm{Res}_{z=\infty}\frac{1}{z^{(p+1)n+1}}(1+z)^{2n}\frac{1}{1+z^p}$$

$$= -\mathrm{Res}_{z=0}\frac{1}{z^2}z^{(p+1)n+1}\frac{(1+z)^{2n}}{z^{2n}}\frac{z^p}{1+z^p}$$

$$= -\mathrm{Res}_{z=0}z^{(p-1)(n+1)}(1+z)^{2n}\frac{1}{1+z^p} = 0.$$

With residues adding to zero and introducing $\rho_k = \exp(\pi i/p + 2\pi i k/p)$ we thus obtain

$$-\sum_{k=0}^{p-1}(-1)^n \frac{1}{\rho_k^{(p+1)n+1}}(1+\rho_k)^{2n}\frac{1}{p\rho_k^{p-1}} = \frac{1}{p}\sum_{k=0}^{p-1}(-1)^n \frac{1}{\rho_k^{pn+n}}(1+\rho_k)^{2n}$$

$$= \frac{1}{p}\sum_{k=0}^{p-1}\left(\frac{1}{\rho_k}+2+\rho_k\right)^n.$$

At this point we can compute a generating function using the fact that

$$\sum_{q\ge 0}\rho^q z^q = \frac{1}{1-\rho z} = -\frac{1}{\rho}\frac{1}{z-1/\rho}$$

and we obtain as a first attempt

$$G_p(z) = \frac{1}{p}\sum_{k=0}^{p-1}\frac{1}{1-2(1+\cos(\pi/p+2\pi k/p))z}.$$

Observe that this correctly represents the cancelation of the pole at $z=-1$ when p is odd, contributing zero when $n\ge 1$ and $1/p$ otherwise. Furthermore note that with $\rho_k = \exp((2k+1)\pi i/p)$ we have

$$\frac{1}{\rho_{p-1-k}} = \exp(-(2(p-1-k)+1)\pi i/p) = \exp((2(k+1-p)-1)\pi i/p)$$

$$= \exp((2(k+1)-1)\pi i/p - 2\pi i) = \exp((2k+1)\pi i/p) = \rho_k$$

so the poles come in pairs with no pole at -1 when p is odd. Therefore the set of poles generated by this sum corresponds to the first $(p-1)/2$ poles when p is odd and the first $p/2$ when p is even. Joining these two we get the degree of the denominator once the sum is computed being $\lfloor p/2\rfloor$.

This first formula enables us to compute a few of these, like for $p=8$ we get (no complex number algebra required, basic trigonometry only)

$$G_8(z) = \frac{1-6z+10z^2-4z^3}{1-8z+20z^2-16z^3+2z^4}.$$

Looking up the coefficients we find for the denominator OEIS A034807 and for the numerator OEIS A011973 which point us to three types of polynomials, Fibonacci polynomials, Dickson polynomials and Lucas polynomials. With these data we are able to state a *conjecture* for the closed form of the generating function, which is

$$G_p(z) = \left(\sum_{q=0}^{\lfloor p/2\rfloor}\frac{p}{p-q}\binom{p-q}{q}(-1)^q z^q\right)^{-1}\sum_{q=0}^{\lfloor (p-1)/2\rfloor}\binom{p-1-q}{q}(-1)^q z^q.$$

To verify this we must show that the poles are at

$$\left(\frac{1}{\rho_k} + 2 + \rho_k\right)^{-1} \quad \text{with residue} \quad -\frac{2}{p}\left(\frac{1}{\rho_k} + 2 + \rho_k\right)^{-1}$$

where the factor two appears because the poles have been paired.

We therefore require the generating functions of the polynomials that appear in $G_p(z)$. Call the numerator $A_p(z)$ and the denominator $B_p(z)$. We first compute the auxiliary generating function

$$\begin{aligned}
Q_1(t,z) &= \sum_{p\ge 0} t^p \sum_{q=0}^{\lfloor p/2\rfloor} \binom{p-q}{q} (-1)^q z^q = \sum_{q\ge 0} (-1)^q z^q \sum_{p\ge 2q} \binom{p-q}{q} t^p \\
&= \sum_{q\ge 0} (-1)^q z^q t^{2q} \sum_{p\ge 0} \binom{p+q}{q} t^p = \sum_{q\ge 0} (-1)^q z^q t^{2q} \frac{1}{(1-t)^{q+1}} \\
&= \frac{1}{1-t} \frac{1}{1+zt^2/(1-t)} = \frac{1}{1-t+zt^2}.
\end{aligned}$$

We then have $A(t,z) = tQ_1(t,z)$. With $p/(p-q) = 1 + q/(p-q)$ we get two pieces for $B(t,z)$, the first is $Q_1(t,z)$ and the second is

$$\begin{aligned}
Q_2(t,z) &= \sum_{p\ge 0} t^p \sum_{q=1}^{\lfloor p/2\rfloor} \binom{p-1-q}{q-1} (-1)^q z^q = \sum_{q\ge 1} (-1)^q z^q \sum_{p\ge 2q} \binom{p-1-q}{q-1} t^p \\
&= \sum_{q\ge 1} (-1)^q z^q t^{2q} \sum_{p\ge 0} \binom{p+q-1}{q-1} t^p = \sum_{q\ge 1} (-1)^q z^q t^{2q} \frac{1}{(1-t)^q} \\
&= -\frac{zt^2/(1-t)}{1+zt^2/(1-t)} = -\frac{zt^2}{1-t+zt^2}
\end{aligned}$$

and hence we have $B(t,z) = Q_1(t,z) + Q_2(t,z)$. This yields the closed form

$$G_p(z) = \frac{[t^p]\frac{t}{1-t+zt^2}}{[t^p]\frac{1-zt^2}{1-t+zt^2}}.$$

Now introducing (we meet a shifted generating function of the Catalan numbers)

$$\alpha(z) = \frac{1+\sqrt{1-4z}}{2} \quad \text{and} \quad \beta(z) = \frac{1-\sqrt{1-4z}}{2}$$

we have a relationship that is analogous to that between Fibonacci and Lucas polynomials, namely,

$$A_p(z) = \frac{1}{\alpha(z)-\beta(z)}(\alpha(z)^p - \beta(z)^p) \quad \text{and} \quad B_p(z) = \alpha(z)^p + \beta(z)^p.$$

We now verify that $B_p(z) = 0$ for z a value from the claimed poles. Using $1/(1/\rho_k + 2 + \rho_k) = \rho_k/(1+\rho_k)^2$ ($\rho_k = -1$ is not included here) we find

$$\alpha(z) = \frac{1 + \sqrt{1 - 4\rho_k/(1+\rho_k)^2}}{2} = \frac{1 + (1-\rho_k)/(1+\rho_k)}{2} = \frac{1}{1+\rho_k}$$

and similarly

$$\beta(z) = \frac{\rho_k}{1+\rho_k}.$$

Raising to the power p we find

$$\alpha(z)^p + \beta(z)^p = \frac{1^p + \rho_k^p}{(1+\rho_k)^p} = \frac{1-1}{(1+\rho_k)^p} = 0.$$

We have located $\lfloor p/2 \rfloor$ distinct zeros here which means given the degree of $B_p(z)$ the poles are all simple. This means we may evaluate the residue by setting $z = \rho_k/(1+\rho_k)^2$ in (differentiate the denominator)

$$\frac{1}{p}\left(\sum_{q=0}^{\lfloor p/2 \rfloor} \frac{1}{p-q}\binom{p-q}{q}(-1)^q q z^{q-1}\right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q}(-1)^q z^q$$

which is

$$\frac{z}{p}\left(\sum_{q=1}^{\lfloor p/2 \rfloor} \binom{p-1-q}{q-1}(-1)^q z^q\right)^{-1} \sum_{q=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1-q}{q}(-1)^q z^q$$

The numerator is $A_p(z)$ and we get

$$\frac{1+\rho_k}{1-\rho_k}\frac{2}{(1+\rho_k)^p} = \frac{2}{(1-\rho_k)(1+\rho_k)^{p-1}}.$$

The denominator is $[t^p]Q_2(t,z)$ which is

$$[t^p]\frac{-zt^2}{1-t+zt^2} = [t^p]\frac{1-zt^2}{1-t+zt^2} - [t^p]\frac{1}{1-t+zt^2}$$

$$= [t^p]\frac{1-zt^2}{1-t+zt^2} - [t^{p+1}]\frac{t}{1-t+zt^2} = B_p(z) - A_{p+1}(z) = -A_{p+1}(z).$$

We get

$$-\frac{1+\rho_k}{1-\rho_k}\frac{1^{p+1} - \rho_k^{p+1}}{(1+\rho_k)^{p+1}} = -\frac{(1+\rho_k)^2}{(1-\rho_k)(1+\rho_k)^{p+1}} = -\frac{1}{(1-\rho_k)(1+\rho_k)^{p-1}}.$$

Joining numerator and denominator and multiplying by z/p finally produces

$$\frac{1}{p}\left(\frac{1}{\rho_k}+2+\rho_k\right)^{-1}\frac{2/(1-\rho_k)/(1+\rho_k)^{p-1}}{-1/(1-\rho_k)/(1+\rho_k)^{p-1}}=-\frac{2}{p}\left(\frac{1}{\rho_k}+2+\rho_k\right)^{-1}$$

as claimed. We have proved that the formula from the Egorychev method matches the conjectured form in terms of a certain class of polynomials that are related to Fibonacci and Lucas polynomials as well as Catalan numbers.

This was math.stackexchange.com problem 2237745.

67 Post Scriptum III: Partial row sums of Pascal's triangle (B_1)

Here we seek to prove that

$$\sum_{k=0}^{n}\binom{2k+1}{k}\binom{m-(2k+1)}{n-k}=\sum_{k=0}^{n}\binom{m+1}{k}.$$

This is

$$[z^n]\sum_{k=0}^{n}\binom{2k+1}{k}z^k(1+z)^{m-(2k+1)}$$
$$=[z^n](1+z)^{m-1}\sum_{k=0}^{n}\binom{2k+1}{k}z^k(1+z)^{-2k}.$$

Here $[z^n]$ enforces the range of the sum and we find

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m-1}}{z^{n+1}}\sum_{k\ge 0}\binom{2k+1}{k}z^k(1+z)^{-2k}\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m-1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w}\sum_{k\ge 0}\frac{(1+w)^{2k}}{w^k}z^k(1+z)^{-2k}\;dw\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m-1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w}\frac{1}{1-z(1+w)^2/w/(1+z)^2}\;dw\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m+1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{w(1+z)^2-z(1+w)^2}\;dw\;dz$$
$$=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{m+1}}{z^{n+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{1+w}{(1-wz)(w-z)}\;dw\;dz.$$

There is no pole at $w=0$ here. Note however that for the geometric series to converge we must have $|z(1+w)^2|<|w(1+z)^2|$. We can achieve this by taking $\gamma=2\epsilon$ so that

$$|z(1+w)^2| \le \epsilon(1+2\epsilon)^2 = 4\epsilon^3 + 4\epsilon^2 + \epsilon\Big|_{\epsilon=1/20} = \frac{242}{4000}$$

and

$$|w(1+z)^2| \ge 2\epsilon(1-\epsilon)^2 = 2\epsilon^3 - 4\epsilon^2 + 2\epsilon\Big|_{\epsilon=1/20} = \frac{361}{4000}.$$

With these values the pole at $w = z$ is inside the contour and we get as the residue

$$\frac{1+z}{1-z^2} = \frac{1}{1-z}.$$

This yields on substitution into the outer integral

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{m+1}}{z^{n+1}} \frac{1}{1-z}\; dz = [z^n]\frac{(1+z)^{m+1}}{1-z}$$

$$= \sum_{k=0}^{n} [z^k](1+z)^{m+1}[z^{n-k}]\frac{1}{1-z} = \sum_{k=0}^{n} {m+1 \choose k}.$$

This is the claim.

Remark. For the pole at $w = 1/z$ to be inside the contour we would need $1/\epsilon < 2\epsilon$ or $1 < 2\epsilon^2$ which does not hold here so this pole does not contribute.

This was math.stackexchange.com problem 3640984.

68 Post Scriptum IV: The Tree function and Eulerian numbers of the second order

We seek to show that the following identity holds:

$$2^{n+1}\sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \frac{1}{2^k} = n![x^n]\frac{1}{1+W(-\exp((x-1)/2)/2)}.$$

We will be using data from Wikipedia on Lambert W and work with the combinatorial branch which is $W_0(z)$.

Recall that

$$W'(z)\frac{z}{W(z)} = \frac{1}{1+W(z)}.$$

We obtain

$$[z^m]\frac{1}{1+W(z)} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^m}\frac{1}{W(z)} W'(z)\; dz.$$

Putting $W(z) = v$ we find

$$\frac{1}{2\pi i}\int_{|v|=\gamma}\frac{1}{v^m\exp(mv)}\frac{1}{v}\;dv = \frac{1}{2\pi i}\int_{|v|=\gamma}\frac{1}{v^{m+1}}\exp(-mv)\;dv = \frac{(-1)^m m^m}{m!}.$$

so that

$$\frac{1}{1+W(z)} = \sum_{m\ge 0}(-1)^m m^m \frac{z^m}{m!}.$$

We get for the original RHS

$$n![x^n]\sum_{m\ge 0}\frac{m^m}{m!}\exp(m(x-1)/2)\frac{1}{2^m}$$

$$= n![x^n]\sum_{m\ge 0}\frac{m^m}{m!}\frac{\exp(-m/2)}{2^m}\exp(mx/2)$$

$$= \sum_{m\ge 0}\frac{m^{m+n}}{m!}\frac{\exp(-m/2)}{2^{m+n}}.$$

First part. Introduce the tree function $T(z)$ from combinatorics where $T(z) = z\exp T(z)$ and $T(z) = -W_0(-z)$. Note that we have by Cayley's theorem that $T(z) = \sum_{m\ge 1} m^{m-1}\frac{z^m}{m!}$. We claim that with $n \ge 1$

$$Q_n(z) = \sum_{m\ge 0} m^{m+n}\frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}}\sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k.$$

This means the RHS is $\frac{1}{2^n}Q_n(\exp(-1/2)/2)$. To verify this last identity note that $Q_{n+1}(z) = z\frac{d}{dz}Q_n(z)$ so we may prove it by induction.

We get for the RHS of the series identity on differentiating and multiplying by z

$$\frac{(2n+1)zT'(z)}{(1-T(z))^{2n+2}}\sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k + \frac{z}{(1-T(z))^{2n+1}}\sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle kT(z)^{k-1}T'(z)$$

Extracting the term $zT'(z)/(1-T(z))^{2n+2}$ in front leaves us with

$$(2n+1)\sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k + (1-T(z))\sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle kT(z)^{k-1}$$

$$= (2n+1)\sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k + \sum_{k=0}^{n-1}\left\langle\!\!\left\langle n \atop k+1 \right\rangle\!\!\right\rangle (k+1)T(z)^k - \sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle kT(z)^k$$

$$= \sum_{k=1}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle (2n+2-(k+1))T(z)^k + \sum_{k=0}^{n-1}\left\langle\!\!\left\langle n \atop k+1 \right\rangle\!\!\right\rangle (k+1)T(z)^k.$$

We may include $k=0$ in the first sum and $k=n$ in the second. Now the Eulerian number recurrence (second order) according to OEIS A349556 is

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle k + \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle (2n-k)$$

We have shown that

$$\begin{aligned} Q_{n+1}(z) &= \frac{zT'(z)}{(1-T(z))^{2n+2}} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n+1 \atop k+1} \right\rangle\!\!\right\rangle T(z)^k \\ &= \frac{zT'(z)}{T(z)(1-T(z))^{2n+2}} \sum_{k=1}^{n+1} \left\langle\!\!\left\langle {n+1 \atop k} \right\rangle\!\!\right\rangle T(z)^k. \end{aligned}$$

Now we just have to verify that

$$\frac{zT'(z)}{T(z)(1-T(z))^{2n+2}} = \frac{1}{(1-T(z))^{2n+3}} \quad \text{or} \quad zT'(z)(1-T(z)) = T(z).$$

The functional equation tells us that $T'(z) = \exp T(z) + z \exp T(z) T'(z)$ so that $T'(z)(1-T(z)) = \exp T(z) = T(z)/z$ which is just what we need. It remains to verify the base case so the induction starts properly. We seek

$$Q_1(z) = \sum_{m\ge 0} m^{m+1} \frac{z^m}{m!} = \frac{T(z)}{(1-T(z))^3}.$$

We verify this by coefficient extraction. We get

$$m![z^m]Q_1(z) = \frac{m!}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{m+1}} \frac{T(z)}{(1-T(z))^3} \; dz.$$

With $T(z) = z + \cdots$ this integral will produce the correct value zero for $m=0$. For $m \ge 1$, we put $T(z) = w$ so that $z = w\exp(-w)$ and $dz = \exp(-w)(1-w)\; dw$ and obtain

$$\begin{aligned} &\frac{m!}{2\pi i} \int_{|w|=\gamma} \frac{\exp((m+1)w)}{w^{m+1}} \frac{w}{(1-w)^3} \exp(-w)(1-w)\; dw \\ &= \frac{m!}{2\pi i} \int_{|w|=\gamma} \frac{\exp(mw)}{w^m} \frac{1}{(1-w)^2} \; dw. \end{aligned}$$

This is

$$\begin{aligned} m! \sum_{q=0}^{m-1} \frac{m^q}{q!}(m-q) &= m! \sum_{q=0}^{m-1} \frac{m^{q+1}}{q!} - m! \sum_{q=1}^{m-1} \frac{m^q}{(q-1)!} \\ &= m! \sum_{q=0}^{m-1} \frac{m^{q+1}}{q!} - m! \sum_{q=0}^{m-2} \frac{m^{q+1}}{q!} = m! \frac{m^m}{(m-1)!} = m^{m+1} \end{aligned}$$

as desired.

Sequel. Note that in the identity for $Q_n(z)$ we have by the definition of the Eulerian numbers that $\left\langle\!\left\langle n \atop 0 \right\rangle\!\right\rangle$ is zero when $n \ge 1$. Therefore we may extend k to include zero (with $n \ge 1$ for the moment) which yields

$$\boxed{Q_n(z) = \sum_{m\ge 0} m^{m+n}\frac{z^m}{m!} = \frac{1}{(1-T(z))^{2n+1}}\sum_{k=0}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle T(z)^k.}$$

Now observe that this will produce $Q_0(z) = \sum_{m\ge 0} m^m \frac{z^m}{m!} = \frac{1}{1-T(z)}$ due to $\left\langle\!\left\langle 0 \atop 0 \right\rangle\!\right\rangle = 1$ which is in fact correct because unlike $Q_n(z)$ with $n \ge 1$, $Q_0(z)$ has a constant term, which is one (this is because $m^{m+n} = 0$ for $m = 0$ and $n \ge 1$ and $m^{m+n} = 1$ for $m = 0$ and $n = 0$). Therefore

$$Q_0(z) = 1 + zT'(z) = 1 + \frac{T(z)}{1-T(z)} = \frac{1}{1-T(z)}$$

as obtained from the boxed version of the main identity, which is seen to hold for all $n \ge 0$.

Conclusion. We are now ready to answer the original question. We have shown that the RHS is $\frac{1}{2^n}Q_n(\exp(-1/2)/2)$. By our formula for $Q_n(z)$ in terms of the tree function we obtain with $T(\exp(-1/2)/2) = \frac{1}{2}$ at last the closed form

$$\frac{1}{2^n}\frac{1}{(1-1/2)^{2n+1}}\sum_{k=0}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle \frac{1}{2^k} = 2^{n+1}\sum_{k=0}^{n} \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle \frac{1}{2^k}$$

which is the LHS and hence the claim.

This was math.stackexchange.com problem 4040942.

69 Egorychev method in formal power series notation

69.1 MSE 2384932

We seek to evaluate

$$\sum_{l=0}^{m} (-4)^l {m\choose l}{2l\choose l}^{-1}\sum_{k=0}^{n}\frac{(-4)^k}{2k+1}{n\choose k}{2k\choose k}^{-1}{k+l\choose l}.$$

We start with the inner term and use the Beta function identity

$$\frac{1}{2k+1}{2k\choose k}^{-1} = \int_0^1 x^k(1-x)^k\; dx.$$

We obtain

$$\int_0^1 [z^l] \sum_{k=0}^{n} \binom{n}{k} (-4)^k x^k (1-x)^k \frac{1}{(1-z)^{k+1}} \; dx$$

$$= [z^l] \frac{1}{1-z} \int_0^1 \left(1 - \frac{4x(1-x)}{1-z}\right)^n \; dx$$

$$= [z^l] \frac{1}{(1-z)^{n+1}} \int_0^1 ((1-2x)^2 - z)^n \; dx$$

$$= \sum_{q=0}^{l} \binom{l-q+n}{n} [z^q] \int_0^1 ((1-2x)^2 - z)^n \; dx$$

$$= \sum_{q=0}^{l} \binom{l-q+n}{n} \binom{n}{q} (-1)^q \int_0^1 (1-2x)^{2n-2q} \; dx$$

$$= \sum_{q=0}^{l} \binom{l-q+n}{n} \binom{n}{q} (-1)^q \left[-\frac{1}{2(2n-2q+1)} (1-2x)^{2n-2q+1} \right]_0^1$$

$$= \sum_{q=0}^{l} \binom{l-q+n}{n} \binom{n}{q} (-1)^q \frac{1}{2n-2q+1}.$$

Now we have

$$\binom{l-q+n}{n} \binom{n}{q} (-1)^q \frac{1}{2n-2q+1}$$

$$= \mathrm{Res}_{z=q} \frac{(-1)^n}{2n+1-2z} \prod_{p=0}^{n-1} (l+n-p-z) \prod_{p=0}^{n} \frac{1}{z-p}.$$

Residues sum to zero and since $\lim_{R\to\infty} 2\pi R \times R^n/R/R^{n+1} = 0$ we may evaluate the sum using the negative of the residue at $z = (2n+1)/2$. We get

$$\frac{1}{2}(-1)^n \prod_{p=0}^{n-1} (l+n-p-(2n+1)/2) \prod_{p=0}^{n} \frac{1}{(2n+1)/2-p}$$

$$= (-1)^n \prod_{p=0}^{n-1} (2l+2n-2p-(2n+1)) \prod_{p=0}^{n} \frac{1}{2n+1-2p}$$

$$= (-1)^n \prod_{p=0}^{n-1} (2l-2p-1) \frac{2^n n!}{(2n+1)!}$$

$$= (-1)^n \frac{1}{2l+1} \prod_{p=-1}^{n-1} (2l-2p-1) \frac{2^n n!}{(2n+1)!}$$

$$
\begin{aligned}
&= (-1)^n \frac{2^n n!}{(2n+1)!} \frac{1}{2l+1} \prod_{p=0}^{n} (2l-2p+1) \\
&= (-1)^n \frac{2^{2n+1} n!}{(2n+1)!} \frac{1}{2l+1} \prod_{p=0}^{n} (l+1/2-p) \\
&= (-1)^n \frac{2^{2n+1} n!(n+1)!}{(2n+1)!} \frac{1}{2l+1} \binom{l+1/2}{n+1}.
\end{aligned}
$$

We obtain for our sum

$$
(-1)^n 2^{2n+1} \binom{2n+1}{n}^{-1} \sum_{l=0}^{m} (-4)^l \binom{m}{l} \frac{1}{2l+1} \binom{2l}{l}^{-1} \binom{l+1/2}{n+1}.
$$

We now work with the remaining sum without the factor in front. We obtain

$$
\begin{aligned}
&\int_0^1 [z^{n+1}]\sqrt{1+z} \sum_{l=0}^{m} \binom{m}{l} (-4)^l x^l (1-x)^l (1+z)^l \; dx \\
&= [z^{n+1}]\sqrt{1+z} \int_0^1 (1-4x(1-x)(1+z))^m \; dx \\
&= [z^{n+1}]\sqrt{1+z} \int_0^1 \sum_{q=0}^{m} \binom{m}{q} (1-2x)^{2m-2q} (-1)^q (4x(1-x))^q z^q \; dx \\
&= \sum_{q=0}^{m} \binom{m}{q} \binom{1/2}{n+1-q} \int_0^1 (1-2x)^{2m-2q} (-1)^q (4x(1-x))^q \; dx \\
&= \sum_{q=0}^{m} \binom{m}{q} \binom{1/2}{n+1-q} \int_0^1 (1-2x)^{2m} \left(1 - \frac{1}{(1-2x)^2}\right)^q \; dx \\
&= \sum_{q=0}^{m} \binom{m}{q} \binom{1/2}{n+1-q} \sum_{p=0}^{q} \binom{q}{p} (-1)^p \int_0^1 (1-2x)^{2m-2p} \; dx \\
&= \sum_{q=0}^{m} \binom{m}{q} \binom{1/2}{n+1-q} \sum_{p=0}^{q} \binom{q}{p} (-1)^p \frac{1}{2m-2p+1}.
\end{aligned}
$$

Re-writing then yields

$$
\sum_{p=0}^{m} (-1)^p \frac{1}{2m-2p+1} \sum_{q=p}^{m} \binom{m}{q} \binom{1/2}{n+1-q} \binom{q}{p}.
$$

Observe that

$$
\binom{m}{q}\binom{q}{p} = \frac{m!}{(m-q)! \times p! \times (q-p)!} = \binom{m}{p}\binom{m-p}{m-q}
$$

so that we find

$$\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2m-2p+1}\sum_{q=p}^{m}\binom{m-p}{m-q}\binom{1/2}{n+1-q}$$

$$=\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2m-2p+1}\sum_{q=0}^{m-p}\binom{m-p}{m-p-q}\binom{1/2}{n+1-p-q}$$

$$=\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2m-2p+1}\sum_{q=0}^{m-p}\binom{m-p}{q}\binom{1/2}{n+1-p-q}.$$

Continuing we obtain

$$\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2m-2p+1}\sum_{q=0}^{m-p}\binom{m-p}{q}[z^{n+1-p}]z^q\sqrt{1+z}$$

$$=\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2m-2p+1}[z^{n+1-p}]\sqrt{1+z}\sum_{q=0}^{m-p}\binom{m-p}{q}z^q$$

$$=\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2m-2p+1}[z^{n+1-p}](1+z)^{m-p+1/2}$$

$$=\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2m-2p+1}\binom{m-p+1/2}{n+1-p}$$

$$=(-1)^m\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2p+1}\binom{p+1/2}{n+1-m+p}$$

$$=(-1)^m\sum_{p=0}^{m}\binom{m}{p}(-1)^p\frac{1}{2}\frac{1}{m-n-1/2}\binom{p-1/2}{n+1-m+p}$$

$$=(-1)^m\frac{1}{2m-2n-1}\sum_{p=0}^{m}\binom{m}{p}(-1)^p\binom{p-1/2}{n+1-m+p}.$$

Concluding with a closed form we establish at last

$$(-1)^m\frac{1}{2m-2n-1}\sum_{p=0}^{m}\binom{m}{p}(-1)^p[z^{n+1-m}]z^{-p}(1+z)^{p-1/2}$$

$$=(-1)^m\frac{1}{2m-2n-1}[z^{n+1-m}](1+z)^{-1/2}\sum_{p=0}^{m}\binom{m}{p}(-1)^pz^{-p}(1+z)^p$$

$$=(-1)^m\frac{1}{2m-2n-1}[z^{n+1-m}](1+z)^{-1/2}\left(1-\frac{1+z}{z}\right)^m$$

$$= \frac{1}{2m-2n-1}[z^{n+1}](1+z)^{-1/2}.$$

We finish by re-introducing the factor in front to obtain

$$(-1)^n 2^{2n+1} \binom{2n+1}{n}^{-1} \frac{1}{2m-2n-1} \binom{-1/2}{n+1}$$

$$= (-1)^n 2^{2n+1} \binom{2n+1}{n}^{-1} \frac{1}{2m-2n-1} \frac{1}{(n+1)!} \prod_{q=0}^{n} (-1/2-q)$$

$$= (-1)^n 2^{n} \binom{2n+1}{n}^{-1} \frac{1}{2m-2n-1} \frac{1}{(n+1)!} \prod_{q=0}^{n} (-1-2q)$$

$$= 2^{n} \binom{2n+1}{n}^{-1} \frac{1}{2n+1-2m} \frac{1}{(n+1)!} \prod_{q=0}^{n} (1+2q)$$

$$= 2^{n} \binom{2n+1}{n}^{-1} \frac{1}{2n+1-2m} \frac{1}{(n+1)!} \frac{(2n+1)!}{2^n n!}.$$

Yes indeed this is

$$\frac{1}{2n+1-2m}.$$

Here I have chosen to document the simple steps as well as the complicated ones to aid all types of readers.

This was math.stackexchange.com problem 2384932.

69.2 MSE 2472978

We seek to verify that

$$\sum_{l=0}^{n} \binom{n}{l}^2 (x+y)^{2l}(x-y)^{2n-2l} = \sum_{l=0}^{n} \binom{2l}{l} \binom{2n-2l}{n-l} x^{2l} y^{2n-2l}.$$

Now we see on the LHS that the powers of x and y always add up to $2n$ and the exponent on x determines the one on y. Extracting the coefficient on $[x^q][y^{2n-q}]$ we obtain

$$\sum_{l=0}^{n} \binom{n}{l}^2 \sum_{p=0}^{q} \binom{2l}{p} (-1)^{2n-2l-(q-p)} \binom{2n-2l}{q-p}$$

$$= \sum_{l=0}^{n} \binom{n}{l}^2 \sum_{p=0}^{q} \binom{2l}{p} (-1)^{q-p} [z^{q-p}](1+z)^{2n-2l}$$

$$= [z^q](-1)^q \sum_{l=0}^{n} \binom{n}{l}^2 (1+z)^{2n-2l} \sum_{p=0}^{q} \binom{2l}{p} (-1)^p z^p.$$

We may extend p to infinity because with $p > q$ there is no contribution to $[z^q]$, getting

$$[z^q](-1)^q \sum_{l=0}^{n} \binom{n}{l}^2 (1+z)^{2n-2l} \sum_{p\ge 0} \binom{2l}{p} (-1)^p z^p$$
$$= [z^q](-1)^q \sum_{l=0}^{n} \binom{n}{l}^2 (1+z)^{2n-2l} (1-z)^{2l}$$
$$= [z^q](-1)^q [w^n](1+w(1-z)^2)^n (1+w(1+z)^2)^n$$
$$= [z^q][w^n](1+w(1-z)^2)^n (1+w(1+z)^2)^n.$$

Re-write this as

$$[z^q][w^n]((w(1+z^2)+1)^2 - 4w^2z^2)^n$$
$$= [z^q][w^n] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} w^{2p} z^{2p} (w(1+z^2)+1)^{2n-2p}$$
$$= [z^q] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} z^{2p} [w^{n-2p}] (w(1+z^2)+1)^{2n-2p}$$
$$= [z^q] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} z^{2p} \binom{2n-2p}{n-2p} (1+z^2)^{n-2p}.$$

We observe at this point that we get zero here when q is odd, which agrees with the target formula. We are thus justified in putting $q = 2l$ to get

$$[z^l] \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} z^p \binom{2n-2p}{n-2p} (1+z)^{n-2p}$$
$$= \sum_{p=0}^{n} \binom{n}{p} (-1)^p 2^{2p} \binom{2n-2p}{n-2p} \binom{n-2p}{l-p}.$$

Note that

$$\binom{n}{p}\binom{2n-2p}{n-2p}\binom{n-2p}{l-p} = \frac{(2n-2p)!}{p! \times (n-p)! \times (l-p)! \times (n-l-p)!}$$
$$= \binom{l}{p} \frac{(2n-2p)!}{(n-p)! \times l! \times (n-l-p)!} = \binom{l}{p}\binom{2n-2p}{n-p}\binom{n-p}{l}.$$

Re-indexing we get for the sum

$$(-1)^n 2^{2n} \sum_{p=0}^{n} \binom{l}{n-p} \binom{2p}{p} \binom{p}{l} (-1)^p 2^{-2p}$$

$$= (-1)^n 2^{2n} \sum_{p=0}^{n} \binom{2p}{p} (-1)^p 2^{-2p} [z^{n-p}](1+z)^l [w^l](1+w)^p$$

$$= (-1)^n 2^{2n} [z^n](1+z)^l [w^l] \sum_{p=0}^{n} \binom{2p}{p} (-1)^p 2^{-2p} z^p (1+w)^p.$$

We may once more extend p to infinity because there is no contribution from the sum term to the coefficient extractor $[z^n]$ when $p > n$, obtaining

$$(-1)^n 2^{2n} [z^n](1+z)^l [w^l] \sum_{p\ge 0} \binom{2p}{p} (-1)^p 2^{-2p} z^p (1+w)^p$$

$$= (-1)^n 2^{2n} [z^n](1+z)^l [w^l] \frac{1}{\sqrt{1+z(1+w)}}$$

$$= (-1)^n 2^{2n} [z^n](1+z)^l [w^l] \frac{1}{\sqrt{1+z+wz}}$$

$$= (-1)^n 2^{2n} [z^n](1+z)^{l-1/2} [w^l] \frac{1}{\sqrt{1+wz/(1+z)}}$$

$$= (-1)^n 2^{2n} [z^n](1+z)^{l-1/2} \binom{2l}{l} (-1)^l 2^{-2l} z^l \frac{1}{(1+z)^l}$$

$$= (-1)^{n-l} 2^{2n-2l} \binom{2l}{l} [z^{n-l}] \frac{1}{\sqrt{1+z}}$$

$$= (-1)^{n-l} 2^{2n-2l} \binom{2l}{l} \binom{2n-2l}{n-l} (-1)^{n-l} 2^{-(2n-2l)}$$

$$= \binom{2l}{l} \binom{2n-2l}{n-l}.$$

This was math.stackexchange.com problem 2472978.

69.3 MSE 2719320

The goal here was to investigate closed forms of

$$\binom{n}{k} \frac{1}{ak+b}.$$

We start by trying to prove the first closed form given to see if a pattern does emerge. We use with c a positive integer

$$\binom{n+c}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+c}$$

Now

$$\binom{n+c}{n}\binom{n}{k} = \frac{(n+c)!}{(c)! \times k! \times (n-k)!} = \binom{n+c}{k+c}\binom{k+c}{k}.$$

Hence we have for the sum

$$\sum_{k=0}^{n} \binom{n+c}{k+c}\binom{k+c}{k}\frac{1}{k+c} = \frac{1}{c}\sum_{k=0}^{n} \binom{n+c}{k+c}\binom{k+c-1}{c-1}.$$

This is

$$\frac{1}{c}\sum_{k=0}^{n} \binom{k+c-1}{c-1}[z^{n-k}]\frac{1}{(1-z)^{k+c+1}} = \frac{1}{c}\sum_{k=0}^{n} \binom{k+c-1}{c-1}[z^n]z^k\frac{1}{(1-z)^{k+c+1}}.$$

Here we get no contribution to $[z^n]$ when $k > n$ so we may continue with

$$\frac{1}{c}[z^n]\frac{1}{(1-z)^{c+1}}\sum_{k\ge 0} \binom{k+c-1}{c-1} z^k \frac{1}{(1-z)^k}$$
$$= \frac{1}{c}[z^n]\frac{1}{(1-z)^{c+1}}\frac{1}{(1-z/(1-z))^c}$$
$$= \frac{1}{c}[z^n]\frac{1}{1-z}\frac{1}{(1-2z)^c}.$$

This is

$$\frac{1}{c}\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}\frac{1}{1-z}\frac{1}{(1-2z)^c}$$
$$= \frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}\frac{1}{z-1}\frac{1}{(z-1/2)^c}.$$

With residues summing to zero we can evaluate this using the residues at $z = 1$, $z = 1/2$ and $z = \infty$. We get for $z = 1$ the residue

$$\frac{(-1)^{c+1}}{c}.$$

For the residue at infinity we find

$$-\frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}\frac{1}{z^2}\frac{1}{(1/z)^{n+1}}\frac{1}{1/z-1}\frac{1}{(1/z-1/2)^c}$$
$$= -\frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}\frac{1}{z^2}z^{n+1}\frac{z}{1-z}\frac{z^c}{(1-z/2)^c}$$
$$= -\frac{(-1)^{c+1}}{c2^c}\mathrm{Res}_{z=0}z^{n+c}\frac{1}{1-z}\frac{1}{(1-z/2)^c} = 0.$$

This also follows by inspection. The residue at $z = 1/2$ requires the use of Leibniz' rule as in

$$\frac{1}{p!}\left(\frac{1}{z^{n+1}}\frac{1}{z-1}\right)^{(p)} = \frac{1}{p!}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^q(n+q)!}{n!z^{n+1+q}}(-1)^{p-q}\frac{(p-q)!}{(z-1)^{p-q+1}}$$

$$= (-1)^p\sum_{q=0}^{p}\binom{n+q}{q}\frac{1}{z^{n+1+q}}\frac{1}{(z-1)^{p-q+1}}.$$

We set $p = c-1$ and $z = 1/2$ and restore the factor in front to get for the residue

$$\frac{(-1)^{c+1}}{c2^c}(-1)^{c-1}\sum_{q=0}^{c-1}\binom{n+q}{q}\frac{1}{(1/2)^{n+1+q}}\frac{(-1)^{c-q}}{(1/2)^{c-q}}$$

$$= \frac{(-1)^c 2^{n+1}}{c}\sum_{q=0}^{c-1}\binom{n+q}{q}(-1)^q.$$

Collecting everything we thus obtain

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+c} = \binom{n+c}{c}^{-1}\frac{(-1)^c}{c}\left(1-2^{n+1}\sum_{q=0}^{c-1}\binom{n+q}{q}(-1)^q\right).$$

This is an improvement in the sense that if n is the variable and c is the constant then we have replaced the sum in n terms (variable) by a sum in c terms (fixed) of polynomials in n. We can make this more explicit by writing

$$\sum_{q=0}^{c-1}\binom{n+q}{q}(-1)^q = \sum_{q=0}^{c-1}\frac{(-1)^q}{q!}\sum_{p=0}^{q}n^p\left[{q+1 \atop p+1}\right]$$

$$= \sum_{p=0}^{c-1}n^p\sum_{q=p}^{c-1}\frac{(-1)^q}{q!}\left[{q+1 \atop p+1}\right].$$

We find

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+c} = \binom{n+c}{c}^{-1}\frac{(-1)^c}{c}\left(1-2^{n+1}\sum_{p=0}^{c-1}n^p\sum_{q=p}^{c-1}\frac{(-1)^q}{q!}\left[{q+1 \atop p+1}\right]\right).$$

With this last result we obtain closed forms for fixed c, e.g. for $c = 5$ it yields

$$\frac{-24+2^{n+1}(n^4+6n^3+23n^2+18n+24)}{(n+5)\times\cdots\times(n+1)}.$$

Addendum. With the purpose of matching conjectures by OP we write

$$\sum_{q=0}^{c-1} \binom{n+q}{q} (-1)^q = \sum_{q=0}^{c-1} \binom{n+q}{q} (-1)^q [z^{c-1}] \frac{z^q}{1-z}$$

$$= [z^{c-1}] \frac{1}{1-z} \sum_{q\ge 0} \binom{n+q}{q} (-1)^q z^q = [z^{c-1}] \frac{1}{1-z} \frac{1}{(1+z)^{n+1}}$$

$$= (-1)^{c-1} [z^{c-1}] \frac{1}{1+z} \frac{1}{(1-z)^{n+1}} = (-1)^{c-1} [z^{c-1}] \frac{1}{1-z^2} \frac{1}{(1-z)^n}.$$

With $c = 2d+1$ where $d \ge 0$ this becomes

$$[z^{2d}] \frac{1}{1-z^2} \frac{1}{(1-z)^n} = \sum_{q=0}^{d} \binom{2q+n-1}{2q}$$

and when $c = 2d$ where $d \ge 1$ it becomes

$$-[z^{2d-1}] \frac{1}{1-z^2} \frac{1}{(1-z)^n} = -\sum_{q=0}^{d-1} \binom{2q+n}{2q+1}.$$

We thus obtain in the first case the closed form

$$\binom{n+2d+1}{2d+1}^{-1} \frac{1}{2d+1} \left(-1 + 2^{n+1} \sum_{q=0}^{d} \binom{2q+n-1}{2q} \right)$$

and in the second case

$$\binom{n+2d}{2d}^{-1} \frac{1}{2d} \left(1 + 2^{n+1} \sum_{q=0}^{d-1} \binom{2q+n}{2q+1} \right).$$

These two confirm the conjectures by OP.
This was math.stackexchange.com problem 2719320.

69.4 MSE 2830860

Starting from (here evidently $n \ge k$ for it to be meaningful).

$$\sum_{j=0}^{n-k} (-1)^j \binom{2k+2j}{j} \binom{n+k+j+1}{n-k-j}$$

$$= (-1)^{n-k} \sum_{j=0}^{n-k} (-1)^j \binom{2n-2j}{n-k-j} \binom{2n-j+1}{j}$$

$$= (-1)^{n-k} \sum_{j=0}^{n-k} (-1)^j \binom{2n-2j}{n-k-j} \binom{2n+1-j}{2n+1-2j}.$$

we write

$$(-1)^{n-k}\sum_{j=0}^{n-k}(-1)^j\binom{2n+1-j}{2n+1-2j}[z^{n-k-j}](1+z)^{2n-2j}$$

$$= (-1)^{n-k}[z^{n-k}](1+z)^{2n}\sum_{j=0}^{n-k}(-1)^j\binom{2n+1-j}{2n+1-2j}z^j(1+z)^{-2j}$$

We get no contribution to the coefficient extractor when $j > n-k$ and hence may continue with

$$(-1)^{n-k}[z^{n-k}](1+z)^{2n}\sum_{j\ge 0}(-1)^j\binom{2n+1-j}{2n+1-2j}z^j(1+z)^{-2j}$$

$$= (-1)^{n-k}[z^{n-k}](1+z)^{2n}\sum_{j\ge 0}(-1)^j z^j(1+z)^{-2j}[w^{2n+1-2j}](1+w)^{2n+1-j}$$

$$= (-1)^{n-k}[z^{n-k}](1+z)^{2n}[w^{2n+1}](1+w)^{2n+1}\sum_{j\ge 0}(-1)^j z^j(1+z)^{-2j}w^{2j}(1+w)^{-j}$$

$$= (-1)^{n-k}[z^{n-k}](1+z)^{2n}[w^{2n+1}](1+w)^{2n+1}\frac{1}{1+zw^2/(1+z)^2/(1+w)}$$

$$= (-1)^{n-k}[z^{n-k}](1+z)^{2n+2}[w^{2n+1}](1+w)^{2n+2}\frac{1}{(1+z)^2(1+w)+zw^2}$$

$$= (-1)^{n-k}[z^{n-k}](1+z)^{2n+2}[w^{2n+1}](1+w)^{2n+2}\frac{1}{(w+1+z)(wz+1+z)}$$

$$= (-1)^{n-k}[z^{n+1-k}](1+z)^{2n+2}[w^{2n+1}](1+w)^{2n+2}\frac{1}{(w+1+z)(w+(1+z)/z)}.$$

Now the inner term is

$$\mathrm{Res}_{w=0}\frac{1}{w^{2n+2}}(1+w)^{2n+2}\frac{1}{(w+1+z)(w+(1+z)/z)}.$$

Residues sum to zero and the residue at infinity is zero since $\lim_{R\to\infty} 2\pi R \times R^{2n+2}/R^{2n+2}/R^2 = 0$. Hence we may compute this from minus the sum of the residues at $-(1+z)$ and $-(1+z)/z$. The first one yields

$$-\frac{1}{(1+z)^{2n+2}}z^{2n+2}\frac{1}{-(1+z)+(1+z)/z}.$$

Replace this in the remaining coefficient extractor to get

$$(-1)^{n+1-k}[z^{n+1-k}]z^{2n+3}\frac{1}{1-z^2} = 0.$$

The second one yields

$$-\frac{z^{2n+2}}{(1+z)^{2n+2}}\frac{1}{z^{2n+2}}\frac{1}{-(1+z)/z+1+z}$$

Once more replace this in the remaining coefficient extractor to get

$$(-1)^{n+1-k}[z^{n+1-k}]\frac{1}{-(1+z)/z+1+z} = (-1)^{n+1-k}[z^{n+1-k}]\frac{z}{z^2-1}$$

$$= -[z^{n+1-k}]\frac{z}{z^2-1} = [z^{n-k}]\frac{1}{1-z^2}.$$

This is

$$[[(n-k) \text{ is even}]] = \frac{1+(-1)^{n-k}}{2}$$

as claimed.

This was math.stackexchange.com problem 2830860.

69.5 MSE 2904333

Starting from

$$\sum_{k=0}^{b-1}\binom{a+k-1}{a-1}p^a(1-p)^k = \sum_{k=a}^{a+b-1}\binom{a+b-1}{k}p^k(1-p)^{a+b-k-1}$$

we simplify to

$$\sum_{k=0}^{b-1}\binom{a+k-1}{a-1}p^a(1-p)^k = \sum_{k=0}^{b-1}\binom{a+b-1}{a+k}p^{a+k}(1-p)^{b-k-1}$$

or

$$\sum_{k=0}^{b-1}\binom{a+k-1}{a-1}(1-p)^k = \sum_{k=0}^{b-1}\binom{a+b-1}{a+k}p^k(1-p)^{b-k-1}.$$

We get for the LHS

$$\sum_{k\ge 0}\binom{a+k-1}{a-1}(1-p)^k[[0\le k\le b-1]]$$

$$= \sum_{k\ge 0}\binom{a+k-1}{a-1}(1-p)^k[z^{b-1}]\frac{z^k}{1-z}$$

$$= [z^{b-1}]\frac{1}{1-z}\sum_{k\ge 0}\binom{a+k-1}{a-1}(1-p)^k z^k$$

$$= [z^{b-1}]\frac{1}{1-z}\frac{1}{(1-(1-p)z)^a}.$$

The RHS is

$$\sum_{k=0}^{b-1} p^k(1-p)^{b-k-1}[z^{b-1-k}]\frac{1}{(1-z)^{a+k+1}}$$

$$= [z^{b-1}]\frac{1}{(1-z)^{a+1}}\sum_{k=0}^{b-1} p^k(1-p)^{b-k-1}\frac{z^k}{(1-z)^k}.$$

There is no contribution to the coefficient extractor in front when $k > b-1$ and may extend k to infinity, getting

$$(1-p)^{b-1}[z^{b-1}]\frac{1}{(1-z)^{a+1}}\sum_{k\ge 0} p^k(1-p)^{-k}\frac{z^k}{(1-z)^k}$$

$$= (1-p)^{b-1}[z^{b-1}]\frac{1}{(1-z)^{a+1}}\frac{1}{1-pz/(1-p)/(1-z)}$$

$$= (1-p)^{b-1}[z^{b-1}]\frac{1}{(1-z)^{a}}\frac{1}{1-z-pz/(1-p)}$$

$$= [z^{b-1}]\frac{1}{(1-(1-p)z)^a}\frac{1}{1-(1-p)z-pz}$$

$$= [z^{b-1}]\frac{1}{1-z}\frac{1}{(1-(1-p)z)^a}.$$

The LHS and the RHS are seen to be the same and we may conclude.

Remark. The first one is the easy one and follows by inspection. The Iverson bracket may be of interest here as an example of the method.

This was math.stackexchange.com problem 2904333.

69.6 MSE 2950043

Starting from

$$(-1)^{n+k}\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=0}^{n-k}(-1)^j\binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j}\left\{ n-k+j \atop j \right\}$$

we introduce the EGF for Stirling numbers of the second kind on the RHS, getting

$$\sum_{j=0}^{n-k}(-1)^j\binom{n-1+j}{n-k+j}\binom{2n-k}{n-k-j}(n-k+j)![z^{n-k+j}]\frac{(\exp(z)-1)^j}{j!}$$

$$= (n-k)![z^{n-k}] \sum_{j=0}^{n-k} (-1)^j \binom{n-1+j}{n-k+j} \binom{2n-k}{n-k-j} \binom{n-k+j}{j} \frac{(\exp(z)-1)^j}{z^j}.$$

Now

$$\binom{n-1+j}{n-k+j}\binom{n-k+j}{j} = \frac{(n-1+j)!}{(k-1)! \times j! \times (n-k)!} = \binom{n-1}{k-1}\binom{n-1+j}{n-1}$$

and we find

$$\frac{(n-1)!}{(k-1)!}[z^{n-k}] \sum_{j=0}^{n-k} (-1)^j \binom{n-1+j}{n-1}\binom{2n-k}{n-k-j} \frac{(\exp(z)-1)^j}{z^j}$$

$$= \frac{(n-1)!}{(k-1)!}[z^{n-k}] \sum_{j=0}^{n-k} (-1)^j \binom{n-1+j}{n-1} \frac{(\exp(z)-1)^j}{z^j} [w^{n-k-j}](1+w)^{2n-k}$$

$$= \frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}] \sum_{j=0}^{n-k} (-1)^j \binom{n-1+j}{n-1} \frac{(\exp(z)-1)^j}{z^j} w^j.$$

Note that there is no contribution to the coefficient extractor $[w^{n-k}]$ when $j > n-k$, so we may write

$$\frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}] \sum_{j\ge 0} (-1)^j \binom{n-1+j}{n-1} \frac{(\exp(z)-1)^j}{z^j} w^j$$

$$= \frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}] \frac{1}{(1+w(\exp(z)-1)/z)^n}$$

$$= \frac{(n-1)!}{(k-1)!}[w^{n-k}](1+w)^{2n-k}[z^{n-k}] \frac{z^n/(\exp(z)-1)^n}{(w+z/(\exp(z)-1))^n}.$$

Working with

$$\mathrm{Res}_{w=0} \frac{1}{w^{n-k+1}}(1+w)^{2n-k}\frac{1}{(w-C)^n}$$

we compute the residues at C and at infinity in order to apply the fact that they must sum to zero. Starting with the first we require (Leibniz rule)

$$\frac{1}{(n-1)!}\left(\frac{1}{w^{n-k+1}}(1+w)^{2n-k}\right)^{(n-1)}$$

$$= \frac{1}{(n-1)!}\sum_{q=0}^{n-1}\binom{n-1}{q}\frac{(n-k+q)!}{(n-k)!}(-1)^q\frac{1}{w^{n-k+1+q}}$$

$$\times \frac{(2n-k)!}{(2n-k-(n-1-q))!}(1+w)^{2n-k-(n-1-q)}$$

$$= \sum_{q=0}^{n-1} \binom{n-k+q}{q} (-1)^q \frac{1}{w^{n-k+1+q}} \binom{2n-k}{n-1-q} (1+w)^{n-k+1+q}$$

$$= \left(\frac{1+w}{w}\right)^{n-k+1} \sum_{q=0}^{n-1} \binom{n-k+q}{q} (-1)^q \binom{2n-k}{n-1-q} \left(\frac{1+w}{w}\right)^q.$$

We have two important observations, the first is that

$$\frac{z^n}{(\exp(z)-1)^n} = 1 + \cdots$$

i.e. no pole at zero and that

$$\left.\frac{1+w}{w}\right|_{w=-z/(\exp(z)-1)} = \frac{1+z-\exp(z)}{z} = -\frac{1}{2}z + \cdots.$$

Hence on substituting into the coefficient extractor on $[z^{n-k}]$ we get for all sum terms

$$[z^{n-k}](1+\cdots) \left(-\frac{1}{2}z + \cdots\right)^{n-k+1} \times \left(-\frac{1}{2}z + \cdots\right)^q = 0,$$

i.e. due to the middle term there is zero contribution from the residue at $w = -z/(\exp(z)-1)$. Returning to the main computation we get for the residue at infinity

$$\mathrm{Res}_{w=\infty} \frac{1}{w^{n-k+1}} (1+w)^{2n-k} \frac{1}{(w-C)^n}$$

$$= -\mathrm{Res}_{w=0} \frac{1}{w^2} w^{n-k+1} (1+1/w)^{2n-k} \frac{1}{(1/w-C)^n}$$

$$= -\mathrm{Res}_{w=0} \frac{1}{w^2} w^{2n-k+1} \frac{(1+w)^{2n-k}}{w^{2n-k}} \frac{1}{(1-Cw)^n}$$

$$= -\mathrm{Res}_{w=0} \frac{1}{w} (1+w)^{2n-k} \frac{1}{(1-Cw)^n} = -1.$$

On flipping the sign and substituting into the coefficient extractor on z we get

$$\frac{(n-1)!}{(k-1)!} [z^{n-k}] \frac{z^n}{(\exp(z)-1)^n}$$

$$= \frac{(n-1)!}{(k-1)!} \mathrm{Res}_{z=0} \frac{1}{z^{n-k+1}} \frac{z^n}{(\exp(z)-1)^n}.$$

Summing we get for the OGF

$$\sum_{k=1}^{n} x^k \frac{(n-1)!}{(k-1)!} \mathrm{Res}_{z=0} \frac{z^{k-1}}{(\exp(z)-1)^n}$$

$$= x(n-1)! \times \mathrm{Res}_{z=0} \frac{1}{(\exp(z)-1)^n} \sum_{k=1}^{n} \frac{x^{k-1} z^{k-1}}{(k-1)!}$$

$$= x(n-1)! \times \mathrm{Res}_{z=0} \frac{\exp(xz)}{(\exp(z)-1)^n}.$$

Now we evaluate the residue for $1 \le x \le n$ an integer. We have

$$\mathrm{Res}_{z=0} \frac{\exp(xz)}{(\exp(z)-1)^n} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{\exp(xz)}{(\exp(z)-1)^n} \; dz$$

$$= \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{\exp((x-1)z)}{(\exp(z)-1)^n} \exp(z) \; dz$$

and putting $\exp(z) = w$ so that $\exp(z)\; dz = dw$ we obtain

$$\frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{w^{x-1}}{(w-1)^n} \; dw$$

$$= \frac{1}{2\pi i} \int_{|w-1|=\gamma} \frac{1}{(w-1)^n} \sum_{q=0}^{x-1} \binom{x-1}{q} (w-1)^q \; dw.$$

This is zero when $x-1 < n-1$ or $x < n$ and it is one when $x = n$. By construction the residue is a polynomial in x of degree $n-1$. We have the $n-1$ roots, they are at $x = 1, 2, \ldots, n-1$ so we know it is

$$Q(x-1)(x-2) \times \cdots \times (x-(n-1)).$$

But we also know that at $x = n$ it evaluates to one, so we must have

$$Q(n-1)(n-2) \times \cdots \times 1 = 1$$

or $Q = 1/(n-1)!$. Restoring the two terms in front we finally obtain

$$x(n-1)! \times \frac{1}{(n-1)!} (x-1)(x-2) \times \cdots \times (x-(n-1))$$

$$= x(x-1)(x-2) \times \cdots \times (x-(n-1)) = \sum_{k=1}^{n} (-1)^{n+k} \left[{n \atop k} \right] x^k$$

which is precisely the Stirling number OGF, first kind, and we are done.
This was math.stackexchange.com problem 2950043.

69.7 MSE 3049572

Starting from the claim

$$\binom{m+n}{s+1} - \binom{n}{s+1} = \sum_{q=0}^{s} \frac{m}{q+1} \binom{m+1+2q}{q} \binom{n-2-2q}{s-q}$$

we observe that

$$\begin{aligned} & \binom{m+1+2q}{q+1} - \binom{m+1+2q}{q} \\ &= \frac{m+1+q}{q+1} \binom{m+1+2q}{q} - \binom{m+1+2q}{q} \\ &= \frac{m}{q+1} \binom{m+1+2q}{q}. \end{aligned}$$

Therefore we have two sums,

$$\sum_{q=0}^{s} \binom{m+1+2q}{q+1} \binom{n-2-2q}{s-q} - \sum_{q=0}^{s} \binom{m+1+2q}{q} \binom{n-2-2q}{s-q}.$$

For the first one we write

$$\sum_{q=0}^{s} [w^{q+1}](1+w)^{m+1+2q} [z^{s-q}](1+z)^{n-2-2q}$$

$$= \mathrm{Res}_{w=0} (1+w)^{m+1} [z^s](1+z)^{n-2} \sum_{q=0}^{s} \frac{1}{w^{q+2}} z^q (1+w)^{2q} (1+z)^{-2q}.$$

We may extend q beyond s because of the coefficient extractor $[z^s]$ in front, getting

$$\mathrm{Res}_{w=0} \frac{1}{w^2} (1+w)^{m+1} [z^s](1+z)^{n-2} \sum_{q\ge 0} z^q w^{-q} (1+w)^{2q} (1+z)^{-2q}$$

$$= \mathrm{Res}_{w=0} (1+w)^{m+1} [z^s](1+z)^{n-2} \frac{1}{w^2} \frac{1}{1 - z(1+w)^2/w/(1+z)^2}$$

$$= \mathrm{Res}_{w=0} (1+w)^{m+1} [z^s](1+z)^{n} \frac{1}{w} \frac{1}{w(1+z)^2 - z(1+w)^2}.$$

Repeat the calculation for the second one to get

$$\mathrm{Res}_{w=0} (1+w)^{m+1} [z^s](1+z)^{n} \frac{1}{w(1+z)^2 - z(1+w)^2}.$$

Now we have

$$\left(\frac{1}{w}-1\right)\frac{1}{w(1+z)^2-z(1+w)^2} = \frac{1}{w-z}\frac{1}{w(1+w)} - \frac{1}{1-wz}\frac{1}{1+w}$$
$$= \frac{1}{1-z/w}\frac{1}{w^2(1+w)} - \frac{1}{1-wz}\frac{1}{1+w}.$$

We thus obtain two components, the first is

$$\mathrm{Res}_{w=0}(1+w)^{m+1}[z^s](1+z)^n\frac{1}{1-z/w}\frac{1}{w^2(1+w)}$$
$$= \mathrm{Res}_{w=0}\frac{1}{w^2}(1+w)^m[z^s](1+z)^n\frac{1}{1-z/w}$$
$$= \mathrm{Res}_{w=0}\frac{1}{w^2}(1+w)^m\sum_{q=0}^{s}\binom{n}{q}\frac{1}{w^{s-q}} = \sum_{q=0}^{s}\binom{n}{q}\mathrm{Res}_{w=0}\frac{1}{w^{s-q+2}}(1+w)^m$$
$$= \sum_{q=0}^{s}\binom{n}{q}[w^{s-q+1}](1+w)^m = [w^{s+1}](1+w)^m\sum_{q=0}^{s}\binom{n}{q}w^q$$
$$= -\binom{n}{s+1} + [w^{s+1}](1+w)^m\sum_{q=0}^{s+1}\binom{n}{q}w^q.$$

We may extend q beyond $s+1$ due to the coefficient extractor in front, to get

$$-\binom{n}{s+1} + [w^{s+1}](1+w)^m\sum_{q\ge 0}\binom{n}{q}w^q = -\binom{n}{s+1} + [w^{s+1}](1+w)^{m+n}$$

This is

$$\binom{m+n}{s+1} - \binom{n}{s+1}.$$

We have the claim, so we just need to prove that the second component will produce zero. We obtain

$$\mathrm{Res}_{w=0}(1+w)^{m+1}[z^s](1+z)^n\frac{1}{1-wz}\frac{1}{1+w}$$
$$= \mathrm{Res}_{w=0}(1+w)^m[z^s](1+z)^n\frac{1}{1-wz}$$
$$= \mathrm{Res}_{w=0}(1+w)^m\sum_{q=0}^{s}\binom{n}{q}w^{s-q} = \sum_{q=0}^{s}\binom{n}{q}\mathrm{Res}_{w=0}w^{s-q}(1+w)^m = 0.$$

This concludes the argument. Having reached the end of the computation we observe that we did not require the full mechanics of the complex residue and a coefficient extractor would have sufficed.

This was math.stackexchange.com problem 3049572.

69.8 MSE 3051713

We seek to evaluate

$$\sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k-1)!!}{(k-q)!} (-1)^k.$$

or alternatively

$$\sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k-1)!}{(k-1)! \times 2^{k-1}} \frac{1}{(k-q)!} (-1)^k$$

$$= \sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k)!}{k! \times 2^k} \frac{1}{(k-q)!} (-1)^k.$$

This is

$$q! \sum_{k=q}^{2n} \binom{2n+k}{2k} \frac{(2k)!}{k! \times k! \times 2^k} \frac{k!}{q! \times (k-q)!} \frac{(-1)^k}{2^k}$$

$$= q! \sum_{k=q}^{2n} \binom{2n+k}{2k} \binom{2k}{k} \binom{k}{q} \frac{(-1)^k}{2^k}.$$

Observe that

$$\binom{2n+k}{2k} \binom{2k}{k} = \frac{(2n+k)!}{(2n-k)! \times k! \times k!} = \binom{2n+k}{2n} \binom{2n}{k}$$

and furthermore

$$\binom{2n}{k} \binom{k}{q} = \frac{(2n)!}{(2n-k)! \times q! \times (k-q)!} = \binom{2n}{q} \binom{2n-q}{k-q}.$$

We get for the sum

$$\binom{2n}{q} q! \sum_{k=q}^{2n} \binom{2n+k}{2n} \binom{2n-q}{k-q} \frac{(-1)^k}{2^k}$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} \sum_{k=0}^{2n-q} \binom{2n+q+k}{2n} \binom{2n-q}{k} \frac{(-1)^k}{2^k}.$$

This becomes

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} \sum_{k=0}^{2n-q} \binom{2n+q+k}{2n} [z^{2n-q-k}](1+z)^{2n-q} \frac{(-1)^k}{2^k}$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} \sum_{k=0}^{2n-q} \binom{2n+q+k}{2n} \frac{(-1)^k}{2^k} z^k.$$

Now we may extend k beyond $2n-q$ because of the coefficient extractor $[z^{2n-q}]$ (no contribution) and get

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} \sum_{k\ge 0} \binom{2n+q+k}{2n} \frac{(-1)^k}{2^k} z^k$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} [w^{2n}](1+w)^{2n+q} \sum_{k\ge 0} (1+w)^k \frac{(-1)^k}{2^k} z^k$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^q} [z^{2n-q}](1+z)^{2n-q} [w^{2n}](1+w)^{2n+q} \frac{1}{1+z(1+w)/2}.$$

Re-write this as

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} [w^{2n}](1+w)^{2n+q} \mathrm{Res}_{z=0} \frac{1}{z^{2n-q+1}} (1+z)^{2n-q} \frac{1}{1+z(1+w)/2}.$$

Working with the residue we apply the substitution $z/(1+z) = v$ or $z = v/(1-v)$ to get

$$\mathrm{Res}_{v=0} \frac{1}{v^{2n-q}} \frac{1-v}{v} \frac{1}{1+(v/(1-v))(1+w)/2} \frac{1}{(1-v)^2}$$

$$= \mathrm{Res}_{v=0} \frac{1}{v^{2n-q+1}} \frac{1}{1-v+v(1+w)/2}$$

$$= \mathrm{Res}_{v=0} \frac{1}{v^{2n-q+1}} \frac{1}{1-v(1-w)/2} = \frac{1}{2^{2n-q}} (1-w)^{2n-q}.$$

Substitute into the remaining coefficient extractor to get

$$\binom{2n}{q} q! \frac{(-1)^q}{2^q} [w^{2n}](1+w)^{2n+q} \frac{1}{2^{2n-q}} (1-w)^{2n-q}$$

$$= \binom{2n}{q} q! \frac{(-1)^q}{2^{2n}} \sum_{p=0}^{2n-q} (-1)^p \binom{2n-q}{p} \binom{2n+q}{2n-p}.$$

Now

$$\binom{2n}{q}\binom{2n-q}{p} = \frac{(2n)!}{q! \times p! \times (2n-q-p)!} = \binom{2n}{p}\binom{2n-p}{q}$$

and

$$\binom{2n-p}{q}\binom{2n+q}{2n-p} = \frac{(2n+q)!}{q! \times (2n-p-q)! \times (p+q)!} = \binom{2n+q}{q}\binom{2n}{p+q}.$$

This yields

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} \sum_{p=0}^{2n-q} (-1)^p \binom{2n}{p} \binom{2n}{p+q}$$

$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} \sum_{p=0}^{2n-q} (-1)^p \binom{2n}{p} [z^{2n-p-q}](1+z)^{2n}$$

$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1+z)^{2n} \sum_{p=0}^{2n-q} (-1)^p \binom{2n}{p} z^p.$$

Now we may extend p beyond $2n-q$ because of the coeffcient extractor $[z^{2n-q}]$ in front. We find

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1+z)^{2n} \sum_{p\ge 0} (-1)^p \binom{2n}{p} z^p$$

$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1+z)^{2n}(1-z)^{2n}$$

$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2n-q}](1-z^2)^{2n}.$$

Concluding we immediately obtain zero when q is odd, and otherwise we find

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{2(n-q/2)}](1-z^2)^{2n}$$

$$= \binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} [z^{n-q/2}](1-z)^{2n}.$$

This is

$$\binom{2n+q}{q} q! \frac{(-1)^q}{2^{2n}} (-1)^{n-q/2} \binom{2n}{n-q/2}$$

or alternatively

$$\frac{(-1)^{n+q/2}}{2^{2n}} \frac{(2n+q)!}{(n-q/2)! \times (n+q/2)!}.$$

This was math.stackexchange.com problem 3051713.

69.9 MSE 3068381

We seek to show that

$$S_n = \sum_{j=n}^{2n} \sum_{k=j+1-n}^{j} (-1)^j 2^{j-k} \binom{2n}{j} \left\{ {j \atop k} \right\} \left[{k \atop j+1-n} \right] = 0.$$

With the usual EGFs we get

$$\sum_{j=n}^{2n} \sum_{k=j+1-n}^{j} (-1)^j 2^{j-k} \binom{2n}{j} j! [z^j] \frac{(\exp(z)-1)^k}{k!}$$
$$\times k! [w^k] \frac{1}{(j+1-n)!} \left(\log \frac{1}{1-w} \right)^{j+1-n}.$$

Now we have

$$\binom{2n}{j} j! \frac{1}{(j+1-n)!} = \frac{(2n)!}{(2n-j)! \times (j+1-n)!} = \frac{(2n)!}{(n+1)!} \binom{n+1}{j+1-n}.$$

This yields for the sum

$$\frac{(2n)!}{(n+1)!} \sum_{j=n}^{2n} \binom{n+1}{j+1-n} (-1)^j 2^j$$
$$\times [z^j] \sum_{k=j+1-n}^{j} 2^{-k} (\exp(z)-1)^k [w^k] \left(\log \frac{1}{1-w} \right)^{j+1-n}$$
$$= \frac{(2n)!}{(n+1)!} (-1)^n 2^n \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j 2^j$$
$$\times [z^{n+j}] \sum_{k=j+1}^{j+n} 2^{-k} (\exp(z)-1)^k [w^k] \left(\log \frac{1}{1-w} \right)^{j+1}.$$

Observe that $(\exp(z)-1)^k = z^k + \cdots$ and hence we may extend the inner sum beyond $j+n$ due to the coefficient extractor $[z^{n+j}]$. We find

$$\frac{(2n)!}{(n+1)!} (-1)^n 2^n \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j 2^j [z^{n+j}]$$
$$\times \sum_{k \ge j+1} 2^{-k} (\exp(z)-1)^k [w^k] \left(\log \frac{1}{1-w} \right)^{j+1}.$$

Furthermore note that $\left(\log \frac{1}{1-w} \right)^{j+1} = w^{j+1} + \cdots$ so that the coefficient extractor $[w^k]$ covers the entire series, producing

$$\frac{(2n)!}{(n+1)!}(-1)^n 2^n \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j 2^j [z^{n+j}] \left(\log \frac{1}{1-(\exp(z)-1)/2}\right)^{j+1}.$$

Working with formal power series we are justified in writing

$$[z^{n+j}] \left(\log \frac{1}{1-(\exp(z)-1)/2}\right)^{j+1} = [z^{n-1}] \frac{1}{z^{j+1}} \left(\log \frac{1}{1-(\exp(z)-1)/2}\right)^{j+1}$$

because the logarithmic term starts at $z^{j+1}/2^{j+1}$. To see this write

$$\frac{\exp(z)-1}{2} + \frac{1}{2}\frac{(\exp(z)-1)^2}{2^2} + \frac{1}{3}\frac{(\exp(z)-1)^3}{2^3} + \cdots$$

We continue

$$\frac{(2n)!}{(n+1)!}(-1)^{n-1} 2^{n-1}$$
$$\times [z^{n-1}] \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^{j+1} 2^{j+1} \frac{1}{z^{j+1}} \left(\log \frac{1}{1-(\exp(z)-1)/2}\right)^{j+1}$$
$$= \frac{(2n)!}{(n+1)!}(-1)^{n-1} 2^{n-1}$$
$$\times [z^{n-1}] \sum_{j=1}^{n+1} \binom{n+1}{j} (-1)^{j} 2^{j} \frac{1}{z^{j}} \left(\log \frac{1}{1-(\exp(z)-1)/2}\right)^{j}.$$

The term for $j = 0$ in the sum is one and hence only contributes to $n = 1$ so that we may write

$$-[[n=1]] + \frac{(2n)!}{(n+1)!}(-1)^{n-1} 2^{n-1}$$
$$\times [z^{n-1}] \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^{j} 2^{j} \frac{1}{z^{j}} \left(\log \frac{1}{1-(\exp(z)-1)/2}\right)^{j}$$
$$= -[[n=1]] + \frac{(2n)!}{(n+1)!}(-1)^{n-1} 2^{n-1}$$
$$\times [z^{n-1}] \left(1 - \frac{2}{z} \log \frac{1}{1-(\exp(z)-1)/2}\right)^{n+1}.$$

Finally observe that

$$\left(1 - \frac{2}{z} \log \frac{1}{1-(\exp(z)-1)/2}\right)^{n+1}$$

$$= \left(1 - \frac{2}{z}\left(\frac{\exp(z)-1}{2} + \frac{1}{2}\frac{(\exp(z)-1)^2}{2^2} + \frac{1}{3}\frac{(\exp(z)-1)^3}{2^3} + \cdots\right)\right)^{n+1}$$

$$= \left(-\frac{3}{4}z - \cdots\right)^{n+1}$$

and furthermore

$$[z^{n-1}]\left((-1)^{n+1}\frac{3^{n+1}}{4^{n+1}}z^{n+1} + \cdots\right) = 0$$

which is the claim.
This was math.stackexchange.com problem 3068381.

69.10 MSE 3138710

We seek to prove that with $n \ge m+2$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{n-j} \binom{n-j}{j} = \binom{n+k+m}{m+1}.$$

This is

$$\binom{m+k}{m+1} + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{n-j} \binom{n-j}{j} = \binom{n+k+m}{m+1}$$

or

$$\binom{m+k}{m+1} + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \frac{n}{j} \binom{n-j-1}{j-1} = \binom{n+k+m}{m+1}$$

Now observe that

$$\binom{n-j-1}{j} = \frac{n-2j}{j}\binom{n-j-1}{j-1} = \frac{n}{j}\binom{n-j-1}{j-1} - 2\binom{n-j-1}{j-1}.$$

We thus get two terms:

$$\binom{m+k}{m+1} + \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{j} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{j}$$

and

$$2\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{j-1}.$$

For the first one we have

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} \binom{n-j-1}{n-2j-1}$$

$$= [z^{n-1}](1+z)^{n-1} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m+j+k}{m-j+1} (1+z)^{-j} z^{2j}.$$

We may extend j to infinity because of the coefficient extractor in front (note that the following representation in the variable w will produce a correct value of zero in the remaining binomial coefficient when $j > m+1$):

$$[z^{n-1}](1+z)^{n-1}[w^{m+1}](1+w)^{m+k} \sum_{j \ge 0} (1+z)^{-j} z^{2j} (1+w)^j w^j$$

$$= [z^{n-1}](1+z)^{n-1}[w^{m+1}](1+w)^{m+k} \frac{1}{1 - z^2 w(1+w)/(1+z)}$$

$$= [z^{n-1}](1+z)^{n}[w^{m+1}](1+w)^{m+k} \frac{1}{1 + z - z^2 w(1+w)}$$

$$= -[z^{n-1}](1+z)^{n}[w^{m+2}](1+w)^{m+k-1} \frac{1}{(z-1/w)(z+1/(1+w))}.$$

Extracting $[z^{n-1}]$ first we get

$$\mathrm{Res}_{z=0} \frac{1}{z^n} (1+z)^n \frac{1}{(z-1/w)(z+1/(1+w))}.$$

We see that the residue at infinity is zero. Residues sum to zero and we get for the residue at $z = 1/w$

$$w^n \frac{(1+w)^n}{w^n} \frac{1}{1/w + 1/(1+w)} = w \frac{(1+w)^{n+1}}{1+2w}.$$

For the residue at $z = -1/(1+w)$ we find

$$-(-1)^n (1+w)^n \frac{w^n}{(1+w)^n} \frac{1}{1/(1+w) + 1/w} = -(-1)^n w^{n+1} (1+w) \frac{1}{1+2w}.$$

Now the coefficient extractor is $[w^{m+2}]$ but we have $n \ge m+2$ so the contribution from this is zero.

It follows that the first sum is given by

$$[w^{m+1}] \frac{(1+w)^{n+k+m}}{1+2w}.$$

Continuing with the second sum we find

$$2\sum_{j=1}^{\lfloor n/2\rfloor}\binom{m+j+k}{m-j+1}\binom{n-j-1}{n-2j}$$

$$= 2[z^n](1+z)^{n-1}\sum_{j=1}^{\lfloor n/2\rfloor}\binom{m+j+k}{m-j+1}(1+z)^{-j}z^{2j}.$$

We may include $j=0$ here because

$$2[z^n](1+z)^{n-1}\binom{m+k}{m+1} = 0,$$

getting

$$2[z^n](1+z)^{n-1}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{m+j+k}{m-j+1}(1+z)^{-j}z^{2j}.$$

We skip forward to the residue computation since the intermediate steps are the same as before. We get for the residue at $z=1/w$

$$2w^{n+1}\frac{(1+w)^n}{w^n}\frac{1}{1/w+1/(1+w)} = 2w^2\frac{(1+w)^{n+1}}{1+2w}.$$

For the residue at $z=-1/(1+w)$ we find

$$-(-1)^{n+1}(1+w)^{n+1}\frac{w^n}{(1+w)^n}\frac{1}{1/(1+w)+1/w} = (-1)^n w^{n+1}(1+w)^2\frac{1}{1+2w}.$$

We note once more that the coefficient extractor is $[w^{m+2}]$ but we have $n \ge m+2$ so the contribution from this is zero. It follows that the second sum is given by

$$[w^{m+1}]2w\frac{(1+w)^{n+k+m}}{1+2w}.$$

Adding the two sums we obtain at last

$$[w^{m+1}]\frac{(1+w)^{n+k+m}}{1+2w} + [w^{m+1}]2w\frac{(1+w)^{n+k+m}}{1+2w} = [w^{m+1}](1+w)^{n+k+m}.$$

or

$$\binom{n+k+m}{m+1}.$$

This was math.stackexchange.com problem 3138710.

69.11 MSE 3196998

As a preliminary, observe that the generating function of the Fibonacci numbers is

$$\frac{z}{1-z-z^2}.$$

so that we have $F_0 = 0$ and $F_1 = F_2 = 1$.

We seek to evaluate

$$\sum_{p=0}^{n}\sum_{q=0}^{n}\binom{n-p}{q}\binom{n-q}{p}$$
$$= \sum_{p=0}^{n}\sum_{q=0}^{n}\binom{n-p}{n-p-q}\binom{n-q}{n-p-q}.$$

Note that on the first line the binomial coefficient $\binom{n}{k} = n^{\underline{k}}/k!$ starts producing non-zero values when $p > n$ and $q > n$. This is not desired here, hence the upper limits. On the second line we use the convention that $\binom{n}{k} = 0$ when $k < 0$, which is also the behavior when residues are used. Continuing we find

$$\sum_{p=0}^{n}\sum_{q=0}^{n}[z^{n-p-q}](1+z)^{n-p}[w^{n-p-q}](1+w)^{n-q}$$
$$= [z^n](1+z)^n[w^n](1+w)^n\sum_{p=0}^{n}\sum_{q=0}^{n} z^{p+q}(1+z)^{-p}w^{p+q}(1+w)^{-q}$$
$$= [z^n](1+z)^n[w^n](1+w)^n\sum_{p=0}^{n} z^p w^p(1+z)^{-p}\sum_{q=0}^{n} z^q w^q(1+w)^{-q}.$$

Here the coefficient extractor controls the range and we may continue with

$$[z^n](1+z)^n[w^n](1+w)^n\sum_{p\ge 0} z^p w^p(1+z)^{-p}\sum_{q\ge 0} z^q w^q(1+w)^{-q}$$
$$= [z^n](1+z)^n[w^n](1+w)^n\frac{1}{1-zw/(1+z)}\frac{1}{1-zw/(1+w)}$$
$$= [z^n](1+z)^{n+1}[w^n](1+w)^{n+1}\frac{1}{1+z-zw}\frac{1}{1+w-zw}.$$

Now we have

$$\frac{1}{1+z-zw}\frac{1}{1+w-zw}$$
$$= \frac{1-w}{1+z-wz}\frac{1}{1+w-w^2} + \frac{w}{1+w-wz}\frac{1}{1+w-w^2}.$$

We get from the first piece treating z first

$$[z^n](1+z)^{n+1}\frac{1-w}{1+z-wz} = [z^n](1+z)^{n+1}\frac{1-w}{1-z(w-1)}$$

$$= (1-w)\sum_{p=0}^{n}\binom{n+1}{n-p}(w-1)^p = -\sum_{p=0}^{n}\binom{n+1}{p+1}(w-1)^{p+1}$$

$$= 1 - \sum_{p=-1}^{n}\binom{n+1}{p+1}(w-1)^{p+1} = 1 - w^{n+1}.$$

The contribution is

$$[w^n](1+w)^{n+1}\frac{1-w^{n+1}}{1+w-w^2} = [w^n](1+w)^{n+1}\frac{1}{1+w-w^2}.$$

The second piece yields

$$[z^n](1+z)^{n+1}\frac{w}{1+w-wz} = \frac{1}{1+w}[z^n](1+z)^{n+1}\frac{w}{1-wz/(1+w)}$$

$$= \frac{w}{1+w}\sum_{p=0}^{n}\binom{n+1}{n-p}\frac{w^p}{(1+w)^p} = \sum_{p=0}^{n}\binom{n+1}{p+1}\frac{w^{p+1}}{(1+w)^{p+1}}$$

$$= -1 + \sum_{p=-1}^{n}\binom{n+1}{p+1}\frac{w^{p+1}}{(1+w)^{p+1}} = -1 + \left(1+\frac{w}{1+w}\right)^{n+1}$$

$$= -1 + \frac{(1+2w)^{n+1}}{(1+w)^{n+1}}.$$

The contribution is

$$[w^n](1+w)^{n+1}\left(-1+\frac{(1+2w)^{n+1}}{(1+w)^{n+1}}\right)\frac{1}{1+w-w^2}.$$

Adding the first and the second contribution we find

$$[w^n](1+2w)^{n+1}\frac{1}{1+w-w^2}$$

$$= \mathrm{Res}_{w=0}\frac{1}{w^{n+1}}(1+2w)^{n+1}\frac{1}{1+w-w^2}.$$

Setting $w/(1+2w) = v$ or $w = v/(1-2v)$ so that $dw = 1/(1-2v)^2\ dv$ we obtain

$$\mathrm{Res}_{v=0}\frac{1}{v^{n+1}}\frac{1}{1+v/(1-2v)-v^2/(1-2v)^2}\frac{1}{(1-2v)^2}$$

$$= \mathrm{Res}_{v=0}\frac{1}{v^{n+1}}\frac{1}{(1-2v)^2+v(1-2v)-v^2}$$

$$= \mathrm{Res}_{v=0} \frac{1}{v^{n+1}} \frac{1}{1-3v+v^2}.$$

We have our answer:

$$[v^n] \frac{1}{1-3v+v^2} = F_{2n+2}.$$

It remains to prove that the coefficient extractor returns the Fibonacci number as claimed. The OGF of even-index Fibonacci numbers is

$$\sum_{n\ge 0} F_{2n} z^{2n} = \frac{1}{2} \frac{z}{1-z-z^2} + \frac{1}{2} \frac{(-z)}{1+z-z^2} = \frac{z^2}{1-3z^2+z^4}.$$

This implies that

$$\sum_{n\ge 0} F_{2n} z^n = \frac{z}{1-3z+z^2}.$$

Therefore

$$F_{2n+2} = [z^{n+1}] \frac{z}{1-3z+z^2} = [z^n] \frac{1}{1-3z+z^2}$$

as required.

This was math.stackexchange.com problem 3196998.

69.12 MSE 3245099

Starting from the claim that $S = 1$ where

$$S = \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^q b^K + a^K b^q}{(a+b)^{q+K}}$$

we get two pieces

$$\frac{b^K}{(a+b)^K} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^q}{(a+b)^q}$$
$$+ \frac{a^K}{(a+b)^K} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{b^q}{(a+b)^q}.$$

This is

$$\frac{b^K}{(a+b)^K} [z^{K-1}] \frac{1}{1-z} \frac{1}{(1-az/(a+b))^K}$$
$$+ \frac{a^K}{(a+b)^K} [z^{K-1}] \frac{1}{1-z} \frac{1}{(1-bz/(a+b))^K}.$$

Call these S_1 and S_2. The first sum is

$$\begin{aligned}
S_1 &= \frac{b^K}{(a+b)^K} \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{1-z} \frac{1}{(1-az/(a+b))^K} \\
&= b^K \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{1-z} \frac{1}{(a+b-az)^K} \\
&= \frac{b^K}{a^K} \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{1-z} \frac{1}{((a+b)/a-z)^K} \\
&= (-1)^{K+1} \frac{b^K}{a^K} \mathrm{Res}_{z=0} \frac{1}{z^K} \frac{1}{z-1} \frac{1}{(z-(a+b)/a)^K}.
\end{aligned}$$

Now residues sum to zero so we compute this from the residues at the poles at $z = 1$ and $z = (a+b)/a$. The residue at infinity is zero by inspection. The residue at $z = 1$ is

$$\begin{aligned}
(-1)^{K+1} \frac{b^K}{a^K} \frac{1}{(1-(a+b)/a)^K} &= (-1)^{K+1} b^K \frac{1}{(a-(a+b))^K} \\
&= (-1)^{K+1} b^K \frac{1}{(-b)^K} = -1.
\end{aligned}$$

For the residue at $z = (a+b)/a$ we require

$$\frac{1}{(K-1)!} \left(\frac{1}{z^K} \frac{1}{z-1} \right)^{(K-1)}$$

$$= \frac{1}{(K-1)!} \sum_{q=0}^{K-1} \binom{K-1}{q} (-1)^q \frac{(K-1+q)!}{(K-1)!} \frac{1}{z^{K+q}} (-1)^{K-1-q} \frac{(K-1-q)!}{(z-1)^{K-q}}$$

$$= (-1)^{K+1} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{1}{z^{K+q}} \frac{1}{(z-1)^{K-q}}.$$

Evaluating the residue we find

$$\begin{aligned}
&(-1)^{K+1} \frac{b^K}{a^K} (-1)^{K+1} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{1}{z^{K+q}} \frac{1}{(z-1)^{K-q}} \Bigg|_{z=(a+b)/a} \\
&= \frac{b^K}{a^K} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^{K+q}}{(a+b)^{K+q}} \frac{1}{((a+b)/a-1)^{K-q}} \\
&= \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^{K+q}}{(a+b)^{K+q}} \frac{b^q}{a^q} \frac{b^{K-q}}{a^{K-q}} \frac{1}{((a+b)/a-1)^{K-q}} \\
&= \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^{K+q}}{(a+b)^{K+q}} \frac{b^q}{a^q}
\end{aligned}$$

$$= \frac{a^K}{(a+b)^K} \sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{b^q}{(a+b)^q} = S_2.$$

We recognise S_2 and hence we have shown that

$$S_1 - 1 + S_2 = 0$$

or

$$\sum_{q=0}^{K-1} \binom{K-1+q}{K-1} \frac{a^q b^K + a^K b^q}{(a+b)^{q+K}} = 1$$

as claimed.

This was math.stackexchange.com problem 3245099.

Remark. This is the formal power series version of the identity by Gosper in section 38.

69.13 MSE 3260307

Starting from the claim (we treat the case r a positive integer)

$$\binom{r+2n-1}{n-1} - \binom{2n-1}{n-1} = S = \sum_{k=1}^{n-1} \binom{2k-1}{k} \binom{r+2(n-k)-1}{r+n-k} \frac{r}{n-k}$$

$$= \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{r+k} \frac{r}{k}$$

$$= \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{k-1} \frac{r}{k}$$

we use the fact that

$$\binom{r+2k-1}{k-1} \frac{r}{k} = \binom{r+2k-1}{k} - \binom{r+2k-1}{k-1}$$

to get two pieces, call them S_1 and S_2 where $S = S_1 - S_2$ and

$$S_1 = \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{k}$$

and

$$S_2 = \sum_{k=1}^{n-1} \binom{2n-2k-1}{n-k} \binom{r+2k-1}{k-1}.$$

We find for S_1

$$\mathrm{Res}_{w=0}(1+w)^{r-1}\sum_{k=1}^{n-1}\binom{2n-2k-1}{n-k}\frac{(1+w)^{2k}}{w^{k+1}}$$

$$= \mathrm{Res}_{w=0}\frac{(1+w)^{r-1}}{w}[z^n](1+z)^{2n-1}\sum_{k=1}^{n-1} z^k(1+z)^{-2k}\frac{(1+w)^{2k}}{w^k}.$$

Including the term at $k=0$ and compensating

$$-\binom{2n-1}{n-1} + \mathrm{Res}_{w=0}\frac{(1+w)^{r-1}}{w}[z^n](1+z)^{2n-1}\sum_{k=0}^{n-1} z^k(1+z)^{-2k}\frac{(1+w)^{2k}}{w^k}.$$

Including the term at $k=n$ and again compensating

$$-\binom{2n-1}{n-1} - \binom{r+2n-1}{n}$$

$$+\mathrm{Res}_{w=0}\frac{(1+w)^{r-1}}{w}[z^n](1+z)^{2n-1}\sum_{k=0}^{n} z^k(1+z)^{-2k}\frac{(1+w)^{2k}}{w^k}.$$

Now we may extend k beyond n owing to the coefficient extractor $[z^n]$ to get

$$-\binom{2n-1}{n-1} - \binom{r+2n-1}{n}$$

$$+\mathrm{Res}_{w=0}\frac{(1+w)^{r-1}}{w}[z^n](1+z)^{2n-1}\frac{1}{1-z(1+w)^2/w/(1+z)^2}$$

$$= -\binom{2n-1}{n-1} - \binom{r+2n-1}{n}$$

$$+\mathrm{Res}_{w=0}(1+w)^{r-1}[z^n](1+z)^{2n+1}\frac{1}{w(1+z)^2-z(1+w)^2}.$$

We get for S_2

$$\mathrm{Res}_{w=0}(1+w)^{r-1}[z^n](1+z)^{2n-1}\sum_{k=1}^{n-1} z^k(1+z)^{-2k}\frac{(1+w)^{2k}}{w^k}.$$

The term $k=0$ contributes zero. Compensating for $k=n$ we find

$$-\binom{r+2n-1}{n-1} + \mathrm{Res}_{w=0}(1+w)^{r-1}[z^n](1+z)^{2n-1}\sum_{k\ge 0} z^k(1+z)^{-2k}\frac{(1+w)^{2k}}{w^k}$$

$$= -\binom{r+2n-1}{n-1} + \mathrm{Res}_{w=0}w(1+w)^{r-1}[z^n](1+z)^{2n+1}\frac{1}{w(1+z)^2-z(1+w)^2}.$$

We therefore have

$$S = S_1 - S_2 = -\binom{2n-1}{n-1} - \binom{r+2n-1}{n} + \binom{r+2n-1}{n-1}$$
$$+\mathrm{Res}_{w=0}(1+w)^{r-1}[z^n](1+z)^{2n}\frac{(1-w)(1+z)}{w(1+z)^2 - z(1+w)^2}.$$

Working with the remaining residue we note that

$$\frac{(1-w)(1+z)}{w(1+z)^2 - z(1+w)^2} = \frac{1}{w}\frac{1}{1-z/w} - \frac{1}{1-zw}.$$

We see on substituting into the residue that we get no contribution from the second term. This leaves

$$\mathrm{Res}_{w=0}\frac{1}{w}(1+w)^{r-1}[z^n](1+z)^{2n}\frac{1}{1-z/w}$$
$$= \mathrm{Res}_{w=0}\frac{1}{w}(1+w)^{r-1}\sum_{q=0}^{n}\binom{2n}{n-q}w^{-q}$$
$$= \sum_{q=0}^{n}\binom{2n}{n-q}\binom{r-1}{q} = [z^n](1+z)^{2n}\sum_{q=0}^{n}\binom{r-1}{q}z^q.$$

The coefficient extractor once more enforces the range and we find

$$[z^n](1+z)^{2n}\sum_{q\ge 0}\binom{r-1}{q}z^q$$
$$= [z^n](1+z)^{2n}(1+z)^{r-1} = [z^n](1+z)^{r+2n-1} = \binom{r+2n-1}{n}.$$

Collecting all four pieces yields

$$S = S_1 - S_2 = -\binom{2n-1}{n-1} - \binom{r+2n-1}{n} + \binom{r+2n-1}{n-1} + \binom{r+2n-1}{n}$$
$$= \binom{r+2n-1}{n-1} - \binom{2n-1}{n-1}$$

which is the claim.

Remark. The next-to-last step may also be done as follows:

$$\mathrm{Res}_{w=0}\frac{1}{w}(1+w)^{r-1}[z^n](1+z)^{2n}\frac{1}{1-z/w}$$
$$= \mathrm{Res}_{w=0}\frac{1}{w}\sum_{q=0}^{r-1}\binom{r-1}{q}w^q[z^n](1+z)^{2n}\frac{1}{1-z/w}$$

$$= [z^n](1+z)^{2n} \sum_{q=0}^{r-1} \binom{r-1}{q} z^q = [z^n](1+z)^{2n}(1+z)^{r-1} = \binom{r+2n-1}{n}.$$

This was math.stackexchange.com problem 3260307.

69.14 MSE 3285142

Starting from (the contribution from $k = 0$ is zero owing to the third binomial coefficient)

$$\sum_{k=1}^{n} \left(-\frac{1}{4}\right)^k \binom{2k}{k}^2 \frac{1}{1-2k} \binom{n+k-2}{2k-2}$$

we seek to show that this is zero when n is odd and

$$\left[\left(\frac{1}{4}\right)^m \binom{2m}{m} \frac{1}{1-2m}\right]^2$$

when $n = 2m$ is even.
We observe that with $k \geq 1$

$$\binom{2k}{k} \frac{1}{1-2k} \binom{n+k-2}{2k-2} = 2\binom{2k-1}{k-1} \frac{1}{1-2k} \binom{n+k-2}{2k-2}$$

$$= -2\binom{2k-2}{k-1} \frac{1}{k} \binom{n+k-2}{2k-2} = -\frac{2}{k} \frac{(n+k-2)!}{(k-1)!^2 \times (n-k)!}$$

$$= -\frac{2}{k} \binom{n+k-2}{k-1} \binom{n-1}{k-1} = -\frac{2}{n} \binom{n}{k} \binom{n+k-2}{k-1}.$$

We get for our sum

$$-\frac{2}{n} \sum_{k=1}^{n} \binom{n}{k} \left(-\frac{1}{4}\right)^k \binom{2k}{k} \binom{n+k-2}{k-1}$$

$$= -\frac{2}{n} \sum_{k=1}^{n} \binom{n}{k} \binom{-1/2}{k} \binom{n+k-2}{n-1}$$

$$= -\frac{2}{n} [z^{n-1}](1+z)^{n-2} \sum_{k=1}^{n} \binom{n}{k} \binom{-1/2}{k} (1+z)^k.$$

The value $k = 0$ contributes zero:

$$-\frac{2}{n} \times \mathrm{Res}_{w=0} \frac{1}{w} (1+w)^{-1/2} [z^{n-1}](1+z)^{n-2} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{w^k} (1+z)^k$$

$$= -\frac{2}{n} \times \mathrm{Res}_{w=0} \frac{1}{w} (1+w)^{-1/2} [z^{n-1}](1+z)^{n-2} (1+(1+z)/w)^n$$

$$= -\frac{2}{n} \times \mathrm{Res}_{w=0} \frac{1}{w^{n+1}} (1+w)^{-1/2} [z^{n-1}] (1+z)^{n-2} (1+w+z)^n$$

$$= -\frac{2}{n} \times \mathrm{Res}_{w=0} \frac{1}{w^{n+1}} (1+w)^{-1/2} [z^{n-1}] (1+z)^{n-2} \sum_{q=0}^{n} {n \choose q} (1+w)^q z^{n-q}$$

$$= -\frac{2}{n} \times \sum_{q=1}^{n} {n \choose q} {q-1/2 \choose n} {n-2 \choose q-1}.$$

Now observe that with $q < n$ (third binomial coefficient is zero when $q = n$)

$${q-1/2 \choose n} = \frac{1}{n!} (q-1/2)^{\underline{n}} = \frac{1}{n!} \prod_{p=0}^{q-1} (q-1/2-p) \prod_{p=q}^{n-1} (q-1/2-p)$$

$$= \frac{1}{n! \times 2^n} \prod_{p=0}^{q-1} (2q-1-2p) \prod_{p=q}^{n-1} (2q-1-2p)$$

$$= \frac{1}{n! \times 2^n} \frac{(2q-1)!}{(q-1)! \times 2^{q-1}} \prod_{p=0}^{n-1-q} (-1-2p)$$

$$= \frac{(-1)^{n-q}}{n! \times 2^n} \frac{(2q-1)!}{(q-1)! \times 2^{q-1}} \frac{(2n-1-2q)!}{(n-1-q)! \times 2^{n-1-q}}$$

$$= \frac{(-1)^{n-q}}{2^{2n-2}} {n \choose q}^{-1} {2q-1 \choose q-1} {2n-1-2q \choose n-q}.$$

We get for our sum

$$-\frac{1}{n \times 2^{2n-3}} \times \sum_{q=1}^{n-1} (-1)^{n-q} {2q-1 \choose q-1} {2n-1-2q \choose n-q} {n-2 \choose q-1}$$

$$= \frac{1}{n \times 2^{2n-3}} \times \sum_{q=0}^{n-2} {n-2 \choose q} (-1)^{n-2-q} {2q+1 \choose q} {2n-3-2q \choose n-q-1}.$$

This becomes

$$\frac{1}{n \times 2^{2n-3}} \times [z^{n-1}] (1+z)^{2n-3} \sum_{q=0}^{n-2} {n-2 \choose q} (-1)^{n-2-q} {2q+1 \choose q} z^q (1+z)^{-2q}$$

$$= \frac{1}{n \times 2^{2n-3}} \mathrm{Res}_{w=0} \frac{1+w}{w} [z^{n-1}] (1+z)^{2n-3}$$

$$\times \sum_{q=0}^{n-2} {n-2 \choose q} (-1)^{n-2-q} \frac{1}{w^q} (1+w)^{2q} z^q (1+z)^{-2q}$$

$$= \frac{1}{n \times 2^{2n-3}} \mathrm{Res}_{w=0} \frac{1+w}{w} [z^{n-1}](1+z)^{2n-3} \left(\frac{z(1+w)^2}{w(1+z)^2} - 1 \right)^{n-2}$$
$$= \frac{1}{n \times 2^{2n-3}} \mathrm{Res}_{w=0} \frac{1+w}{w^{n-1}} [z^{n-1}](1+z) \left(z(1+w)^2 - w(1+z)^2 \right)^{n-2}$$
$$= \frac{1}{n \times 2^{2n-3}} \mathrm{Res}_{w=0} \frac{1+w}{w^{n-1}} [z^{n-1}](1+z)(z-w)^{n-2}(1-wz)^{n-2}.$$

The first piece in z is

$$[z^{n-1}](z-w)^{n-2}(1-wz)^{n-2}$$
$$= \sum_{q=1}^{n-2} \binom{n-2}{q} (-1)^{n-2-q} w^{n-2-q} \binom{n-2}{n-1-q} (-1)^{n-1-q} w^{n-1-q}$$
$$= -\sum_{q=1}^{n-2} \binom{n-2}{q} \binom{n-2}{q-1} w^{2n-3-2q}.$$

Here we require

$$([w^{n-2}] + [w^{n-3}]) w^{2n-3-2q}$$

We get $q = (n-1)/2$ in the first case and $q = n/2$ in the second. As this is a pair of an integer and a fraction clearly only one of these extractors can return a non-zero value.

The second piece in z is

$$[z^{n-2}](z-w)^{n-2}(1-wz)^{n-2}$$
$$= \sum_{q=0}^{n-2} \binom{n-2}{q} (-1)^{n-2-q} w^{n-2-q} \binom{n-2}{n-2-q} (-1)^{n-2-q} w^{n-2-q}$$
$$= \sum_{q=0}^{n-2} \binom{n-2}{q} \binom{n-2}{q} w^{2n-4-2q}.$$

Solving for q again we require

$$([w^{n-2}] + [w^{n-3}]) w^{2n-4-2q}$$

getting $q = n/2 - 1$ and $q = (n-1)/2$.

Supposing that n is odd i.e. $n = 2m+1$ we thus have

$$-\binom{2m-1}{m}\binom{2m-1}{m-1} + \binom{2m-1}{m}\binom{2m-1}{m} = 0,$$

and we have proved the second part of the claim. On the other hand with $n = 2m$ even we collect

$$-\binom{2m-2}{m}\binom{2m-2}{m-1} + \binom{2m-2}{m-1}\binom{2m-2}{m-1}$$

$$= \binom{2m-2}{m-1}^2 \left(1 - \frac{m-1}{m}\right) = \frac{m^2}{(2m-1)^2} \binom{2m-1}{m}^2 \frac{1}{m}$$

$$= \frac{m^2}{(2m-1)^2} \frac{m^2}{(2m)^2} \binom{2m}{m}^2 \frac{1}{m} = \frac{1}{4} \frac{m}{(2m-1)^2} \binom{2m}{m}^2.$$

Restoring the factor in front we obtain

$$\frac{1}{n \times 2^{2n-3}} \frac{1}{4} \frac{m}{(2m-1)^2} \binom{2m}{m}^2 = \frac{1}{2^{2n}} \frac{1}{(2m-1)^2} \binom{2m}{m}^2$$

$$= \frac{1}{2^{4m}} \frac{1}{(1-2m)^2} \binom{2m}{m}^2$$

This is

$$\left[\left(\frac{1}{4}\right)^m \binom{2m}{m} \frac{1}{1-2m}\right]^2$$

as was to be shown.

This was math.stackexchange.com problem 3285142.

69.15 MSE 3333597

We seek to verify that

$$\sum_{n=0}^{N} \sum_{k=0}^{N} \frac{(-1)^{n+k}}{n+k+1} \binom{N}{n} \binom{N}{k} \binom{N+n}{n} \binom{N+k}{k} = \frac{1}{2N+1}.$$

Now we have

$$\binom{N}{k} \binom{N+n}{n} = \frac{(N+n)!}{(N-k)! \times k! \times n!} = \binom{N+n}{n+k} \binom{n+k}{k}.$$

We get for the LHS

$$\sum_{n=0}^{N} \sum_{k=0}^{N} \frac{(-1)^{n+k}}{n+k+1} \binom{N+n}{n+k} \binom{N}{n} \binom{N+k}{k} \binom{n+k}{k}$$

$$= \sum_{n=0}^{N} \frac{1}{N+n+1} \sum_{k=0}^{N} (-1)^{n+k} \binom{N+n+1}{n+k+1} \binom{N}{n} \binom{N+k}{k} \binom{n+k}{k}$$

$$= \sum_{n=0}^{N} \frac{1}{N+n+1} \binom{N}{n} \sum_{k=0}^{N} (-1)^{n+k} \binom{N+n+1}{N-k} \binom{N+k}{k} \binom{n+k}{k}$$

$$= \sum_{n=0}^{N} \frac{1}{N+n+1} [z^N] (1+z)^{N+n+1} \binom{N}{n} \sum_{k=0}^{N} (-1)^{n+k} z^k \binom{N+k}{N} \binom{n+k}{n}.$$

Now the coefficient extractor controls the range and we continue with

$$\sum_{n=0}^N \frac{1}{N+n+1} [z^N](1+z)^{N+n+1} {N\choose n}$$

$$\times \sum_{k\ge 0} (-1)^{n+k} z^k {N+k\choose N} \mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \frac{1}{(1-w)^{k+1}}$$

$$= \sum_{n=0}^N \frac{1}{N+n+1} [z^N](1+z)^{N+n+1} {N\choose n} \mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \frac{1}{1-w}$$

$$\times \sum_{k\ge 0} (-1)^{n+k} z^k {N+k\choose N} \frac{1}{(1-w)^k}$$

$$= \sum_{n=0}^N \frac{(-1)^n}{N+n+1} [z^N](1+z)^{N+n+1} {N\choose n}$$

$$\times \mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \frac{1}{1-w} \frac{1}{(1+z/(1-w))^{N+1}}$$

$$= \sum_{n=0}^N \frac{(-1)^n}{N+n+1} [z^N](1+z)^{N+n+1} {N\choose n}$$

$$\times \mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \frac{(1-w)^N}{(1-w+z)^{N+1}}$$

$$= \sum_{n=0}^N \frac{(-1)^n}{N+n+1} [z^N](1+z)^{n} {N\choose n}$$

$$\times \mathrm{Res}_{w=0} \frac{1}{w^{n+1}} \frac{(1-w)^N}{(1-w/(1+z))^{N+1}}$$

$$= \sum_{n=0}^N \frac{(-1)^n}{N+n+1} [z^N](1+z)^{n} {N\choose n}$$

$$\times \sum_{k=0}^n (-1)^k {N\choose k} {n-k+N\choose N} \frac{1}{(1+z)^{n-k}}$$

$$= \sum_{n=0}^N \frac{(-1)^n}{N+n+1} {N\choose n}$$

$$\times \sum_{k=0}^n (-1)^k {N\choose k} {n-k+N\choose N} [z^N](1+z)^k.$$

Now for the coefficient extractor to be non-zero we must have $k \ge N$ which happens just once, namely when $n = N$ and $k = N$. We get

$$\frac{(-1)^N}{2N+1}\binom{N}{N}(-1)^N\binom{N}{N}\binom{N-N+N}{N}.$$

This expression does indeed simplify to

$$\frac{1}{2N+1}$$

as claimed.

This was math.stackexchange.com problem 3333597.

69.16 MSE 3342361

We seek to verify that

$$\sum_{k=3}^n (-1)^k \binom{n}{k} \sum_{j=1}^{k-2} \binom{j(n+1)+k-3}{n-2} = (-1)^{n-1}\left[\binom{n}{2} - \binom{2n+1}{n-2}\right].$$

where $n \geq 3$. Now for

$$\sum_{k=3}^n (-1)^k \binom{n}{k} \binom{k-3}{n-2}$$

to be non-zero we would need $k-3 \geq n-2$ or $k \geq n+1$, which is not in the range, so it is zero and we may work with

$$\sum_{k=3}^n (-1)^k \binom{n}{k} \sum_{j=0}^{k-2} \binom{j(n+1)+k-3}{n-2}$$

$$= \sum_{k=3}^n (-1)^k \binom{n}{k} \sum_{j\geq 0} \binom{j(n+1)+k-3}{n-2} [[0 \leq j \leq k-2]]$$

$$= \sum_{k=3}^n (-1)^k \binom{n}{k} \sum_{j\geq 0} \binom{j(n+1)+k-3}{n-2} \mathrm{Res}_{z=0} \frac{1}{z^{k-1}} \frac{z^j}{1-z}$$

$$= \mathrm{Res}_{z=0} \frac{z}{1-z} \sum_{k=3}^n (-1)^k \binom{n}{k} \frac{1}{z^k} \sum_{j\geq 0} \binom{j(n+1)+k-3}{n-2} z^j$$

$$= \mathrm{Res}_{z=0} \frac{z}{1-z} \sum_{k=3}^n (-1)^k \binom{n}{k} \frac{1}{z^k} \sum_{j\geq 0} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} (1+w)^{j(n+1)+k-3} z^j$$

$$= \mathrm{Res}_{z=0} \frac{z}{1-z} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \sum_{k=3}^n (-1)^k \binom{n}{k} \frac{1}{z^k} (1+w)^{k-3} \sum_{j\geq 0} (1+w)^{j(n+1)} z^j$$

$$= \mathrm{Res}_{z=0} \frac{z}{1-z} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{1-z(1+w)^{n+1}} \sum_{k=3}^{n} (-1)^k \binom{n}{k} \frac{1}{z^k} (1+w)^{k-3}$$

$$= \mathrm{Res}_{z=0} \frac{z}{1-z} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{(1+w)^3} \frac{1}{1-z(1+w)^{n+1}}$$

$$\times \sum_{k=3}^{n} (-1)^k \binom{n}{k} \frac{1}{z^k} (1+w)^k.$$

We compute this by lowering the index to $k=0$ and subtracting the values for $k=0,1$ and $k=2$ from this completed sum. **First** (piece A), extending to $k=0$ we find

$$\mathrm{Res}_{z=0} \frac{z}{1-z} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{(1+w)^3} \frac{1}{1-z(1+w)^{n+1}} \left(1 - \frac{1+w}{z}\right)^n$$

$$= \mathrm{Res}_{z=0} \frac{1}{z^n} \frac{z}{1-z} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{(1+w)^3} \frac{1}{1-z(1+w)^{n+1}} (z-1-w)^n.$$

We introduce $z/(1+w-z) = v$ so that $z = v(1+w)/(1+v)$ and $dz = (1+w)/(1+v)^2\; dv$ as well as $z/(1-z) = v(1+w)/(1-vw)$ to get

$$\mathrm{Res}_{v=0} \frac{(-1)^n}{v^n} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{(1+w)^3} \frac{v(1+w)}{1-vw} \frac{1}{1-v(1+w)^{n+2}/(1+v)} \frac{1+w}{(1+v)^2}$$

$$= \mathrm{Res}_{v=0} \frac{(-1)^n}{v^{n-1}} \frac{1}{1+v} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{1+w} \frac{1}{1-vw} \frac{1}{1-v((1+w)^{n+2}-1)}.$$

Observe that

$$\frac{1}{1+v} \frac{1}{1-vw} = \frac{1}{1+w} \frac{1}{1+v} + \frac{w}{1+w} \frac{1}{1-vw}.$$

We thus have piece A_1 :

$$\mathrm{Res}_{v=0} \frac{(-1)^n}{v^{n-1}} \frac{1}{1+v} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{(1+w)^2} \frac{1}{1-v((1+w)^{n+2}-1)}$$

$$= \mathrm{Res}_{w=0} \frac{(-1)^n}{w^{n-1}} \frac{1}{(1+w)^2} \sum_{q=0}^{n-2} (-1)^{n-2-q} ((1+w)^{n+2}-1)^q$$

$$= \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{(1+w)^2} \sum_{q=0}^{n-2} (1-(1+w)^{n+2})^q$$

$$= \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{(1+w)^2} \frac{1-(1-(1+w)^{n+2})^{n-1}}{(1+w)^{n+2}}$$

$$= [w^{n-2}]\frac{1-(-(n+2)w-\cdots-w^{n+2})^{n-1}}{(1+w)^{n+4}} = (-1)^{n-2}\binom{n-2+n+3}{n-2}$$

$$= (-1)^n\binom{2n+1}{n-2}.$$

We have one correct piece. Continuing with A_2 (which we conjecture to be zero) we find

$$\mathrm{Res}_{v=0}\frac{(-1)^n}{v^{n-1}}\mathrm{Res}_{w=0}\frac{1}{w^{n-2}}\frac{1}{(1+w)^2}\frac{1}{1-vw}\frac{1}{1-v((1+w)^{n+2}-1)}$$

$$= \mathrm{Res}_{w=0}\frac{(-1)^n}{w^{n-2}}\frac{1}{(1+w)^2}\sum_{q=0}^{n-2} w^{n-2-q}((1+w)^{n+2}-1)^q$$

$$= \mathrm{Res}_{w=0}\frac{(-1)^n}{w^{n-2}}\frac{1}{(1+w)^2}\sum_{q=0}^{n-2} w^{n-2-q}((n+2)w+\cdots+w^{n+2})^q$$

$$= \mathrm{Res}_{w=0}\frac{(-1)^n}{w^{n-2}}\frac{1}{(1+w)^2}\sum_{q=0}^{n-2}((n+2)^q w^{n-2}+\cdots+w^{(n+1)q+n-2})$$

$$= \mathrm{Res}_{w=0}\frac{(-1)^n}{(1+w)^2}\sum_{q=0}^{n-2}((n+2)^q+\cdots+w^{(n+1)q}) = 0.$$

Continuing with the **second** piece B which corresponds to $k=0$

$$\mathrm{Res}_{z=0}\frac{z}{1-z}\mathrm{Res}_{w=0}\frac{1}{w^{n-1}}\frac{1}{(1+w)^3}\frac{1}{1-z(1+w)^{n+1}}.$$

This is zero by inspection because there is no pole at $z=0$. More formally,

$$\mathrm{Res}_{w=0}\frac{1}{w^{n-1}}\frac{1}{(1+w)^3}$$

$$\times\mathrm{Res}_{z=0} z(1+z+z^2+\cdots)(1+z(1+w)^{n+1}+z^2(1+w)^{2n+2}+\cdots) = 0.$$

For the **third** piece C which corresponds to $k=1$ we get a factor of $-n(1+w)/z$ for

$$-n\mathrm{Res}_{w=0}\frac{1}{w^{n-1}}\frac{1}{(1+w)^2}$$

$$\times\mathrm{Res}_{z=0}(1+z+z^2+\cdots)(1+z(1+w)^{n+1}+z^2(1+w)^{2n+2}+\cdots) = 0.$$

The factor for the **fourth** piece D is $\binom{n}{2}(1+w)^2/z^2$:

$$\binom{n}{2}\mathrm{Res}_{w=0}\frac{1}{w^{n-1}}\frac{1}{1+w}$$

$$\times \mathrm{Res}_{z=0} \frac{1}{z}(1+z+z^2+\cdots)(1+z(1+w)^{n+1}+z^2(1+w)^{2n+2}+\cdots)$$

$$= \binom{n}{2} \mathrm{Res}_{w=0} \frac{1}{w^{n-1}} \frac{1}{1+w} = (-1)^n \binom{n}{2}.$$

Subtracting B, C and D from A we finally obtain

$$(-1)^n \left[\binom{2n+1}{n-2} - \binom{n}{2} \right].$$

This was math.stackexchange.com problem 3342361.

69.17 MSE 3383557

We seek to show that

$$n \sum_{k=0}^{n} \frac{(-1)^k}{2n-k} \binom{2n-k}{k} x^k y^{2n-2k} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{2k} y^{2k} (y^2-4x)^{n-k}.$$

We compare the coefficient on $[x^q]$ of the LHS and the RHS where $0 \le q \le n$ and show that they are equal. We must therefore show that

$$n \frac{(-1)^q}{2n-q} \binom{2n-q}{q} y^{2n-2q} = [x^q] \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{2k} y^{2k} (y^2-4x)^{n-k}.$$

The RHS is

$$[x^q] \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{2n-2k} y^{2n-2k} (y^2-4x)^k$$

$$= \frac{1}{2^{2n}} \sum_{k=q}^{n} \binom{2n}{2n-2k} y^{2n-2k} [x^q] (y^2-4x)^k$$

$$= \frac{1}{2^{2n}} \sum_{k=q}^{n} \binom{2n}{2k} y^{2n-2k} \binom{k}{q} (-4)^q y^{2k-2q}$$

$$= y^{2n-2q} \frac{1}{2^{2n}} \sum_{k=q}^{n} \binom{2n}{2k} \binom{k}{q} (-4)^q.$$

We have reduced the claim to

$$n \frac{(-1)^q}{2n-q} \binom{2n-q}{q} = \frac{1}{2^{2n}} \sum_{k=q}^{n} \binom{2n}{2k} \binom{k}{q} (-4)^q.$$

The RHS is

$$\frac{1}{2^{2n}}\sum_{k=q}^{n}\binom{k}{q}(-4)^q[z^{2n-2k}](1+z)^{2n}$$
$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n}](1+z)^{2n}\sum_{k=q}^{n}\binom{k}{q}z^{2k}.$$

Now when k exceeds n we get zero from the coefficient extractor, which enforces the range:

$$\frac{(-1)^q}{2^{2n-2q}}[z^{2n}](1+z)^{2n}\sum_{k\ge q}\binom{k}{q}z^{2k}$$
$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n}]z^{2q}(1+z)^{2n}\sum_{k\ge 0}\binom{k+q}{q}z^{2k}$$
$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n}]z^{2q}(1+z)^{2n}\frac{1}{(1-z^2)^{q+1}}$$
$$= \frac{(-1)^q}{2^{2n-2q}}[z^{2n-2q}](1+z)^{2n-q-1}\frac{1}{(1-z)^{q+1}}$$
$$= \frac{(-1)^q}{2^{2n-2q}}\sum_{p=0}^{2n-q-1}\binom{2n-q-1}{p}\binom{2n-2q-p+q}{q}$$
$$= \frac{(-1)^q}{2^{2n-2q}}\sum_{p=0}^{2n-q-1}\binom{2n-q-1}{2n-q-1-p}\binom{2n-q-p}{q}.$$

Then we have

$$\binom{2n-q-1}{2n-q-1-p}\binom{2n-q-p}{q} = \frac{(2n-q-1)!(2n-q-p)}{p!\times q!\times(2n-2q-p)!}$$
$$= \frac{1}{2n-q}\frac{(2n-q)!(2n-q-p)}{p!\times q!\times(2n-2q-p)!}$$
$$= \frac{1}{2n-q}\binom{2n-q}{q}\binom{2n-2q}{p}(2n-q-p).$$

Substituting we find (here we have included the value for $p = 2n-q$, which is zero):

$$\frac{(-1)^q}{2^{2n-2q}}\frac{1}{2n-q}\binom{2n-q}{q}\sum_{p=0}^{2n-q}\binom{2n-2q}{p}(2n-q-p).$$

Working with the remaining sum we note that $(2n-2q)^{\underline{p}} = 0$ when $p > 2n-2q$ and $2n-q \ge 2n-2q$ so we may continue with

$$\sum_{p=0}^{2n-2q} \binom{2n-2q}{p} (2n-q-p) = (2n-q)2^{2n-2q} - \sum_{p=1}^{2n-2q} \binom{2n-2q}{p} p$$

$$= (2n-q)2^{2n-2q} - (2n-2q) \sum_{p=1}^{2n-2q} \binom{2n-2q-1}{p-1}$$

$$= (2n-q)2^{2n-2q} - (2n-2q)2^{2n-2q-1} = (2n-q)2^{2n-2q} - (n-q)2^{2n-2q}$$

$$= n2^{2n-2q}.$$

Substituting we at last obtain

$$n\frac{(-1)^q}{2n-q}\binom{2n-q}{q}$$

which was to be shown.
This was math.stackexchange.com problem 3383557.

69.18 MSE 3441855

We start as follows:

$$\sum_{k=0}^{n} (-1)^k 4^{n-k} \binom{2n-k}{k} = \sum_{k=0}^{n} (-1)^k 4^{n-k} \binom{2n-k}{2n-2k}$$

$$= \sum_{k=0}^{n} (-1)^k 4^{n-k} [z^{2n-2k}](1+z)^{2n-k}$$

$$= [z^{2n}](1+z)^{2n} \sum_{k=0}^{n} (-1)^k 4^{n-k} z^{2k} (1+z)^{-k}.$$

Now when $k > n$ we get zero contribution due to the coefficient extractor $[z^{2n}]$ and the factor z^{2k}, so this enforces the range of the sum and we may continue with

$$[z^{2n}](1+z)^{2n} \sum_{k\ge 0} (-1)^k 4^{n-k} z^{2k} (1+z)^{-k}$$

$$= 4^n [z^{2n}](1+z)^{2n} \frac{1}{1+z^2/(1+z)/4}$$

$$= 4^{n+1} [z^{2n}](1+z)^{2n+1} \frac{1}{4+4z+z^2} = 4^{n+1} [z^{2n}](1+z)^{2n+1} \frac{1}{(z+2)^2}.$$

This is

$$4^{n+1} \mathrm{Res}_{z=0} \frac{1}{z^{2n+1}} (1+z)^{2n+1} \frac{1}{(z+2)^2}.$$

We introduce $z/(1+z) = w$ so that $z = w/(1-w)$ and $dz = 1/(1-w)^2\; dw$, to obtain

$$\begin{aligned}
&4^{n+1}\mathrm{Res}_{w=0} \frac{1}{w^{2n+1}} \frac{1}{(w/(1-w)+2)^2} \frac{1}{(1-w)^2} \\
&= 4^{n+1}\mathrm{Res}_{w=0} \frac{1}{w^{2n+1}} \frac{1}{(2-w)^2} \\
&= 4^{n+1}[w^{2n}] \frac{1}{(2-w)^2} = 4^n [w^{2n}] \frac{1}{(1-w/2)^2} = 4^n (2n+1) \frac{1}{2^{2n}} \\
&= 2n+1.
\end{aligned}$$

Remark. This can also be done using the fact that residues sum to zero, which starting from the residue in z we see that the residue at infinity is zero, so our sum is

$$\begin{aligned}
&-4^{n+1}\mathrm{Res}_{z=-2} \frac{1}{z^{2n+1}} (1+z)^{2n+1} \frac{1}{(z+2)^2} \\
&= -4^{n+1} \left(\frac{1}{z^{2n+1}} (1+z)^{2n+1} \right)' \Bigg|_{z=-2} \\
&= -4^{n+1} \left(-\frac{2n+1}{z^{2n+2}} (1+z)^{2n+1} + \frac{(2n+1)}{z^{2n+1}} (1+z)^{2n} \right) \Bigg|_{z=-2} \\
&= (2n+1) \times 4^{n+1} \left(\frac{(-1)^{2n+1}}{(-2)^{2n+2}} - \frac{(-1)^{2n}}{(-2)^{2n+1}} \right) \\
&= (2n+1) \times 2^{2n+2} \left(-\frac{1}{2^{2n+2}} + \frac{1}{2^{2n+1}} \right) = 2n+1.
\end{aligned}$$

This was math.stackexchange.com problem 3441855.

69.19 MSE 3577193

We seek to show that

$$\sum_{k=0}^{l} \binom{k}{m} \binom{k}{n} = \sum_{k=0}^{n} (-1)^k \binom{l+1}{m+k+1} \binom{l-k}{n-k}.$$

The RHS is

$$[z^n] \sum_{k=0}^{n} (-1)^k \binom{l+1}{m+k+1} z^k (1+z)^{l-k}.$$

The coefficient extractor enforces the range:

$$[z^n] \sum_{k\ge 0} (-1)^k \binom{l+1}{l-m-k} z^k (1+z)^{l-k}$$

$$= [z^n](1+z)^l[w^{l-m}](1+w)^{l+1}\sum_{k\ge 0}(-1)^k w^k z^k (1+z)^{-k}$$

$$= [z^n](1+z)^l[w^{l-m}](1+w)^{l+1}\frac{1}{1+wz/(1+z)}$$

$$= [z^n](1+z)^{l+1}[w^{l-m}](1+w)^{l+1}\frac{1}{1+z+wz}$$

$$= [z^n](1+z)^{l+1}[w^{l-m}](1+w)^{l+1}\frac{1}{1+z(1+w)}$$

$$= [z^n](1+z)^{l+1}[w^{l-m}]\sum_{k\ge 0}(-1)^k z^k (1+w)^{k+l+1}$$

$$= [z^n](1+z)^{l+1}\sum_{k\ge 0}(-1)^k z^k \binom{k+l+1}{l-m}.$$

This is

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{l+1}{n-k}\binom{k+l+1}{l-m}.}$$

The LHS is

$$\sum_{k\ge 0}[[0\le k\le l]][z^m](1+z)^k[w^n](1+w)^k$$

$$= [z^m][w^n]\sum_{k\ge 0}(1+z)^k(1+w)^k[v^l]\frac{v^k}{1-v}$$

$$= [z^m][w^n][v^l]\frac{1}{1-v}\sum_{k\ge 0}(1+z)^k(1+w)^k v^k$$

$$= [z^m][w^n][v^l]\frac{1}{1-v}\frac{1}{1-(1+z)(1+w)v}$$

$$= [z^m][w^n][v^l]\frac{1}{v-1}\frac{1/(1+z)/(1+w)}{v-1/(1+z)/(1+w)}.$$

The inner term is

$$\mathrm{Res}_{v=0}\frac{1}{v^{l+1}}\frac{1}{v-1}\frac{1/(1+z)/(1+w)}{v-1/(1+z)/(1+w)}.$$

Residues sum to zero and the residue at infinity in v is zero. The contribution from minus the residue at $v = 1/(1+z)/(1+w)$ is

$$-[z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/(1+z)/(1+w)}{1/(1+z)/(1+w)-1}$$

$$= -[z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/(1+z)}{1/(1+z)-(1+w)}$$
$$= [z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/(1+z)}{w+z/(1+z)}$$
$$= [z^m](1+z)^{l+1}[w^n](1+w)^{l+1}\frac{1/z}{w(1+z)/z+1}.$$

Now with l, m, n positive integers we must have $l \ge n, m$ or else there is no contribution to $k^{\underline{m}}k^{\underline{n}}$. This means we continue with

$$[z^m](1+z)^{l+1}\sum_{k=0}^{n}\binom{l+1}{k}\frac{1}{z}(-1)^{n-k}\frac{(1+z)^{n-k}}{z^{n-k}}$$
$$= \sum_{k=0}^{n}(-1)^{n-k}\binom{l+1}{k}\binom{l+1+n-k}{m+1+n-k}.$$

This is
$$\boxed{\sum_{k=0}^{n}(-1)^{n-k}\binom{l+1}{k}\binom{l+1+n-k}{l-m}.}$$

We have the same closed form for LHS and RHS, thus proving the claim.

For a full proof we also need to show that the contribution from $v = 1$ is zero. We get

$$[z^m][w^n]\frac{1/(1+z)/(1+w)}{1-1/(1+z)/(1+w)} = [z^m][w^n]\frac{1}{(1+z)(1+w)-1}$$
$$= [z^m][w^n]\frac{1}{z+w+zw} = [z^{m+1}][w^n]\frac{1}{1+w(1+z)/z}$$
$$= [z^{m+1}](-1)^n\frac{(1+z)^n}{z^n} = (-1)^n\binom{n}{n+m+1} = 0.$$

This was math.stackexchange.com problem 3577193.

69.20 MSE 3583191

Goal here is

$$\sum_{j=0}^{k}\binom{2n}{2j}\binom{n-j}{k-j} = \frac{4^k n}{n+k}\binom{n+k}{n-k}.$$

Start as follows:

$$\sum_{j=0}^{k}\binom{2n}{2j}\binom{n-j}{k-j} = \sum_{j=0}^{k}\binom{2n}{2k-2j}\binom{n-k+j}{j}$$

$$= [z^{2k}](1+z)^{2n} \sum_{j=0}^{k} z^{2j} \binom{n-k+j}{j}.$$

Here the coefficient extractor enforces the range:

$$[z^{2k}](1+z)^{2n} \sum_{j\ge 0} z^{2j} \binom{n-k+j}{j}$$

$$= [z^{2k}](1+z)^{2n} \frac{1}{(1-z^2)^{n-k+1}} = [z^{2k}](1+z)^{n+k-1} \frac{1}{(1-z)^{n-k+1}}.$$

This is

$$\mathrm{Res}_{z=0} \frac{1}{z^{2k+1}} (1+z)^{n+k-1} \frac{1}{(1-z)^{n-k+1}}$$

$$= (-1)^{n-k+1} \mathrm{Res}_{z=0} \frac{1}{z^{2k+1}} (1+z)^{n+k-1} \frac{1}{(z-1)^{n-k+1}}.$$

Now the residue at infinity is zero so this is minus the residue at one:

$$(-1)^{n-k} \mathrm{Res}_{z=1} \frac{1}{(1+(z-1))^{2k+1}} (2+(z-1))^{n+k-1} \frac{1}{(z-1)^{n-k+1}}$$

$$= (-1)^{n-k} \sum_{j=0}^{n-k} \binom{n+k-1}{j} 2^{n+k-1-j} (-1)^{n-k-j} \binom{n-k-j+2k}{2k}$$

$$= 2^{n+k-1} \sum_{j=0}^{n-k} \binom{n+k-1}{j} 2^{-j} (-1)^{j} \binom{n+k-j}{n-k-j}.$$

Coefficient extractor enforces range:

$$2^{n+k-1} [z^{n-k}](1+z)^{n+k} \sum_{j\ge 0} \binom{n+k-1}{j} 2^{-j} (-1)^j \frac{z^j}{(1+z)^j}$$

$$= 2^{n+k-1} [z^{n-k}](1+z)^{n+k} \left(1 - \frac{z}{2(1+z)}\right)^{n+k-1}$$

$$= [z^{n-k}](1+z)(2+z)^{n+k-1}$$

$$= [z^{n-k}](2+z)^{n+k-1} + [z^{n-k-1}](2+z)^{n+k-1}$$

$$= \binom{n+k-1}{n-k} 2^{n+k-1-(n-k)} + \binom{n+k-1}{n-k-1} 2^{n+k-1-(n-k-1)}$$

$$= \frac{1}{2} 4^k \frac{2k}{n+k} \binom{n+k}{n-k} + \frac{n-k}{n+k} 4^k \binom{n+k}{n-k}$$

$$= \frac{4^k n}{n+k} \binom{n+k}{n-k}.$$

This was math.stackexchange.com problem 3583191.

69.21 MSE 3592240

We seek to verify that

$$\sum_{q=m}^{n-k} (-1)^{q-m} \binom{k-1+q}{k-1} \left\{ q \atop m \right\} \left[n \atop q+k \right] = \binom{n-1}{m} \left[n-m \atop k \right].$$

Using the standard EGFs the LHS becomes

$$\sum_{q=m}^{n-k} (-1)^{q-m} \binom{k-1+q}{k-1} q![z^q] \frac{(\exp(z)-1)^m}{m!} n![w^n] \frac{1}{(q+k)!} \left(\log \frac{1}{1-w}\right)^{q+k}$$

$$= \frac{n!}{(k-1)! \times m!} [w^n] \sum_{q=m}^{n-k} (-1)^{q-m} [z^q] (\exp(z)-1)^m \frac{1}{q+k} \left(\log \frac{1}{1-w}\right)^{q+k}$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \sum_{q=m}^{n-k} (-1)^{q-m} [z^q] (\exp(z)-1)^m \left(\log \frac{1}{1-w}\right)^{q+k-1} \frac{1}{1-w}$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times \sum_{q=m}^{n-k} (-1)^{q-m} [z^{q+k-1}] z^{k-1} (\exp(z)-1)^m \left(\log \frac{1}{1-w}\right)^{q+k-1}$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times \sum_{q=m+k-1}^{n-1} (-1)^{q-(k-1)-m} [z^q] z^{k-1} (\exp(z)-1)^m \left(\log \frac{1}{1-w}\right)^q.$$

Now as $\log \frac{1}{1-w} = w + \cdots$ when $q > n-1$ there is no contribution from the logarithmic power term due to the coefficient extractor $[w^{n-1}]$ so we find

$$(-1)^{m+(k-1)} \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times \sum_{q \ge m+k-1} (-1)^q \left(\log \frac{1}{1-w}\right)^q [z^q] z^{k-1} (\exp(z)-1)^m.$$

Note that $z^{k-1}(\exp(z)-1)^m = z^{m+k-1} + \cdots$ which means that the remaining sum / coefficient etractor pair covers the entire series and we get

$$(-1)^{m+(k-1)} \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times (-1)^{k-1} \left(\log \frac{1}{1-w}\right)^{k-1} \left(\exp\left(-\log \frac{1}{1-w}\right) - 1\right)^m$$

$$= (-1)^{m+(k-1)} \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1}] \frac{1}{1-w}$$

$$\times (-1)^{k-1} \left(\log \frac{1}{1-w} \right)^{k-1} (-w)^m$$

$$= \frac{(n-1)!}{(k-1)! \times m!} [w^{n-1-m}] \frac{1}{1-w} \left(\log \frac{1}{1-w} \right)^{k-1}$$

$$= \frac{(n-1)!}{m!} [w^{n-1-m}] \frac{1}{1-w} \frac{1}{(k-1)!} \left(\log \frac{1}{1-w} \right)^{k-1}$$

$$= \frac{(n-1)!}{m!} (n-m) [w^{n-m}] \frac{1}{k!} \left(\log \frac{1}{1-w} \right)^{k}$$

$$= \frac{(n-1)!}{m! \times (n-1-m)!} (n-m)! [w^{n-m}] \frac{1}{k!} \left(\log \frac{1}{1-w} \right)^{k}$$

$$= \binom{n-1}{m} \left[n-m \atop k \right].$$

This is the claim.
This was math.stackexchange.com problem 3592240.

69.22 MSE 3604802

We seek to evaluate

$$S(N) = \sum_{q=0}^{N} (-1)^q \binom{2q}{q} \binom{N+q}{N-q} \frac{q^2}{(q+1)^2}$$

or alternatively

$$S(N) = \sum_{q=0}^{N} (-1)^q \frac{(N+q)!}{(N-q)!(q-1)!^2} \frac{1}{(q+1)^2}.$$

This is

$$S(N) = \sum_{q=0}^{N} q^2 (-1)^q \frac{(N+q)!}{(N-q)!(q+1)!^2}$$

$$= \sum_{q=0}^{N} q^2 (-1)^q \binom{N+1}{q+1} \frac{(N+q)!}{(N+1)!(q+1)!}$$

$$= \frac{1}{N(N+1)} \sum_{q=0}^{N} q^2 (-1)^q \binom{N+1}{q+1} \frac{(N+q)!}{(N-1)!(q+1)!}$$

$$= \frac{1}{N(N+1)} \sum_{q=0}^{N} q^2 (-1)^q \binom{N+1}{q+1} \binom{N+q}{q+1}.$$

We continue with

$$\frac{1}{N(N+1)} \sum_{q=0}^{N} q^2 (-1)^q \binom{N+1}{N-q} \binom{N+q}{q+1}$$

$$= \frac{1}{N(N+1)} [z^N](1+z)^{N+1} \sum_{q=0}^{N} q^2 (-1)^q z^q \binom{N+q}{q+1}.$$

Here the coefficient extractor enforces the upper limit of the sum:

$$\frac{1}{N(N+1)} [z^N](1+z)^{N+1} \sum_{q\ge 0} q^2 (-1)^q z^q \binom{N+q}{N-1}$$

$$= \frac{1}{N(N+1)} [z^N](1+z)^{N+1} [w^{N-1}](1+w)^N \sum_{q\ge 0} q^2 (-1)^q z^q (1+w)^q$$

$$= \frac{1}{N(N+1)} [z^N](1+z)^{N+1} [w^{N-1}](1+w)^N \frac{-z(1+w)(1-z(1+w))}{(1+z(1+w))^3}$$

$$= -\frac{1}{N(N+1)} [z^{N-1}](1+z)^{N+1} [w^{N-1}](1+w)^{N+1} \frac{1-z(1+w)}{(1+z(1+w))^3}.$$

We have two pieces here, the first one is

$$-\frac{1}{N(N+1)} [z^{N-1}](1+z)^{N+1} [w^{N-1}](1+w)^{N+1} \frac{1}{(1+z(1+w))^3}$$

$$= -\frac{1}{N(N+1)} [z^{N-1}](1+z)^{N-2} [w^{N-1}](1+w)^{N+1} \frac{1}{(1+zw/(1+z))^3}.$$

The inner term is

$$\sum_{q=0}^{N-1} \binom{N+1}{N-1-q} (-1)^q \binom{q+2}{2} \frac{z^q}{(1+z)^q}.$$

Now

$$\binom{N+1}{N-1-q}\binom{q+2}{2} = \frac{(N+1)!}{(N-1-q)! \times q! \times 2!} = \binom{N+1}{2}\binom{N-1}{q}$$

and we find for the inner term

$$\binom{N+1}{2} \sum_{q=0}^{N-1} \binom{N-1}{q} (-1)^q \frac{z^q}{(1+z)^q} = \binom{N+1}{2} \left(1 - \frac{z}{1+z}\right)^{N-1}$$

$$= \binom{N+1}{2} \frac{1}{(1+z)^{N-1}}.$$

Substitute into the outer term to get

$$-\frac{1}{N(N+1)} [z^{N-1}](1+z)^{N-2} \binom{N+1}{2} \frac{1}{(1+z)^{N-1}}$$

$$= -\frac{1}{2} [z^{N-1}] \frac{1}{1+z} = \frac{1}{2}(-1)^N.$$

The second piece is

$$\frac{1}{N(N+1)} [z^{N-2}](1+z)^{N-2} [w^{N-1}](1+w)^{N+2} \frac{1}{(1+zw/(1+z))^3}.$$

For this piece we obtain

$$\frac{1}{N(N+1)} [z^{N-2}](1+z)^{N-2} \sum_{q=0}^{N-1} \binom{N+2}{N-1-q} (-1)^q \binom{q+2}{2} \frac{z^q}{(1+z)^q}.$$

The remaining coefficient extractor cancels the term for $q = N-1$:

$$\frac{1}{N(N+1)} [z^{N-2}](1+z)^{N-2} \sum_{q=0}^{N-2} \binom{N+2}{N-1-q} (-1)^q \binom{q+2}{2} \frac{z^q}{(1+z)^q}$$

$$= \frac{1}{N(N+1)} \sum_{q=0}^{N-2} \binom{N+2}{N-1-q} (-1)^q \binom{q+2}{2}$$

$$= -\frac{1}{N(N+1)} (-1)^{N-1} \binom{N+1}{2} + \frac{1}{N(N+1)} \sum_{q=0}^{N-1} \binom{N+2}{N-1-q} (-1)^q \binom{q+2}{2}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)} \sum_{q=0}^{N-1} \binom{N+2}{N-1-q} (-1)^q \binom{q+2}{2}.$$

Continuing, with the coefficient extractor enforcing the range,

$$\frac{1}{2}(-1)^N + \frac{1}{N(N+1)} [z^{N-1}](1+z)^{N+2} \sum_{q \ge 0} z^q (-1)^q \binom{q+2}{2}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)} [z^{N-1}](1+z)^{N+2} \frac{1}{(1+z)^3}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)} [z^{N-1}](1+z)^{N-1}$$

$$= \frac{1}{2}(-1)^N + \frac{1}{N(N+1)}.$$

Collecting the contributions from the two pieces we obtain at last

$$\boxed{(-1)^N + \frac{1}{N(N+1)}.}$$

This was math.stackexchange.com problem 3604802.

69.23 MSE 3619182

We seek to verify that

$$\sum_{k=0}^n {n\choose k}^2 \sum_{l=0}^k {k\choose l}{n\choose l}{2n-l\choose n} = \sum_{k=0}^n {n\choose k}^2 {n+k\choose k}^2.$$

Starting with the inner term on the LHS we have

$$\sum_{l=0}^k {k\choose k-l}{n\choose l}{2n-l\choose n}$$

$$= [z^k](1+z)^k \sum_{l=0}^k z^l {n\choose l}{2n-l\choose n}$$

$$= [z^k](1+z)^k [w^n](1+w)^{2n} \sum_{l=0}^k z^l {n\choose l}(1+w)^{-l}.$$

The coefficient extractor $[z^k]$ enforces the upper limit of the sum and we find

$$[z^k](1+z)^k [w^n](1+w)^{2n} \sum_{l\ge 0} z^l {n\choose l}(1+w)^{-l}$$

$$= [z^k](1+z)^k [w^n](1+w)^{2n} \left(1+\frac{z}{1+w}\right)^n$$

$$= [z^k](1+z)^k [w^n](1+w)^n (1+w+z)^n.$$

We get from the outer sum

$$\sum_{k=0}^n {n\choose k}^2 [z^k](1+z)^k [w^n](1+w)^n (1+w+z)^n$$

$$= \sum_{k=0}^n {n\choose k}^2 [z^n] z^k (1+z)^{n-k} [w^n](1+w)^n (1+w+z)^n$$

$$= [z^n](1+z)^n [w^n](1+w)^n (1+w+z)^n \sum_{k=0}^n {n\choose k}^2 z^k (1+z)^{-k}$$

$$= [z^n](1+z)^n[w^n](1+w)^n(1+w+z)^n[v^n](1+v)^n \sum_{k=0}^{n} \binom{n}{k} v^k z^k (1+z)^{-k}$$

$$= [z^n](1+z)^n[w^n](1+w)^n(1+w+z)^n[v^n](1+v)^n \left(1+\frac{vz}{1+z}\right)^n$$

$$= [z^n][w^n](1+w)^n(1+w+z)^n[v^n](1+v)^n(1+z+vz)^n.$$

Extracting the coefficient on $[z^n]$ we obtain

$$\sum_{k=0}^{n} ([z^{n-k}][w^n](1+w)^n(1+w+z)^n)([z^k][v^n](1+v)^n(1+z(1+v))^n)$$

$$= \sum_{k=0}^{n} \left(\binom{n}{n-k}[w^n](1+w)^{n+k}\right)\left(\binom{n}{k}[v^n](1+v)^{n+k}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2.$$

This is the claim.

This was math.stackexchange.com problem 3619182.

69.24 MSE 3638162

Suppose we seek to verify that

$$\sum_{k=1}^{a} (-1)^{a-k} \binom{a}{k}\binom{b+k}{b+1} = \binom{b}{a-1}.$$

We get

$$\sum_{k=1}^{a} [z^{a-k}](-1)^{a-k} \frac{1}{(1-z)^{k+1}} \binom{b+k}{b+1}$$

$$= [z^a]\frac{1}{1-z} \sum_{k=1}^{a} z^k (-1)^{a-k} \frac{1}{(1-z)^k} \binom{b+k}{b+1}.$$

Here the coefficient extractor enforces the upper limit of the sum and we find

$$[z^a]\frac{1}{1-z} \sum_{k\ge 1} z^k (-1)^{a-k} \frac{1}{(1-z)^k} \binom{b+k}{b+1}$$

$$= [z^a]\frac{1}{1-z}(-1)^{a-1}\frac{z}{1-z} \sum_{k\ge 0} z^k (-1)^k \frac{1}{(1-z)^k} \binom{b+1+k}{b+1}$$

$$= [z^{a-1}]\frac{(-1)^{a-1}}{(1-z)^2}\frac{1}{(1+z/(1-z))^{b+2}} = [z^{a-1}]\frac{(-1)^{a-1}}{(1-z)^2}(1-z)^{b+2}$$

$$= [z^{a-1}](-1)^{a-1}(1-z)^b = [z^{a-1}](1+z)^b = \binom{b}{a-1}.$$

This is the claim.

This was math.stackexchange.com problem 3638162.

69.25 MSE 3661349

We seek to show that

$$\sum_{q=0}^{k} (-1)^{q-j} \binom{n+q}{q} \binom{n+k-q}{k-q} \binom{2n}{n+j-q} = \binom{2n}{n}$$

where $0 \le j \le k$.

The LHS is

$$(-1)^j [w^{n+j}](1+w)^{2n} \sum_{q=0}^{k} (-1)^q \binom{n+q}{q} w^q \binom{n+k-q}{k-q}$$

$$= (-1)^j [w^{n+j}](1+w)^{2n} [z^k] \frac{1}{(1+wz)^{n+1}} \frac{1}{(1-z)^{n+1}}.$$

The inner term is

$$\mathrm{Res}_{z=0} \frac{1}{z^{k+1}} \frac{1}{(1+wz)^{n+1}} \frac{1}{(1-z)^{n+1}}.$$

Residues sum to zero and the residue at infinity is zero by inspection. We get for the residue at $z=1$

$$(-1)^{n+1} \mathrm{Res}_{z=1} \frac{1}{z^{k+1}} \frac{1}{(1+wz)^{n+1}} \frac{1}{(z-1)^{n+1}}$$

$$= (-1)^{n+1} \mathrm{Res}_{z=1} \frac{1}{(1+(z-1))^{k+1}} \frac{1}{(1+w+w(z-1))^{n+1}} \frac{1}{(z-1)^{n+1}}$$

$$= \frac{(-1)^{n+1}}{(1+w)^{n+1}} \mathrm{Res}_{z=1} \frac{1}{(1+(z-1))^{k+1}} \frac{1}{(1+w(z-1)/(1+w))^{n+1}} \frac{1}{(z-1)^{n+1}}$$

$$= \frac{(-1)^{n+1}}{(1+w)^{n+1}} \sum_{q=0}^{n} \binom{n+q}{q} (-1)^q \frac{w^q}{(1+w)^q} (-1)^{n-q} \binom{k+n-q}{k}$$

$$= -\sum_{q=0}^{n} \binom{n+q}{q} \frac{w^q}{(1+w)^{n+1+q}} \binom{k+n-q}{k}.$$

Substitute into the coefficient extractor in w to get

$$-(-1)^j \sum_{q=0}^{n} \binom{n+q}{q} \binom{k+n-q}{k} [w^{n+j-q}](1+w)^{n-1-q}.$$

Now with $0 \le q \le n-1$ and $j \ge 0$ we have $[w^{n+j-q}](1+w)^{n-1-q} = 0$. This leaves $q = n$ which yields

$$-(-1)^j \binom{2n}{n}\binom{k}{k}[w^j]\frac{1}{1+w} = -\binom{2n}{n}.$$

This is the claim. We have the result if we can show that the residue at $z = -1/w$ makes for a zero contribution. We get

$$\frac{1}{w^{n+1}}\mathrm{Res}_{z=-1/w}\frac{1}{z^{k+1}}\frac{1}{(z+1/w)^{n+1}}\frac{1}{(1-z)^{n+1}}.$$

This requires

$$\frac{1}{n!}\left(\frac{1}{z^{k+1}}\frac{1}{(1-z)^{n+1}}\right)^{(n)} = \frac{1}{n!}\sum_{q=0}^{n}\binom{n}{q}\frac{(-1)^q(k+q)!}{z^{k+1+q}\times k!}\frac{(n+n-q)!}{(1-z)^{n+1+n-q}\times n!}$$

$$= \sum_{q=0}^{n}\binom{k+q}{k}(-1)^q\frac{1}{z^{k+1+q}}\binom{2n-q}{n}\frac{1}{(1-z)^{2n+1-q}}.$$

Evaluate at $z = -1/w$ and restore the factor in front:

$$\frac{1}{w^{n+1}}\sum_{q=0}^{n}\binom{k+q}{k}(-1)^{k+1}w^{k+1+q}\binom{2n-q}{n}\frac{1}{(1+1/w)^{2n+1-q}}.$$

Applying the coefficient extractor in w we get

$$(-1)^j[w^{n+j}](1+w)^{2n}\frac{1}{w^{n+1}}w^{k+1+q}\frac{w^{2n+1-q}}{(1+w)^{2n+1-q}}$$
$$= (-1)^j[w^{n+j}](1+w)^{q-1}w^{n+k+1} = (-1)^j[w^j](1+w)^{q-1}w^{k+1} = 0$$

because $j \le k$. This concludes the argument.
This was math.stackexchange.com problem 3661349.

69.26 MSE 3706767

We seek to verify that

$$S_{n,m} = \sum_{k=m}^{n}\binom{k+m}{2m}\binom{2n+1}{n+k+1} = \binom{n}{m}4^{n-m}.$$

The LHS is

$$\sum_{k=0}^{n-m}\binom{k+2m}{2m}\binom{2n+1}{n+m+k+1}$$

$$= \sum_{k=0}^{n-m} \binom{k+2m}{2m} [z^{n-m-k}] \frac{1}{(1-z)^{n+m+k+2}}$$

$$= [z^{n-m}] \frac{1}{(1-z)^{n+m+2}} \sum_{k=0}^{n-m} \binom{k+2m}{2m} \frac{z^k}{(1-z)^k}.$$

Now when $k > n-m$ there is no contribution to the coefficient extractor and we may continue with

$$[z^{n-m}] \frac{1}{(1-z)^{n+m+2}} \sum_{k\ge 0} \binom{k+2m}{2m} \frac{z^k}{(1-z)^k}$$

$$= [z^{n-m}] \frac{1}{(1-z)^{n+m+2}} \frac{1}{(1-z/(1-z))^{2m+1}}$$

$$= [z^{n-m}] \frac{1}{(1-z)^{n-m+1}} \frac{1}{(1-2z)^{2m+1}}.$$

This yields

$$S_{n,m} = \mathrm{Res}_{z=0} \frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \frac{1}{(1-2z)^{2m+1}}.$$

Residues sum to zero and the residue at infinity is zero by inspection. We get for the residue at $z = 1$

$$\mathrm{Res}_{z=1} \frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \frac{1}{(1-2z)^{2m+1}}.$$

Setting $z = 1 - u$ we get

$$-\mathrm{Res}_{u=0} \frac{1}{(1-u)^{n-m+1}} \frac{1}{u^{n-m+1}} \frac{1}{(1-2(1-u))^{2m+1}}$$

$$= -\mathrm{Res}_{u=0} \frac{1}{(1-u)^{n-m+1}} \frac{1}{u^{n-m+1}} \frac{1}{(2u-1)^{2m+1}}$$

$$= \mathrm{Res}_{u=0} \frac{1}{(1-u)^{n-m+1}} \frac{1}{u^{n-m+1}} \frac{1}{(1-2u)^{2m+1}} = S_{n,m}.$$

Continuing with the residue at $z = 1/2$ we find

$$-\frac{1}{2^{2m+1}} \mathrm{Res}_{z=1/2} \frac{1}{z^{n-m+1}} \frac{1}{(1-z)^{n-m+1}} \frac{1}{(z-1/2)^{2m+1}}$$

$$= -\frac{1}{2^{2m+1}} \mathrm{Res}_{z=1/2} \frac{1}{(1/2+(z-1/2))^{n-m+1}} \frac{1}{(1/2-(z-1/2))^{n-m+1}} \times \frac{1}{(z-1/2)^{2m+1}}$$

$$= -\frac{1}{2^{2m+1}} \mathrm{Res}_{z=1/2} \frac{1}{(1/4-(z-1/2)^2)^{n-m+1}} \frac{1}{(z-1/2)^{2m+1}}$$

$$= -\frac{1}{2^{2m+1}} \mathrm{Res}_{z=1/2} \frac{4^{n-m+1}}{(1-4(z-1/2)^2)^{n-m+1}} \frac{1}{(z-1/2)^{2m+1}}$$

$$= -\frac{2^{2n-2m+2}}{2^{2m+1}} [(z-1/2)^{2m}] \frac{1}{(1-4(z-1/2)^2)^{n-m+1}}$$

$$= -\frac{2^{2n-2m+2}}{2^{2m+1}} [(z-1/2)^{m}] \frac{1}{(1-4(z-1/2))^{n-m+1}}$$

$$= -\frac{2^{2n-2m+2}}{2^{2m+1}} \binom{m+n-m}{n-m} 2^{2m}.$$

We have shown that

$$S_{n,m} + S_{n,m} - 2^{2n-2m+1}\binom{n}{m} = 0$$

which is at last

$$\boxed{S_{n,m} = \binom{n}{m} 4^{n-m}.}$$

This was math.stackexchange.com problem 3706767.

69.27 MSE 3737197

We seek to show that

$$\sum_{j=0}^{k} \binom{k}{j}\binom{j/2}{n}(-1)^{n+k-j} = \frac{k}{n}(-1)^k 2^{k-2n}\binom{2n-k-1}{n-1}$$

where $n \ge k \ge 0$. We get for the even component

$$\sum_{p=0}^{\lfloor k/2 \rfloor} \binom{k}{2p}\binom{p}{n}(-1)^{n+k} = 0$$

because $n > p$ and $p \ge 0$. This leaves the odd component

$$-(-1)^{n+k} \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2p+1}\binom{p+1/2}{n}.$$

Now we have

$$\binom{p+1/2}{n} = \frac{1}{n!}\prod_{q=0}^{n-1}(p+1/2-q) = \frac{1}{2^n n!}\prod_{q=0}^{n-1}(2p+1-2q)$$

$$= \frac{1}{2^n n!}\prod_{q=0}^{p}(2p+1-2q) \prod_{q=p+1}^{n-1}(2p+1-2q)$$

$$= \frac{1}{2^n n!} \frac{(2p+2)!}{2^{p+1}(p+1)!} (-1)^{n-p-1} \prod_{q=p+1}^{n-1} (2q-2p-1)$$

$$= \frac{1}{2^n n!} \frac{(2p+2)!}{2^{p+1}(p+1)!} (-1)^{n-p-1} \frac{(2n-2p-2)!}{2^{n-p-1}(n-p-1)!}$$

$$= \frac{(-1)^{n-p-1}(2n)!}{2^{2n} n!^2} {2n \choose 2p+2}^{-1} {n \choose p+1}$$

$$= \frac{(-1)^{n-p-1}}{2^{2n}} {2n \choose n} {2n \choose 2p+2}^{-1} {n \choose p+1}.$$

where $p < n$. It will be helpful to re-write this as

$$\frac{p+1}{n} \frac{(-1)^{n-p-1}}{2^{2n}} {2n \choose n} {2n-1 \choose 2p+1}^{-1} {n \choose p+1}$$

$$= \frac{(-1)^{n-p-1}}{2^{2n}} {2n \choose n} {2n-1 \choose 2p+1}^{-1} {n-1 \choose p}.$$

We thus get for our sum

$$\frac{(-1)^k}{2^{2n}} {2n \choose n} \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} (-1)^p {k \choose 2p+1} {2n-1 \choose 2p+1}^{-1} {n-1 \choose p}.$$

Now observe that

$${k \choose 2p+1} {2n-1 \choose 2p+1}^{-1} = \frac{k!}{(k-2p-1)!} \frac{(2n-2p-2)!}{(2n-1)!}$$

$$= {2n-1 \choose k}^{-1} {2n-2p-2 \choose k-2p-1}.$$

This yields for the sum

$$\frac{(-1)^k}{2^{2n}} {2n \choose n} {2n-1 \choose k}^{-1} \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} (-1)^p {2n-2p-2 \choose k-2p-1} {n-1 \choose p}.$$

Now to treat the remaining sum we have

$$[z^k](1+z)^{2n-2} \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} (-1)^p z^{2p+1} (1+z)^{-2p} {n-1 \choose p}.$$

The coefficient extractor enforces the upper limit $\lfloor (k-1)/2 \rfloor \geq p$ so we may continue with

$$[z^k](1+z)^{2n-2} \sum_{p \geq 0} (-1)^p z^{2p+1} (1+z)^{-2p} {n-1 \choose p}$$

$$= [z^k](1+z)^{2n-2} z \left(1 - \frac{z^2}{(1+z)^2}\right)^{n-1}$$
$$= [z^k] z(1+2z)^{n-1}.$$

This means for $k=0$ the sum is zero. For $k \ge 1$ we get including the factor in front

$$\boxed{\frac{(-1)^k}{2^{2n}} \binom{2n}{n} \binom{2n-1}{k}^{-1} \binom{n-1}{k-1} 2^{k-1}.}$$

To simplify this we expand the binomial coefficients

$$\frac{(-1)^k}{2^{2n-k+1}} \frac{(2n)! \times k! \times (2n-1-k)! \times (n-1)!}{n! \times n! \times (2n-1)! \times (k-1)! \times (n-k)!}$$
$$= \frac{(-1)^k}{2^{2n-k+1}} \frac{(2n) \times k \times (2n-1-k)!}{n \times n! \times (n-k)!}$$
$$= \frac{(-1)^k}{2^{2n-k}} \frac{k \times (2n-1-k)!}{n! \times (n-k)!}.$$

This yields at last

$$\boxed{\frac{(-1)^k}{2^{2n-k}} \frac{k}{n} \binom{2n-1-k}{n-1}.}$$

This was math.stackexchange.com problem 3737197.

69.28 MSE 3825092

We seek to show that

$$S(n) = \sum_{k=1}^{n} (-1)^{n-k} k^n \binom{n+1}{n-k} = 1.$$

Replacing the binomial coefficient first we find

$$[z^n](1+z)^{n+1} \sum_{k=1}^{n} (-1)^{n-k} k^n z^k$$

The coefficient extractor enforces the range and hence we have

$$[z^n](1+z)^{n+1} \sum_{k\ge 1} (-1)^{n-k} k^n z^k$$
$$= n![w^n][z^n](1+z)^{n+1} \sum_{k\ge 1} (-1)^{n-k} \exp(kw) z^k.$$

The contribution from $k=0$ is zero due to $[w^n]$ where $n \ge 1$ so this is

$$n![w^n][z^n](1+z)^{n+1}\sum_{k\ge 0}(-1)^{n-k}\exp(kw)z^k$$

$$= n!(-1)^n[w^n][z^n](1+z)^{n+1}\frac{1}{1+\exp(w)z}.$$

The coefficient extractor in z is

$$\mathrm{Res}_{z=0}\frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{1+\exp(w)z}.$$

Now we put $z/(1+z) = v$ so that $z = v/(1-v)$ and $dz = 1/(1-v)^2\; dv$ so that we find

$$\mathrm{Res}_{v=0}\frac{1}{v^{n+1}}\frac{1}{1+\exp(w)v/(1-v)}\frac{1}{(1-v)^2}$$

$$= \mathrm{Res}_{v=0}\frac{1}{v^{n+1}}\frac{1}{1-v(1-\exp(w))}\frac{1}{1-v}$$

$$= \sum_{k=0}^{n}(1-\exp(w))^k.$$

Observe that $(1-\exp(w))^k = (-1)^k w^k + \cdots$ so this is taking into account $[w^n]$

$$n!(-1)^n[w^n]\sum_{k\ge 0}(1-\exp(w))^k$$

$$= n!(-1)^n[w^n]\frac{1}{1-(1-\exp(w))} = n!(-1)^n[w^n]\exp(-w) = 1.$$

This is the claim.

This was math.stackexchange.com problem 3825092.

69.29 MSE 3845061

We seek to show that

$$\sum_{q=a+1}^{n}\binom{q-1}{a}\binom{n-q}{k-a} = \binom{n}{k+1}$$

or alternatively

$$\sum_{q=0}^{n}\binom{q}{a}\binom{n-q}{b} = \binom{n+1}{a+b+1}.$$

where $k \ge a$ for the binomial coefficient to be defined, and $n \ge a+1$ or alternatively

$$\sum_{q=0}^{n-a-1} \binom{q+a}{a} \binom{n-a-1-q}{k-a} = \binom{n}{k+1}.$$

The LHS is

$$[z^{k-a}](1+z)^{n-a-1} \sum_{q\ge 0} \binom{q+a}{a} (1+z)^{-q} [[q \le n-a-1]]$$

$$= [z^{k-a}](1+z)^{n-a-1} \sum_{q\ge 0} \binom{q+a}{a} (1+z)^{-q} [w^{n-a-1}] \frac{w^q}{1-w}$$

$$= [z^{k-a}](1+z)^{n-a-1} [w^{n-a-1}] \frac{1}{1-w} \sum_{q\ge 0} \binom{q+a}{a} (1+z)^{-q} w^q$$

$$= [z^{k-a}](1+z)^{n-a-1} [w^{n-a-1}] \frac{1}{1-w} \frac{1}{(1-w/(1+z))^{a+1}}$$

$$= [z^{k-a}](1+z)^{n} [w^{n-a-1}] \frac{1}{1-w} \frac{1}{(1+z-w)^{a+1}}.$$

This is

$$[z^{k-a}](1+z)^n (-1)^a \mathrm{Res}_{w=0} \frac{1}{w^{n-a}} \frac{1}{w-1} \frac{1}{(w-(1+z))^{a+1}}.$$

Now the residue at infinity for w is zero by inspection, residues sum to zero and the residue at $w=1$ yields

$$[z^{k-a}](1+z)^n (-1)^a \frac{1}{(-1)^{a+1} z^{a+1}} = -\binom{n}{k+1}.$$

This is the claim if we can show that the contribution from the pole at $w = 1+z$ is zero. We get (Leibniz rule)

$$\frac{1}{a!} \left(\frac{1}{w^{n-a}} \frac{1}{w-1} \right)^{(a)} = \frac{1}{a!} \sum_{q=0}^{a} \binom{a}{q} \frac{(-1)^q (n-1-a+q)!}{(n-1-a)! \times w^{n-a+q}} \frac{(-1)^{a-q}(a-q)!}{(w-1)^{a+1-q}}$$

$$= (-1)^a \sum_{q=0}^{a} \binom{n-1-a+q}{q} \frac{1}{w^{n-a+q}} \frac{1}{(w-1)^{a+1-q}}.$$

We thus obtain for the contribution

$$[z^{k-a}](1+z)^n \sum_{q=0}^{a} \binom{n-1-a+q}{q} \frac{1}{(1+z)^{n-a+q}} \frac{1}{z^{a+1-q}}$$

$$= \sum_{q=0}^{a} \binom{n-1-a+q}{q} [z^{k+1-q}](1+z)^{a-q} = 0$$

because $a \ge q$ and $k+1 > a$. This concludes the argument.

This was math.stackexchange.com problem 3845061.

69.30 MSE 3885278

Introduction

The identity

$$\sum_{k\ge 0} \frac{(2k+1)^2}{(p+k+1)(q+k+1)} \binom{2p}{p-k}\binom{2q}{q-k} = \frac{1}{p+q+1}\binom{2p+2q}{p+q}$$

is identical to

$$\sum_{k=0}^{\min(p,q)} (2k+1)^2 \binom{2p+1}{p+k+1}\binom{2q+1}{q+k+1} = \frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}$$

or

$$\sum_{k=0}^{\min(p,q)} (2k+1)^2 \binom{2p+1}{p-k}\binom{2q+1}{q-k} = \frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}.$$

The LHS is

$$S = [z^p](1+z)^{2p+1}[w^q](1+w)^{2q+1}\sum_{k=0}^{\min(p,q)}(2k+1)^2 z^k w^k.$$

The two coefficient extractors enforce the upper limit of the sum:

$$[z^p](1+z)^{2p+1}[w^q](1+w)^{2q+1}\sum_{k\ge 0}(2k+1)^2 z^k w^k$$

$$= [z^p](1+z)^{2p+1}[w^q](1+w)^{2q+1}\frac{z^2w^2+6zw+1}{(1-zw)^3}$$

$$= -[z^p]\frac{1}{z^3}(1+z)^{2p+1}[w^q](1+w)^{2q+1}\frac{z^2w^2+6zw+1}{(w-1/z)^3}$$

$$= -[z^{p+3}](1+z)^{2p+1}[w^q](1+w)^{2q+1}\frac{z^2w^2+6zw+1}{(w-1/z)^3}.$$

The coefficient extractor in w is

$$\mathrm{Res}_{w=0}\frac{1}{w^{q+1}}(1+w)^{2q+1}\frac{z^2w^2+6zw+1}{(w-1/z)^3}.$$

Residue at infinity

Now residues sum to zero and the residue at infinity is given by

$$-\mathrm{Res}_{w=0} \frac{1}{w^2} w^{q+1} \frac{(1+w)^{2q+1}}{w^{2q+1}} \frac{z^2/w^2+6z/w+1}{(1/w-1/z)^3}$$
$$= -\mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+2}} \frac{z^2w+6zw^2+w^3}{(1-w/z)^3}$$
$$= -\mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \frac{z^2+6zw+w^2}{(1-w/z)^3}.$$

Next applying the coefficient extractor in z we find

$$\mathrm{Res}_{z=0} \frac{(1+z)^{2p+1}}{z^{p+4}} \mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \frac{z^2+6zw+w^2}{(1-w/z)^3}$$
$$= \mathrm{Res}_{z=0} \frac{(1+z)^{2p+1}}{z^{p+2}} \mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \frac{1+6w/z+w^2/z^2}{(1-w/z)^3}$$
$$= \mathrm{Res}_{z=0} \frac{(1+z)^{2p+1}}{z^{p+2}} \mathrm{Res}_{w=0} \frac{(1+w)^{2q+1}}{w^{q+1}} \sum_{k\ge 0} (2k+1)^2 \frac{w^k}{z^k}$$
$$= \sum_{k\ge 0} (2k+1)^2 \binom{2p+1}{p+k+1} \binom{2q+1}{q-k} = S.$$

This means that S is minus half the residue at $w = 1/z$, substituted into the coefficient extractor in z.

Residue at $w = 1/z$

The residue at $w = 1/z$ is

$$\mathrm{Res}_{w=1/z} \frac{1}{w^{q+1}} (1+w)^{2q+1} \frac{z^2w^2+6zw+1}{(w-1/z)^3}$$
$$= \mathrm{Res}_{w=1/z} \frac{1}{w^{q+1}} (1+w)^{2q+1} \left(\frac{8}{(w-1/z)^3} + \frac{8z}{(w-1/z)^2} + \frac{z^2}{w-1/z} \right).$$

Evaluating the three pieces in turn we start with

$$8\frac{1}{2} \left(\frac{(1+w)^{2q+1}}{w^{q+1}} \right)'' = 4(q+1)(q+2) \frac{(1+w)^{2q+1}}{w^{q+3}}$$
$$-8(q+1)(2q+1) \frac{(1+w)^{2q}}{w^{q+2}} + 4(2q+1)(2q) \frac{(1+w)^{2q-1}}{w^{q+1}}.$$

Evaluate at $w = 1/z$ to get

$$4(q+1)(q+2) \frac{(1+z)^{2q+1}}{z^{q-2}}$$

$$-8(q+1)(2q+1)\frac{(1+z)^{2q}}{z^{q-2}} + 4(2q+1)(2q)\frac{(1+z)^{2q-1}}{z^{q-2}}.$$

Substituting into the coefficient extractor in z we find

$$-4(q+1)(q+2)\binom{2p+2q+2}{p+q+1}$$
$$+8(q+1)(2q+1)\binom{2p+2q+1}{p+q+1} - 4(2q+1)(2q)\binom{2p+2q}{p+q+1}.$$

Continuing with the middle piece we have

$$8z\left(\frac{(1+w)^{2q+1}}{w^{q+1}}\right)' = -8z(q+1)\frac{(1+w)^{2q+1}}{w^{q+2}} + 8z(2q+1)\frac{(1+w)^{2q}}{w^{q+1}}.$$

Evaluate at $w = 1/z$ to get

$$-8(q+1)\frac{(1+z)^{2q+1}}{z^{q-2}} + 8(2q+1)\frac{(1+z)^{2q}}{z^{q-2}}.$$

The coefficient extractor now yields

$$8(q+1)\binom{2p+2q+2}{p+q+1} - 8(2q+1)\binom{2p+2q+1}{p+q+1}.$$

The third and last piece produces

$$\frac{(1+z)^{2q+1}}{z^{q-2}}$$

which when substituted into the coefficient extractor yields

$$-\binom{2p+2q+2}{p+q+1}.$$

Collecting the three pieces

We get

$$-(2q+1)^2\binom{2p+2q+2}{p+q+1} + 8q(2q+1)\binom{2p+2q+1}{p+q+1} - 8q(2q+1)\binom{2p+2q}{p+q+1}$$
$$= -(2q+1)^2\binom{2p+2q+2}{p+q+1} + 8q(2q+1)\binom{2p+2q}{p+q}$$
$$= -2(2q+1)^2\binom{2p+2q+1}{p+q} + 8q(2q+1)\binom{2p+2q}{p+q}$$
$$= -2(2q+1)^2\frac{2p+2q+1}{p+q+1}\binom{2p+2q}{p+q} + 8q(2q+1)\binom{2p+2q}{p+q}$$

$$= -2\frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}.$$

Halve this value and flip the sign to obtain the coveted

$$\boxed{\frac{(2p+1)(2q+1)}{p+q+1}\binom{2p+2q}{p+q}.}$$

This was math.stackexchange.com problem 3885278.

69.31 MSE 3559223

We seek to evaluate

$$G_{n,j} = \sum_{k=1}^{n} \frac{k^j (-1)^{n-k}\binom{n}{k}}{\frac{1}{2}n(n+1)-k}.$$

With this in mind we introduce the function

$$F_n(z) = n!\frac{z^{j-1}}{\frac{1}{2}n(n+1)-z}\prod_{q=1}^{n}\frac{1}{z-q}.$$

This has the property that the residue at $z = k$ where $1 \le k \le n$ is the desired sum term. We find

$$\begin{aligned}\mathrm{Res}_{z=k} F_n(z) &= n!\frac{k^{j-1}}{\frac{1}{2}n(n+1)-k}\prod_{q=1}^{k-1}\frac{1}{k-q}\prod_{q=k+1}^{n}\frac{1}{k-q} \\ &= n!\frac{k^j}{\frac{1}{2}n(n+1)-k}\frac{1}{k}\frac{1}{(k-1)!}\frac{(-1)^{n-k}}{(n-k)!} \\ &= \frac{k^j}{\frac{1}{2}n(n+1)-k}(-1)^{n-k}\binom{n}{k}.\end{aligned}$$

We will evaluate this using the fact that residues sum to zero and if $(n+1)-(j-1) \ge 2$ or $n \ge j$ the residue at infinity is zero, so we have in this case

$$G_{n,j} = -\mathrm{Res}_{z=\frac{1}{2}n(n+1)} F_n(z) = n!\frac{(\frac{1}{2}n(n+1))^{j-1}}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)}.$$

We thus have

$$\boxed{G_{n,1} = \frac{n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)}}$$

and

$$\boxed{G_{n,n} = \frac{(\frac{1}{2}n(n+1))^{n-1}n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)}.}$$

When $j > n$ we must use the formula

$$G_{n,j} = -\mathrm{Res}_{z=\frac{1}{2}n(n+1)} F_n(z) - \mathrm{Res}_{z=\infty} F_n(z).$$

We have

$$-\mathrm{Res}_{z=\infty} F_n(z) = \mathrm{Res}_{z=0} \frac{1}{z^2} F_n(1/z)$$

$$= n! \times \mathrm{Res}_{z=0} \frac{1}{z^2} \frac{1}{z^{j-1}} \frac{1}{\frac{1}{2}n(n+1) - 1/z} \prod_{q=1}^{n} \frac{1}{1/z - q}$$

$$= n! \times \mathrm{Res}_{z=0} \frac{1}{z^{j+1}} \frac{z}{\frac{1}{2}n(n+1)z - 1} \prod_{q=1}^{n} \frac{z}{1 - qz}$$

$$= n! \times \mathrm{Res}_{z=0} \frac{1}{z^{j-n}} \frac{1}{\frac{1}{2}n(n+1)z - 1} \prod_{q=1}^{n} \frac{1}{1 - qz}.$$

In particular when $j = n+1$ we just need the constant term and find

$$n! \frac{1}{\frac{1}{2}n(n+1) \times 0 - 1} \prod_{q=1}^{n} \frac{1}{1 - q \times 0} = -n!$$

we thus have

$$\boxed{G_{n,n+1} = \frac{(\frac{1}{2}n(n+1))^{n}n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)} - n!.}$$

The general case for $j > n$ is

$$n! \times \mathrm{Res}_{z=0} \frac{1}{z^{j}} \frac{1}{\frac{1}{2}n(n+1)z - 1} \prod_{q=1}^{n} \frac{z}{1 - qz}$$

which yields

$$-n! \sum_{q=0}^{j-1} \left(\frac{1}{2}n(n+1)\right)^q \left\{ {j-1-q \atop n} \right\}$$

so that the closed form is (here we must have $j - 1 - q \ge n$)

$$\boxed{G_{n,j} = \frac{(\frac{1}{2}n(n+1))^{j-1}n!}{\prod_{q=1}^{n}(\frac{1}{2}n(n+1)-q)} - [[j > n]] n! \sum_{q=0}^{j-1-n} \left(\frac{1}{2}n(n+1)\right)^q \left\{ {j-1-q \atop n} \right\}.}$$

This was math.stackexchange.com problem 3559223.

69.32 MSE 3926409

Suppose we seek an alternate representation of

$$\sum_{p=q}^{k}(-1)^p\binom{k}{p}(q-p)^k.$$

This is

$$\sum_{p=0}^{k}(-1)^p\binom{k}{p}(q-p)^k - \sum_{p=0}^{q-1}(-1)^p\binom{k}{p}(q-p)^k.$$

We get for the first piece

$$k![z^k]\sum_{p=0}^{k}(-1)^p\binom{k}{p}\exp((q-p)z)$$

$$= k![z^k]\exp(qz)\sum_{p=0}^{k}(-1)^p\binom{k}{p}\exp(-pz)$$

$$= k![z^k]\exp(qz)(1-\exp(-z))^k.$$

Now $(1-\exp(-z))^k = z^k + \cdots$ so this evaluates to $k!$. We thus have

$$k! - \sum_{p=0}^{q-1}(-1)^p\binom{k}{p}(q-p)^k.$$

Using an Iverson bracket we get for the sum component

$$[w^{q-1}]\frac{1}{1-w}\sum_{p\ge 0}(-1)^p\binom{k}{p}(q-p)^k w^p$$

$$= k![z^k][w^{q-1}]\frac{1}{1-w}\exp(qz)(1-w\exp(-z))^k$$

$$= k!\mathrm{Res}_{z=0}\frac{1}{z^{k+1}}\mathrm{Res}_{w=0}\frac{1}{w^q}\frac{1}{1-w}\exp(qz)(1-w\exp(-z))^k.$$

We now apply Jacobi's Residue Formula. We put $w = v\exp((1-v)u)$ and $z = (1-v)u$. The scalar to obtain a non-zero constant term in u and v for z and w is u for z and v for w. Using the determinant of the Jacobian we obtain

$$\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}^{-1}\begin{vmatrix}1-v & -u\\ v(1-v)\exp((1-v)u) & \exp((1-v)u) - uv\exp((1-v)u)\end{vmatrix}$$

$$= \exp((1-v)u)\begin{vmatrix}1-v & -u\\ v(1-v) & 1-uv\end{vmatrix}$$

$$= \exp((1-v)u)(1 - uv - v + uv^2 + uv - uv^2)$$
$$= \exp((1-v)u)(1-v).$$

Doing the substitution we find

$$k!\mathrm{Res}_{u=0} \frac{1}{u^{k+1}} \frac{1}{(1-v)^{k+1}} \mathrm{Res}_{v=0} \frac{1}{v^q} \frac{1}{\exp(q(1-v)u)}$$
$$\times \frac{1}{1 - v\exp((1-v)u)} \exp(q(1-v)u)(1 - v\exp((1-v)u)\exp(-(1-v)u))^k$$
$$\times \exp((1-v)u)(1-v)$$
$$= k!\mathrm{Res}_{u=0} \frac{1}{u^{k+1}} \frac{1}{(1-v)^{k+1}} \mathrm{Res}_{v=0} \frac{1}{v^q} \frac{1}{1 - v\exp((1-v)u)} (1-v)^k$$
$$\times \exp((1-v)u)(1-v)$$
$$= k!\mathrm{Res}_{u=0} \frac{1}{u^{k+1}} \mathrm{Res}_{v=0} \frac{1}{v^q} \frac{1}{1 - v\exp((1-v)u)} \exp((1-v)u)$$
$$= k!\mathrm{Res}_{u=0} \frac{1}{u^{k+1}} \mathrm{Res}_{v=0} \frac{1}{v^q} \frac{1}{\exp((v-1)u) - v}.$$

Consider on the other hand the quantity

$$\sum_{p=0}^{q-1} \left\langle {k \atop p} \right\rangle.$$

This is

$$k![z^k] \sum_{p=0}^{q-1} [w^p] \frac{1-w}{\exp((w-1)z) - w}$$
$$= k![z^k][w^{q-1}] \frac{1}{1-w} \frac{1-w}{\exp((w-1)z) - w}$$
$$= k![z^k][w^{q-1}] \frac{1}{\exp((w-1)z) - w}$$
$$= k!\mathrm{Res}_{z=0} \frac{1}{z^{k+1}} \mathrm{Res}_{w=0} \frac{1}{w^q} \frac{1}{\exp((w-1)z) - w}.$$

This is the same as the sum term and we conclude the argument having shown that

$$\sum_{p=q}^{k} (-1)^p \binom{k}{p} (q-p)^k = k! - \sum_{p=0}^{q-1} \left\langle {k \atop p} \right\rangle$$

which is

$$\boxed{\sum_{p=q}^{k} (-1)^p \binom{k}{p} (q-p)^k = \sum_{p=q}^{k} \left\langle {k \atop p} \right\rangle.}$$

The reference for Jacobi's Residue Formula is Theorem 3 in [Ges87].

This was math.stackexchange.com problem 3926409.

69.33 MSE 3942039

We seek to verify that

$$\sum_{k=0}^{n} (-1)^k \frac{2^{n-k}\binom{n}{k}}{(m+k+1)\binom{m+k}{k}} = \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k+1}.$$

We can re-write this as

$$\frac{m!n!}{(n+m+1)!} 2^n \sum_{k=0}^{n} (-1)^k 2^{-k} \binom{n+m+1}{n-k} = \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k+1}$$

or

$$2^n \sum_{k=0}^{n} (-1)^k 2^{-k} \binom{n+m+1}{n-k} = (m+1)\binom{n+m+1}{n} \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k+1}.$$

We get for the LHS

$$2^n \sum_{k=0}^{n} (-1)^k 2^{-k} \binom{n+m+1}{m+k+1} = 2^n \sum_{k=0}^{n} (-1)^k 2^{-k} [z^{n-k}] \frac{1}{(1-z)^{m+k+2}}$$

$$= 2^n [z^n] \frac{1}{(1-z)^{m+2}} \sum_{k=0}^{n} (-1)^k 2^{-k} z^k \frac{1}{(1-z)^k}.$$

Here the coefficient extractor enforces the range and we find

$$2^n [z^n] \frac{1}{(1-z)^{m+2}} \sum_{k\ge 0} (-1)^k 2^{-k} z^k \frac{1}{(1-z)^k} = 2^n [z^n] \frac{1}{(1-z)^{m+2}} \frac{1}{1+z/(1-z)/2}$$

$$= [z^n] \frac{1}{(1-2z)^{m+2}} \frac{1}{1+z/(1-2z)} = [z^n] \frac{1}{(1-2z)^{m+1}} \frac{1}{1-z}.$$

On the other hand we have

$$\binom{n+m+1}{n}\binom{n}{k} = \frac{(n+m+1)!}{(m+1)! \times k! \times (n-k)!} = \binom{n+m+1}{n-k}\binom{m+k+1}{m+1}$$

which gives for the RHS

$$\sum_{k=0}^{n} \binom{n+m+1}{n-k} \frac{m+1}{m+k+1} \binom{m+k+1}{m+1} = \sum_{k=0}^{n} \binom{n+m+1}{m+k+1} \binom{m+k}{m}$$

$$= \sum_{k=0}^{n} \binom{m+k}{m} [z^{n-k}] \frac{1}{(1-z)^{m+k+2}} = [z^n] \sum_{k=0}^{n} \binom{m+k}{m} \frac{1}{(1-z)^{m+k+2}} z^k.$$

We once more have the coefficient extractor enforcing the range and we get

$$[z^n] \frac{1}{(1-z)^{m+2}} \sum_{k\ge 0} \binom{m+k}{m} \frac{1}{(1-z)^k} z^k$$

$$= [z^n] \frac{1}{(1-z)^{m+2}} \frac{1}{(1-z/(1-z))^{m+1}} = [z^n] \frac{1}{1-z} \frac{1}{(1-2z)^{m+1}}.$$

The LHS is the same as the RHS which concludes the argument. The coefficient extractor evaluates to

$$\boxed{\sum_{k=0}^{n} \binom{k+m}{m} 2^k.}$$

This was math.stackexchange.com problem 3942039.

69.34 MSE 3956698

The sum in the problem statement here is

$$\sum_{k\ge 1} \left[\binom{\lfloor \frac{k}{2} \rfloor}{m} + \binom{\lceil \frac{k}{2} \rceil}{m} \right] \binom{n-1}{k-1}$$

$$= 2 \sum_{k\ge 0} \binom{k}{m} \binom{n-1}{2k-1} + \sum_{k\ge 0} \binom{k}{m} \binom{n-1}{2k} + \sum_{k\ge 0} \binom{k+1}{m} \binom{n-1}{2k}$$

which we seek to prove is equal to

$$2^{n-2m} \binom{n-m}{m-1} \frac{n+1}{m}$$

where we will take $m \ge 1$. We get for the first term

$$2 \sum_{k\ge 0} \binom{k}{m} \binom{n-1}{n-2k} = 2[z^n](1+z)^{n-1} \sum_{k\ge 0} \binom{k}{m} z^{2k}$$

$$= 2[z^n](1+z)^{n-1} \sum_{k\ge m} \binom{k}{m} z^{2k} = 2[z^{n-2m}](1+z)^{n-1} \sum_{k\ge 0} \binom{k+m}{m} z^{2k}$$

$$= 2[z^{n-2m}](1+z)^{n-1}\frac{1}{(1-z^2)^{m+1}}.$$

The second term is

$$\sum_{k\ge 0}\binom{k}{m}\binom{n-1}{n-1-2k} = [z^{n-1}](1+z)^{n-1}\sum_{k\ge 0}\binom{k}{m}z^{2k}$$

$$= [z^{n-2m-1}](1+z)^{n-1}\frac{1}{(1-z^2)^{m+1}}.$$

The third term is

$$\sum_{k\ge 0}\binom{k+1}{m}\binom{n-1}{n-1-2k} = [z^{n-1}](1+z)^{n-1}\sum_{k\ge 0}\binom{k+1}{m}z^{2k}$$

$$= [z^{n-1}](1+z)^{n-1}\sum_{k\ge m-1}\binom{k+1}{m}z^{2k} = [z^{n-2m+1}](1+z)^{n-1}\sum_{k\ge 0}\binom{k+m}{m}z^{2k}$$

$$= [z^{n-2m+1}](1+z)^{n-1}\frac{1}{(1-z^2)^{m+1}}.$$

Adding these together we get

$$[z^{n-2m+1}](1+z^2+2z)(1+z)^{n-1}\frac{1}{(1-z^2)^{m+1}} = [z^{n-2m+1}](1+z)^{n+1}\frac{1}{(1-z^2)^{m+1}}$$

$$= [z^{n-2m+1}](1+z)^{n-m}\frac{1}{(1-z)^{m+1}}.$$

The coefficient extractor now yields

$$\sum_{q=0}^{n+1-2m}\binom{n-m}{q}\binom{n+1-2m-q+m}{m} = \sum_{q=0}^{n+1-2m}\binom{n-m}{q}\binom{n+1-m-q}{m}$$

$$= \sum_{q=0}^{n+1-2m}\binom{n-m}{q}\frac{n+1-m-q}{m}\binom{n-m-q}{m-1}.$$

Now

$$\binom{n-m}{q}\binom{n-m-q}{m-1} = \frac{(n-m)!}{q!\times(m-1)!\times(n+1-2m-q)!}$$

$$= \binom{n-m}{m-1}\binom{n+1-2m}{q}.$$

We get for the sum

$$\frac{1}{m}\binom{n-m}{m-1}\sum_{q=0}^{n+1-2m}(n+1-m-q)\binom{n+1-2m}{q}$$

$$= \frac{1}{m}\binom{n-m}{m-1}\sum_{q=0}^{n+1-2m}(n+1-2m-q)\binom{n+1-2m}{q}$$

$$+\binom{n-m}{m-1}\sum_{q=0}^{n+1-2m}\binom{n+1-2m}{q}$$

$$= \frac{1}{m}\binom{n-m}{m-1}\sum_{q=0}^{n+1-2m} q\binom{n+1-2m}{q} + \binom{n-m}{m-1}2^{n+1-2m}$$

$$= \frac{n+1-2m}{m}\binom{n-m}{m-1}\sum_{q=1}^{n+1-2m}\binom{n-2m}{q-1} + \binom{n-m}{m-1}2^{n+1-2m}$$

$$= \frac{n+1-2m}{m}\binom{n-m}{m-1}2^{n-2m} + \binom{n-m}{m-1}2^{n+1-2m}.$$

This simplifies to

$$\boxed{\frac{n+1}{m}\binom{n-m}{m-1}2^{n-2m}.}$$

Addendum. Following the hint by OP in view of the intermediate closed form we see that we can simplify the three terms first. We get

$$2\sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k-1} + \sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k} + \sum_{k\ge m-1}\binom{k+1}{m}\binom{n-1}{2k}$$

$$= 2\sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k-1} + \sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k} + \sum_{k\ge m}\binom{k}{m}\binom{n-1}{2k-2}$$

$$= \sum_{k\ge m}\binom{k}{m}\binom{n}{2k} + \sum_{k\ge m}\binom{k}{m}\binom{n}{2k-1} = \sum_{k\ge m}\binom{k}{m}\binom{n+1}{2k}.$$

We then find

$$\sum_{k\ge m}\binom{k}{m}\binom{n+1}{n+1-2k} = [z^{n+1}](1+z)^{n+1}\sum_{k\ge m}\binom{k}{m}z^{2k}$$

$$= [z^{n+1-2m}](1+z)^{n+1}\sum_{k\ge 0}\binom{k+m}{m}z^{2k} = [z^{n+1-2m}](1+z)^{n+1}\frac{1}{(1-z^2)^{m+1}}.$$

From this point on the computation continues as before.

This was math.stackexchange.com problem 3956698.

69.35 MSE 3993530

We seek to verify that (with $n \ge 1$, $n = 0$ holds by inspection)

$$\sum_{k=0}^n \left\langle n \atop k \right\rangle x^{n-k} = (1-x)^n \sum_{k=0}^n {n \brace k} k! \left(\frac{x}{1-x}\right)^k.$$

We get using standard EGFs for the RHS

$$n![z^n](1-x)^n \sum_{k=0}^n \frac{(\exp(z)-1)^k}{k!} k! \left(\frac{x}{1-x}\right)^k$$

$$= n![z^n](1-x)^n \sum_{k=0}^n (\exp(z)-1)^k \left(\frac{x}{1-x}\right)^k.$$

Now because $\exp(z) - 1 = z + \cdots$ we have $(\exp(z)-1)^k = z^k + \cdots$ so when $k > n$ there is no contribution to the coefficient extractor and we get

$$n![z^n](1-x)^n \sum_{k\ge 0} (\exp(z)-1)^k \left(\frac{x}{1-x}\right)^k$$

$$= n![z^n](1-x)^n \frac{1}{1-(\exp(z)-1)x/(1-x)}$$

$$= n![z^n](1-x)^n \frac{1-x}{1-x-(\exp(z)-1)x}$$

$$= n![z^n](1-x)^n \frac{1-x}{1-x\exp(z)}$$

$$= n![z^n] \frac{1-x}{1-x\exp(z(1-x))}.$$

On the other hand we have for the LHS by the mixed GF of the Eulerian numbers

$$n![z^n] \sum_{k=0}^n x^{n-k} [w^k] \frac{w-1}{w-\exp((w-1)z)}$$

Now we have $\left\langle n \atop k \right\rangle = 0$ when $k \ge n$ so this is

$$n![z^n] x^n \sum_{k\ge 0} x^{-k} [w^k] \frac{w-1}{w-\exp((w-1)z)}$$

$$= n![z^n] x^n \frac{1/x-1}{1/x-\exp((1/x-1)z)}$$

$$= n![z^n] x^n \frac{1-x}{1-x\exp((1/x-1)z)}$$

$$= n![z^n]\frac{1-x}{1-x\exp((1/x-1)zx)}$$

$$= n![z^n]\frac{1-x}{1-x\exp((1-x)z)}.$$

The LHS is the same as the RHS and we have the claim.
Addendum. We have

$$n![z^n][w^k]\frac{w-1}{w-\exp((w-1)z)}$$

$$= n![z^n][w^{k+1}]\frac{w-1}{1-\exp((w-1)z)/w}$$

$$= n![z^n][w^{k+1}](w-1)\sum_{q\ge 0}\frac{1}{w^q}\exp(q(w-1)z)$$

$$= [w^{k+1}]\sum_{q\ge 0}\frac{1}{w^q}q^n(w-1)^{n+1} = \sum_{q\ge 0}[w^{k+1+q}]q^n(w-1)^{n+1}$$

$$= (-1)^{n-k}\sum_{q=1}^{n-k}(-1)^q q^n\binom{n+1}{k+1+q}.$$

This justifies that $\left\langle {n \atop k} \right\rangle = 0$ when $k \ge n$ and hence the two coefficient extractors combined return zero in that case as claimed.

This was math.stackexchange.com problem 3993530.

69.36 MSE 4008277

We seek to show that

$$\sum_{k=0}^{r} k^p \binom{m}{k}\binom{n}{r-k} = \sum_{j=0}^{p} m^{\underline{j}}\binom{m+n-j}{m+n-r}\left\{ {p \atop j} \right\}.$$

The LHS is

$$p![w^p]\sum_{k=0}^{r}\exp(kw)\binom{m}{k}\binom{n}{r-k}$$

$$= p![w^p][z^r](1+z)^n\sum_{k=0}^{r}\exp(kw)\binom{m}{k}z^k.$$

Now the coefficient extractor enforces the upper limit of the range and we may continue with

$$p![w^p][z^r](1+z)^n\sum_{k\ge 0}\exp(kw)\binom{m}{k}z^k$$

$$
\begin{aligned}
&= p![w^p][z^r](1+z)^n(1+z\exp(w))^m \\
&= p![w^p][z^r](1+z)^n(1+z+z(\exp(w)-1))^m \\
&= p![w^p][z^r](1+z)^n\sum_{j=0}^m \binom{m}{j}(1+z)^{m-j}z^j(\exp(w)-1)^j \\
&= [z^r]\sum_{j=0}^m \binom{m}{j}(1+z)^{m+n-j}z^j j!\left\{ p \atop j \right\} \\
&= \sum_{j=0}^m \binom{m}{j}\binom{m+n-j}{r-j} j!\left\{ p \atop j \right\}.
\end{aligned}
$$

Note that if $m > p$ the values with $m \ge j > p$ produce a zero Stirling number so we may lower m to p. If $m < p$ the values with $p \ge j > m$ produce a zero binomial coefficient and we may raise m to p. We thus obtain

$$
\boxed{\sum_{j=0}^p \binom{m}{j}\binom{m+n-j}{m+n-r} j!\left\{ p \atop j \right\}.}
$$

a sum with p non-zero terms except for $p=0$, when it has one term. (We could also use $\min(m,p)$ as the upper limit but we want to emphasize the dependence on p.) Note that in the initial sum for it to be non-zero with non-negative k we must have $m \ge k$ and $n \ge r-k$ or $k \ge r-n$ so that $m \ge k \ge r-n$ and for the range not to be empty we must have $m \ge r-n$ or $m+n-r \ge 0$ which ensures that the middle binomial coefficient in the boxed form is well defined. Observe that with $p=0$ we obtain $\binom{m+n}{m+n-r} = \binom{m+n}{r}$ which is Vandermonde. A slight variation is

$$
\boxed{\sum_{j=0}^p m^{\underline{j}}\binom{m+n-j}{m+n-r}\left\{ p \atop j \right\}.}
$$

Remark. We may keep the $\binom{m+n-j}{r-j}$ if we remember that it originates with $[z^r](1+z)^{m+n-j}z^j$ and hence is zero when $j > r$.

This was math.stackexchange.com problem 4008277.

69.37 MSE 4031272

We seek

$$
\binom{m+k}{k}^2 = \sum_{q=0}^m \binom{k}{m-q}^2\binom{2k+q}{q}
$$

Starting with the RHS we find

$$\sum_{q=0}^{m} \binom{k}{q}^2 \binom{2k+m-q}{m-q}$$

$$= \sum_{q=0}^{m} \binom{k}{q} [z^k] z^q (1+z)^k [w^m] w^q (1+w)^{2k+m-q}.$$

Now we may extend q to infinity because the coefficient extractor $[w^m]$ enforces the upper limit. We get

$$[z^k](1+z)^k [w^m](1+w)^{2k+m} \sum_{q\ge 0} \binom{k}{q} z^q w^q (1+w)^{-q}$$

$$= [z^k](1+z)^k [w^m](1+w)^{2k+m} (1+zw/(1+w))^k$$

$$= [z^k](1+z)^k [w^m](1+w)^{k+m} (1+w+zw)^k$$

Re-expanding we find

$$[z^k](1+z)^k [w^m](1+w)^{k+m} \sum_{q=0}^{k} \binom{k}{q} w^q (1+z)^q.$$

We may set the upper limit of the sum to m. (If $k < m$ the values $k < q \le m$ produce zero from the binomial coefficient and we may raise q to m. If $k > m$ the values $m < q \le k$ produce zero by the coefficient extractor $[w^m]$ and we may lower q to m.) We get

$$[z^k](1+z)^k [w^m](1+w)^{k+m} \sum_{q=0}^{m} \binom{k}{q} w^q (1+z)^q$$

$$= \sum_{q=0}^{m} \binom{k}{q} \binom{k+q}{k} \binom{k+m}{m-q}.$$

Now observe that

$$\binom{k+q}{k}\binom{k+m}{m-q} = \frac{(k+m)!}{k! \times q! \times (m-q)!} = \binom{m+k}{k}\binom{m}{m-q}.$$

This yields for our sum

$$\binom{m+k}{k} \sum_{q=0}^{m} \binom{k}{q} \binom{m}{m-q}.$$

Using Vandermonde we obtain at last

$$\boxed{\binom{m+k}{k}^2.}$$

This was math.stackexchange.com problem 4031272 and this identity is the Li Shanlan identity.

69.38 MSE 4034224

We seek to show that with $0 \le k \le n$ the following identity holds: (two alternate representations of second order Eulerian numbers)

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{2n+1}{k-j} \left\{ n+j \atop j \right\} = \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k} (-1)^{j} \binom{2n+1}{j} \left[2n-k-j+1 \atop n-k-j+1 \right].$$

We will start with the LHS. The chapter 6.2 on Eulerian Numbers of *Concrete Mathematics* by Knuth et al. [GKP89] proposes the formula

$$\left\{ n \atop m \right\} = (-1)^{n-m+1} \frac{n!}{(m-1)!} \sigma_{n-m}(-m)$$

where $\sigma_n(x)$ is a Stirling polynomial and we have the identity

$$\left(\frac{1}{z} \log \frac{1}{1-z} \right)^x = x \sum_{n \ge 0} \sigma_n(x+n) z^n.$$

We get

$$[z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^x = x \sigma_{n-m}(x+n-m)$$

and hence

$$[z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n} = -n \sigma_{n-m}(-m)$$

which implies that for $n \ge m \ge 1$

$$\boxed{\left\{ n \atop m \right\} = (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n}.}$$

This gives for the LHS

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{2n+1}{k-j} (-1)^n \frac{(n+j-1)!}{(j-1)!} [z^n] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n-j}$$

$$= (-1)^{n-k+1} n! [z^n] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n-1} [w^{k-1}] (1+w)^{2n+1}$$

$$\times \sum_{j=1}^{k} \binom{n+j-1}{n} (-1)^{j-1} w^{j-1} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-j+1}.$$

Now the coefficient extractor in w enforces the upper limit of the sum and we may extend j to infinity, getting

$$(-1)^{n-k+1} n! [z^n] \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-n-1} [w^{k-1}] (1+w)^{2n+1} \frac{1}{(1 + w/(\frac{1}{z} \log \frac{1}{1-z}))^{n+1}}$$

$$= (-1)^{n-k+1} n! [z^n] [w^{k-1}] (1+w)^{2n+1} \frac{1}{(w + \frac{1}{z} \log \frac{1}{1-z})^{n+1}}.$$

Continuing,

$$(-1)^{n-k+1} n! [z^n] [w^{n+k}] (1+w)^{2n+1} \frac{1}{(1 + \frac{1}{w} \frac{1}{z} \log \frac{1}{1-z})^{n+1}}$$

$$= (-1)^{n-k+1} n! [z^n] [w^{n+k}] (1+w)^{2n+1} \sum_{q \ge 0} \binom{n+q}{n} (-1)^q \frac{1}{w^q} \left(\frac{1}{z} \log \frac{1}{1-z}\right)^q$$

$$= (-1)^{n-k+1} n! [z^n] \sum_{j=n+k}^{2n+1} \binom{2n+1}{j} \binom{n+j-(n+k)}{n} (-1)^{j-(n+k)}$$

$$\times \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{j-(n+k)}$$

$$= (-1)^{n-k+1} n! [z^n] \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} \binom{n+j}{n} (-1)^j \left(\frac{1}{z} \log \frac{1}{1-z}\right)^j$$

$$= (-1)^{n-k+1} n! \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} \binom{n+j}{n} (-1)^j [z^{n+j}] \left(\log \frac{1}{1-z}\right)^j$$

$$= (-1)^{n-k+1} n! \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} \binom{n+j}{n} (-1)^j$$

$$\times \frac{j!}{(n+j)!} \times (n+j)! [z^{n+j}] \frac{1}{j!} \left(\log \frac{1}{1-z}\right)^j$$

$$= (-1)^{n-k+1} \sum_{j=0}^{n-k+1} \binom{2n+1}{j+n+k} (-1)^j \left[{n+j \atop j}\right]$$

$$= (-1)^{n-k+1} \sum_{j=0}^{n-k+1} \binom{2n+1}{2n-j+1} (-1)^{n-k-j+1} \left[{2n-k-j+1 \atop n-k-j+1}\right]$$

$$= \sum_{j=0}^{n-k+1} \binom{2n+1}{j} (-1)^j \left[{2n-k-j+1 \atop n-k-j+1}\right].$$

The Stirling number is zero for $j = n-k+1$ and we get at last

$$\sum_{j=0}^{n-k} \binom{2n+1}{j} (-1)^j \left[2n-k-j+1 \atop n-k-j+1 \right].$$

This is the RHS and we have the claim.

This was math.stackexchange.com problem 4034224.

69.39 MSE 4037172

We seek to show that with $0 \le k \le n$ the following identity holds: (two alternate representations of second order Eulerian numbers)

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \left\{\!\!\left\{ n+j \atop j \right\}\!\!\right\} = \left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k+1} (-1)^{n-k-j+1} \binom{n-j}{k-1} \left[\!\!\left[n+j \atop j \right]\!\!\right]$$

where we have associated Stirling numbers of the first and second kind.

Now from the combinatorial meaning of these numbers (cancel fixed points resp. singleton sets) we have that

$$\left[\!\!\left[n \atop k \right]\!\!\right] = \sum_{q=0}^{k} (-1)^q \binom{n}{q} \left[n-q \atop k-q \right]$$

and

$$\left\{\!\!\left\{ n \atop k \right\}\!\!\right\} = \sum_{q=0}^{k} (-1)^q \binom{n}{q} \left\{ n-q \atop k-q \right\}.$$

Consult OEIS A008306 and OEIS A008299 for more information. We will only use the second of these but we show the pair to illustrate the similarity in their construction (PIE). The combinatorial classes for these are $\textsc{SET}(\mathcal{U} \times \textsc{CYC}_{\ge 2}(\mathcal{Z}))$ and $\textsc{SET}(\mathcal{U} \times \textsc{SET}_{\ge 2}(\mathcal{Z}))$.

We start with the LHS and obtain

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \sum_{q=0}^{j} (-1)^q \binom{n+j}{q} \left\{ n+j-q \atop j-q \right\}.$$

With $n \ge 1$ this is

$$(-1)^k \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=1}^{j} (-1)^q \binom{n+j}{j-q} \left\{ n+q \atop q \right\}.$$

Recall e.g. from *Concrete Mathematics* chapter 6.2. [GKP89] that

$$\boxed{\left\{ n \atop m \right\} = (-1)^{n-m} \frac{(n-1)!}{(m-1)!} [z^{n-m}] \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{-n}.}$$

We find for the LHS

$$(-1)^k \sum_{j=1}^k \binom{n-j}{k-j} \sum_{q=1}^j (-1)^q \binom{n+j}{j-q} (-1)^n \frac{(n+q-1)!}{(q-1)!} [z^n] \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-n-q}$$

$$= (-1)^{n-k+1} n! [z^n] \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-n-1} \sum_{j=1}^k \binom{n-j}{k-j}$$

$$\times \sum_{q=1}^j (-1)^{q-1} \binom{n+j}{j-q} \binom{n+q-1}{q-1} \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-q+1}$$

$$= (-1)^{n-k+1} n! [z^n] \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-n-1} \sum_{j=1}^k \binom{n-j}{k-j}$$

$$\times [w^{j-1}](1+w)^{n+j} \sum_{q=1}^j (-1)^{q-1} w^{q-1} \binom{n+q-1}{q-1} \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-q+1}.$$

Now the coefficient extractor enforces the upper limit of the inner sum and we may extend q to infinity, getting

$$(-1)^{n-k+1} n! [z^n] \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-n-1} \sum_{j=1}^k \binom{n-j}{k-j}$$

$$\times [w^{j-1}](1+w)^{n+j} \frac{1}{(1+w/(\frac{1}{z}\log\frac{1}{1-z}))^{n+1}}$$

$$= (-1)^{n-k+1} n! [z^n] \sum_{j=1}^k \binom{n-j}{k-j} [w^{j-1}](1+w)^{n+j} \frac{1}{(\frac{1}{z}\log\frac{1}{1-z} + w)^{n+1}}.$$

The inner term is

$$[w^{j-1}](1+w)^{n+j} \frac{1}{(\frac{1}{z}(\log\frac{1}{1-z} - z) + 1 + w)^{n+1}}$$

$$= [w^{j-1}](1+w)^{j-1} \frac{1}{(1 + \frac{1}{1+w}\frac{1}{z}(\log\frac{1}{1-z} - z))^{n+1}}.$$

Re-expanding the series,

$$(-1)^{n-k+1} n! [z^n] \sum_{j=1}^k \binom{n-j}{k-j} [w^{j-1}](1+w)^{j-1}$$

$$\times \sum_{q=j}^n \binom{n+q}{n} (-1)^q \frac{1}{(1+w)^q} \left(\frac{1}{z}(\log\frac{1}{1-z} - z)\right)^q.$$

The upper limit on the inner sum results from $[z^n]$ because $\frac{1}{z}(\log\frac{1}{1-z}-z) = \frac{1}{2}z+\cdots$ and the lower one from the fact that $[w^{j-1}](1+w)^{j-1-q} = 0$ when $1 \le q \le j-1$; $q=0$ produces a constant. Continuing,

$$(-1)^{n-k+1} n! [z^n] \sum_{j=1}^{k} \binom{n-j}{k-j} [w^{j-1}]$$
$$\times \sum_{q=j}^{n} \binom{n+q}{n} (-1)^q \frac{1}{(1+w)^{q-(j-1)}} \left(\frac{1}{z}(\log\frac{1}{1-z}-z)\right)^q$$
$$= (-1)^{n-k} n! [z^n] \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} \binom{n+q}{n} (-1)^{q-j} \binom{q-1}{q-j} \left(\frac{1}{z}(\log\frac{1}{1-z}-z)\right)^q$$
$$= (-1)^{n-k} n! \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} \binom{n+q}{n} (-1)^{q-j} \binom{q-1}{q-j} [z^{n+q}] \left(\log\frac{1}{1-z}-z\right)^q$$
$$= (-1)^{n-k} n! \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} \binom{n+q}{n} (-1)^{q-j} \binom{q-1}{q-j}$$
$$\times \frac{q!}{(n+q)!} \times (n+q)! [z^{n+q}] \frac{1}{q!} \left(\log\frac{1}{1-z}-z\right)^q$$
$$= (-1)^{n-k} \sum_{j=1}^{k} \binom{n-j}{k-j} \sum_{q=j}^{n} (-1)^{q-j} \binom{q-1}{q-j} \left[\!\left[n+q \atop q \right]\!\right].$$

It remains to simplify the binomial coefficients:

$$(-1)^{n-k} \sum_{j=1}^{k} [u^k] u^j (1+u)^{n-j} \sum_{q=j}^{n} (-1)^{q-j} \binom{q-1}{q-j} \left[\!\left[n+q \atop q \right]\!\right].$$

We see that we may raise j to n owing to $[u^k]$:

$$(-1)^{n-k} \sum_{j=1}^{n} [u^k] u^j (1+u)^{n-j} \sum_{q=j}^{n} (-1)^{q-j} \binom{q-1}{q-j} \left[\!\left[n+q \atop q \right]\!\right]$$
$$= (-1)^{n-k} \sum_{q=1}^{n} \left[\!\left[n+q \atop q \right]\!\right] [u^k] (1+u)^n \sum_{j=1}^{q} \binom{q-1}{q-j} (-1)^{q-j} u^j (1+u)^{-j}.$$

The inner term is

$$[u^k](1+u)^n \frac{u^q}{(1+u)^q} \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j \frac{(1+u)^j}{u^j}$$
$$= [u^k](1+u)^{n-q} u^q \left(1 - \frac{1+u}{u}\right)^{q-1}$$

$$= [u^k](1+u)^{n-q}u(-1)^{q-1} = (-1)^{q-1}[u^{k-1}](1+u)^{n-q}.$$

This yields

$$\sum_{q=0}^{n-k+1} (-1)^{n-k-q+1} \binom{n-q}{k-1} \left[\!\!\left[{n+q \atop q} \right]\!\!\right]$$

which is the claim. (Here we must have $n-q \ge k-1$ or $n-k+1 \ge q$ else the binomial coefficient vanishes and we may lower the upper limit from n to $n-k+1$.)

This was math.stackexchange.com problem 4037172.

69.40 MSE 4037946

We seek to show that with $0 \le k \le n$ the following identity holds:

$$\left[{n \atop n-k} \right] - \left\{ {n \atop n-k} \right\} = \sum_{j=0}^{k} \left(\binom{n+j-1}{2k} - \binom{n+k-j}{2k} \right) \left\langle\!\!\left\langle {k \atop j} \right\rangle\!\!\right\rangle.$$

Recall from the previous example the identity

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \left\{\!\!\left\{ {n+j \atop j} \right\}\!\!\right\} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{j=0}^{n-k+1} (-1)^{n-k-j+1} \binom{n-j}{k-1} \left[\!\!\left[{n+j \atop j} \right]\!\!\right]$$

We get for the first piece

$$\sum_{j=1}^{k} \binom{n+j-1}{2k} \sum_{p=0}^{k-j+1} (-1)^{k-j-p+1} \binom{k-p}{j-1} \left[\!\!\left[{k+p \atop p} \right]\!\!\right]$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] (-1)^{k-p+1} \sum_{j=1}^{k+1-p} (-1)^j \binom{n+j-1}{2k} \binom{k-p}{j-1}$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] (-1)^{k-p} [z^{2k}](1+z)^n \sum_{j=1}^{k+1-p} (-1)^{j-1} (1+z)^{j-1} \binom{k-p}{j-1}$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] (-1)^{k-p} [z^{2k}](1+z)^n (1-(1+z))^{k-p}$$

$$= \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] [z^{k+p}](1+z)^n = \sum_{p=0}^{k} \left[\!\!\left[{k+p \atop p} \right]\!\!\right] \binom{n}{k+p}.$$

Now this last piece evaluates combinatorially to $\left[{n \atop n-k} \right]$ when written as $\left[\!\left[{k+p \atop p} \right]\!\right] \binom{n}{n-k-p}$ namely we choose $n-k-p$ fixed points and split the remaining

$k+p$ elements into p cycles of size at least two for a total of $n-k$ cycles. Here we must have $k+p \ge 2p$ or $p \le k$. (We have classified by the number of fixed points).

We get for the second piece

$$\sum_{j=1}^{k} \binom{n+k-j}{2k} \sum_{p=0}^{j} (-1)^{j-p} \binom{k-p}{j-p} \left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\} (-1)^p \sum_{j=p}^{k} (-1)^j \binom{n+k-j}{2k} \binom{k-p}{j-p}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\} \sum_{j=0}^{k-p} (-1)^j \binom{n+k-j-p}{2k} \binom{k-p}{j}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\} [z^{2k}](1+z)^{n+k-p} \sum_{j=0}^{k-p} (-1)^j (1+z)^{-j} \binom{k-p}{j}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\} [z^{2k}](1+z)^{n+k-p} \left(1 - \frac{1}{1+z}\right)^{k-p}$$

$$= \sum_{p=0}^{k} \left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\} [z^{2k}](1+z)^{n} z^{k-p} = \sum_{p=0}^{k} \left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\} \binom{n}{k+p}.$$

With this piece we get exactly the same reasoning as with the first one, namely it evaluates to $\left\{ {n \atop n-k} \right\}$. We write it as $\left\{\!\!\left\{ {k+p \atop p} \right\}\!\!\right\} \binom{n}{n-k-p}$ in choosing the number of singletons, of which there are $n-k-p$. The remaining $k+p$ elements are distributed into p disjoint sets of at least two elements for a total of $n-k$ sets. We once more have the condition that $k+p \ge 2p$ or $p \le k$. (We have classified by the number of singleton sets.)

This was math.stackexchange.com problem 4037946.

69.41 MSE 4055292

In trying to verify the identity

$$\sum_{k=0}^{2n} (-1)^k \binom{n+k}{k}^{-1} \binom{2n}{k} \binom{2k}{k} = 1$$

we see that

$$\binom{n+k}{k}^{-1} \binom{2n}{k} = \frac{(2n)!/(2n-k)!}{(n+k)!/n!} = \binom{3n}{n}^{-1} \binom{3n}{2n-k}$$

so that we seek to prove

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{2n-k} \binom{2k}{k} = \binom{3n}{n}.$$

The LHS is

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{k} \binom{4n-2k}{2n-k} = [z^{2n}](1+z)^{4n} \sum_{k=0}^{2n} (-1)^k \binom{3n}{k} \frac{z^k}{(1+z)^{2k}}$$

Here the coefficient extractor enforces the range of the sum and we find

$$[z^{2n}](1+z)^{4n} \sum_{k\ge 0} (-1)^k \binom{3n}{k} \frac{z^k}{(1+z)^{2k}} = [z^{2n}](1+z)^{4n} \left(1 - \frac{z}{(1+z)^2}\right)^{3n}$$

$$= [z^{2n}] \frac{1}{(1+z)^{2n}} (1+z+z^2)^{3n}.$$

Expanding the second powered term

$$[z^{2n}] \frac{1}{(1+z)^{2n}} \sum_{q=0}^{3n} \binom{3n}{q} (1+z)^{3n-q} z^{2q}$$

The coefficient extractor sets the upper limit of the sum to n and we get (note that the powers of $1+z$ do not have a pole at zero hence the expansion about zero starts with z^{2q} and there is no contribution to $[z^{2n}]$ when $q > n$):

$$[z^{2n}] \sum_{q=0}^{n} \binom{3n}{q} (1+z)^{n-q} z^{2q} = \sum_{q=0}^{n} \binom{3n}{q} \binom{n-q}{2n-2q} = \binom{3n}{n}.$$

Observe that the the power $n-q$ to which $1+z$ is raised is a non-negative integer and hence we are justified in writing $[z^{2n}]z^{2q}(1+z)^{n-q} = [z^{2n-2q}](1+z)^{n-q} = \binom{n-q}{2n-2q}$. The only q in the range $0 \le q \le n$ where this binomial coefficient is not zero is $q = n$, producing a contribution of $\binom{3n}{n}$ and we have the claim.

This was math.stackexchange.com problem 4055292.

69.42 MSE 4054024

We seek to verify the identity

$$\sum_{k=1}^{n} \binom{2n-2k}{n-k} \frac{H_{2k} - 2H_k}{2n-2k-1} \binom{2k}{k} = \frac{1}{n} \left[4^n - 3 \binom{2n-1}{n} \right].$$

Preliminary. We get for the first piece in H_{2k} call it A that

$$\sum_{k=1}^{n} \binom{2n-2k}{n-k} \frac{1}{2n-2k-1} \binom{2k}{k} [z^{2k}] \frac{1}{1-z} \log \frac{1}{1-z}$$

$$= \sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{2k-1} \binom{2n-2k}{n-k} [z^{2n-2k}] \frac{1}{1-z} \log \frac{1}{1-z}$$

We may raise k to n because the function in z has no constant term:

$$[z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} \binom{2n-2k}{n-k} z^{2k}$$

Now the coefficient extractor enforces the upper limit of the sum and we get (in fact expansions start at z^{2k+1} which cancels $k=n$ already)

$$[z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} \sum_{k\ge 0} \binom{2k}{k} \frac{1}{2k-1} \binom{2n-2k}{n-k} z^{2k}$$

$$= -[z^{2n}] \frac{1}{1-z} \log \frac{1}{1-z} [w^n] \sqrt{1-4wz^2} \frac{1}{\sqrt{1-4w}}.$$

The same method yields for the second piece in H_k call it B

$$-[z^{n}] \frac{1}{1-z} \log \frac{1}{1-z} [w^n] \sqrt{1-4wz} \frac{1}{\sqrt{1-4w}}.$$

First part. Continuing with piece B

$$[w^n] \sqrt{1+\frac{4w(1-z)}{1-4w}} = -[w^n] \sum_{k\ge 0} \binom{2k}{k} \frac{1}{2k-1} (-1)^k \frac{w^k(1-z)^k}{(1-4w)^k}$$

$$= -\sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k [w^{n-k}] \frac{(1-z)^k}{(1-4w)^k}$$

$$= -4^n \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k (1-z)^k 4^{-k} \binom{n-1}{k-1}$$

and extracting the coefficient in $[z^n]$

$$4^n \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k 4^{-k} \binom{n-1}{k-1} [z^n] (1-z)^{k-1} \log \frac{1}{1-z}$$

$$= 4^n \sum_{k=1}^{n} \binom{2k}{k} \frac{1}{2k-1} (-1)^k 4^{-k} \binom{n-1}{k-1} \sum_{q=0}^{k-1} (-1)^q \binom{k-1}{q} \frac{1}{n-q}.$$

Now

$$\binom{n-1}{k-1}\binom{k-1}{q} = \frac{(n-1)!}{(n-k)! \times q! \times (k-1-q)!} = \binom{n-1}{q}\binom{n-1-q}{k-1-q}$$

Switching the order of the summation,

$$4^n \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{(-1)^q}{n-q} \sum_{k=q+1}^{n} \binom{n-1-q}{k-1-q}\binom{2k}{k}\frac{1}{2k-1}(-1)^k 4^{-k}$$

$$= \frac{4^n}{n} \sum_{q=0}^{n-1} \binom{n}{q} (-1)^q \sum_{k=q+1}^{n} \binom{n-1-q}{k-1-q}\binom{2k}{k}\frac{1}{2k-1}(-1)^k 4^{-k}$$

$$= -\frac{4^n}{n} \sum_{q=0}^{n-1} \binom{n}{q} (-1)^q \sum_{k=q+1}^{n} \binom{n-1-q}{k-1-q}[z^k]\sqrt{1+z}.$$

The inner sum is

$$\sum_{k=0}^{n-1-q} \binom{n-1-q}{k} [z^{k+q+1}]\sqrt{1+z} = \sum_{k=0}^{n-1-q} \binom{n-1-q}{k} [z^{n-k}]\sqrt{1+z}$$

$$= [z^n]\sqrt{1+z} \sum_{k=0}^{n-1-q} \binom{n-1-q}{k} z^k = [z^n]\sqrt{1+z}(1+z)^{n-1-q}.$$

Substitute into the outer sum to get

$$-\frac{4^n}{n}[z^n]\sqrt{1+z}\sum_{q=0}^{n-1}\binom{n}{q}(-1)^q(1+z)^{n-1-q} = -\frac{4^n}{n}[z^n]\frac{1}{\sqrt{1+z}}(-(-1)^n + z^n)$$

$$= -\frac{4^n}{n}\left(-4^{-n}\binom{2n}{n} + 1\right) = -\frac{4^n}{n} + \binom{2n}{n}\frac{1}{n}.$$

Second part. Here we may recycle the first segment from the easy piece B and obtain for piece A

$$4^n \sum_{k=1}^{n} \binom{2k}{k}\frac{1}{2k-1}(-1)^k 4^{-k}\binom{n-1}{k-1}[z^{2n}](1-z^2)^{k-1}(1+z)\log\frac{1}{1-z}.$$

The coefficient extractor in z has two parts, the first of which is

$$\sum_{q=0}^{k-1}(-1)^q\binom{k-1}{q}\frac{1}{2n-2q}$$

which contributes half the value of the piece B. The second is

$$\sum_{q=0}^{k-1} (-1)^q \binom{k-1}{q} \frac{1}{2n-1-2q}.$$

This yields

$$-4^n [z^n] \sqrt{1+z} \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{(-1)^q}{2n-1-2q} (1+z)^{n-1-q}$$
$$= -4^n [z^n] \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{(-1)^q}{2n-1-2q} (1+z)^{n-1/2-q}$$
$$= -4^n \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{(-1)^q}{2n-1-2q} (n-1/2-q)^{\underline{n}}/n!.$$

We have for the falling factorial

$$\prod_{p=0}^{n-1} (n-1/2-q-p) = \frac{1}{2^n} \prod_{p=0}^{n-1} (2n-1-2q-2p)$$
$$= \frac{1}{2^n} \prod_{p=-(n-1)}^{0} (1-2q-2p) = \frac{(-1)^n}{2^n} \prod_{p=q-(n-1)}^{q} (2p-1)$$
$$= \frac{(-1)^{n+1}}{2^n} \frac{(2q-1)!}{2^{q-1}(q-1)!} \prod_{p=q-(n-1)}^{-1} (2p-1).$$

With $2q-2(n-1)-1 = 2q-2n+1$ this finally becomes

$$\frac{(-1)^q}{2^n} \frac{(2q-1)!}{2^{q-1}(q-1)!} \frac{(2n-1-2q)!}{2^{n-1-q}(n-1-q)!}$$
$$= \frac{(-1)^q}{2^{2n-1}} \frac{(2q)!}{q!} \frac{(2n-1-2q)!}{(n-1-q)!}.$$

This was for $1 \le q \le n-1$. We get for $q=0$

$$\frac{1}{2^n} \prod_{p=-(n-1)}^{0} (1-2p) = \frac{1}{2^n} \frac{(2n-1)!}{2^{n-1}(n-1)!}$$

and we see that the generic term in four factorials represents this case correctly as well.

Returning to the sum we obtain

$$-\frac{2}{n} \sum_{q=0}^{n-1} \binom{2q}{q} \binom{2n-2-2q}{n-1-q}$$

$$= -\frac{2}{n}[z^{n-1}]\frac{1}{\sqrt{1-4z}}\frac{1}{\sqrt{1-4z}} = -\frac{2}{n}[z^{n-1}]\frac{1}{1-4z} = -\frac{2}{n}4^{n-1} = -\frac{1}{2}\frac{4^n}{n}.$$

Conclusion. We now collect the three pieces with A first then B :

$$-\frac{1}{2}\frac{4^n}{n} - \frac{1}{2}\frac{4^n}{n} + \frac{1}{2}\frac{1}{n}\binom{2n}{n}$$

$$+2\frac{4^n}{n} - 2\frac{1}{n}\binom{2n}{n} = \frac{4^n}{n} - \frac{3}{2}\frac{1}{n}\binom{2n}{n} = \frac{4^n}{n} - 3\frac{1}{n}\binom{2n-1}{n-1}.$$

This is indeed

$$\boxed{\frac{1}{n}\left[4^n - 3\binom{2n-1}{n}\right].}$$

This was math.stackexchange.com problem 4054024.

69.43 MSE 4088666

We to verify that (two equivalent answers to the Hertzsprung problem):

$$n! + \sum_{k=1}^{n}(-1)^k(n-k)!\sum_{q=1}^{k}2^q\binom{k-1}{q-1}\binom{n-k}{q} = [w^n]\sum_{k\ge 0}k!\left[w\frac{1-w}{1+w}\right]^k.$$

We have for the inner sum

$$\sum_{q=1}^{k}2^q\binom{k-1}{k-q}\binom{n-k}{q} = [z^k](1+z)^{k-1}\sum_{q=1}^{k}2^q\binom{n-k}{q}z^q.$$

Now the coefficient extractor enforces the upper limit of the sum and we find

$$[z^k](1+z)^{k-1}\sum_{q\ge 1}2^q\binom{n-k}{q}z^q = [z^k](1+z)^{k-1}(-1+(1+2z)^{n-k}).$$

The constant term does not contribute because $[z^k](1+z)^{k-1} = 0$ with $k \ge 1$ so this is

$$[z^k](1+z)^{k-1}(1+2z)^{n-k} = \mathrm{Res}_{z=0}\frac{1}{z^{k+1}}(1+z)^{k-1}(1+2z)^{n-k}.$$

Now we put $z/(1+z) = w$ and hence $z = w/(1-w)$ and $1+z = 1/(1-w)$ and $dz = 1/(1-w)^2\; dw$ to get

$$\mathrm{Res}_{w=0}\frac{1}{w^k}\frac{1-w}{w}(1-w)\frac{(1+w)^{n-k}}{(1-w)^{n-k}}\frac{1}{(1-w)^2} = \mathrm{Res}_{w=0}\frac{1}{w^{k+1}}\frac{(1+w)^{n-k}}{(1-w)^{n-k}}.$$

We thus have for the outer sum

$$\sum_{k=1}^{n}(-1)^k(n-k)![w^k]\frac{(1+w)^{n-k}}{(1-w)^{n-k}} = \sum_{k=1}^{n}(n-k)![w^k]\frac{(1-w)^{n-k}}{(1+w)^{n-k}}$$
$$= \sum_{k=0}^{n-1} k![w^{n-k}]\frac{(1-w)^k}{(1+w)^k} = [w^n]\sum_{k=0}^{n-1} k!w^k\frac{(1-w)^k}{(1+w)^k}.$$

Restoring the $n!$ in front yields

$$n! + [w^n]\left[-n!w^n\frac{(1-w)^n}{(1+w)^n} + \sum_{k=0}^{n} k!w^k\frac{(1-w)^k}{(1+w)^k}\right]$$
$$= [w^n]\sum_{k=0}^{n} k!w^k\frac{(1-w)^k}{(1+w)^k}.$$

The coefficient extractor once more enforces the upper limit of the range and we find

$$\boxed{[w^n]\sum_{k\ge 0} k!\left[w\frac{1-w}{1+w}\right]^k}$$

as claimed.

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69.44 MSE 4084763

We seek to evaluate

$$\sum_{q=0}^{n}\binom{n}{q}q^k,$$

k a positive integer. We get

$$k![z^k]\sum_{q=0}^{n}\binom{n}{q}\exp(qz) = k![z^k](\exp(z)+1)^n$$
$$= k![z^k]\sum_{q=0}^{n}\binom{n}{q}(\exp(z)-1)^q 2^{n-q} = \sum_{q=0}^{n}\binom{n}{q}q!\left\{k \atop q\right\}2^{n-q}$$
$$= \sum_{q=0}^{n} n^{\underline{q}}\left\{k \atop q\right\}2^{n-q}.$$

Now we may set the upper limit to k. If $n > k$ we may lower to k because the extra range $k < q \le n$ produces zero from the Stirling number. If $n < k$ we may raise to k because the extra range $n < q \le k$ produces zero from the falling factorial. We get

$$\boxed{\sum_{q=1}^{k} n^{\underline{q}} {k \brace q} 2^{n-q}.}$$

In this way we obtain e.g. for $k = 4$

$$2^{n-1} n^{\underline{1}} {4 \brace 1} + 2^{n-2} n^{\underline{2}} {4 \brace 2} + 2^{n-3} n^{\underline{3}} {4 \brace 3} + 2^{n-4} n^{\underline{4}} {4 \brace 4}.$$

Now the Stirling numbers can be evaluated by inspection:

$$2^{n-1} n^{\underline{1}} \times 1 + 2^{n-2} n^{\underline{2}} \times \left(\frac{1}{2}\binom{4}{2} + \binom{4}{1}\right) + 2^{n-3} n^{\underline{3}} \times \binom{4}{2} + 2^{n-4} n^{\underline{4}} \times 1.$$

We find at last

$$2^{n-1} n^{\underline{1}} + 7 \times 2^{n-2} n^{\underline{2}} + 6 \times 2^{n-3} n^{\underline{3}} + 2^{n-4} n^{\underline{4}}.$$

We may expand the falling factorial if desired:

$$2^{n-1} \times n + 7 \times 2^{n-2} \times n(n-1)$$
$$+6 \times 2^{n-3} \times n(n-1)(n-2) + 2^{n-4} \times n(n-1)(n-2)(n-3).$$

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69.45 MSE 4095795

We seek to evaluate

$$\sum_{r=0}^{n} r^k.$$

We may also express this in terms of Stirling numbers of the second kind and falling factorials. We start with

$$\sum_{r=0}^{n} r^k = k![z^k] \sum_{r=0}^{n} \exp(rz) = k![z^k] \frac{\exp((n+1)z) - 1}{\exp(z) - 1}$$
$$= k![z^k] \frac{1}{\exp(z) - 1} \sum_{q=1}^{n+1} \binom{n+1}{q} (\exp(z) - 1)^q$$
$$= k![z^k] \sum_{q=1}^{n+1} \binom{n+1}{q} (\exp(z) - 1)^{q-1}$$

$$= k![z^k] \sum_{q=1}^{n+1} (n+1)^{\underline{q}} \frac{1}{q} \frac{(\exp(z)-1)^{q-1}}{(q-1)!}$$
$$= \sum_{q=1}^{n+1} (n+1)^{\underline{q}} \frac{1}{q} \left\{ k \atop q-1 \right\}.$$

Note that we may set the upper limit of the sum to $k+1$. If $n+1 > k+1$ we may lower to $k+1$ because the removed terms from the range $k+2 \le q \le n+1$ produce zero by the Stirling number. If $k+1 > n+1$ we may raise to $k+1$ because the extra terms from the range $n+2 \le q \le k+1$ produce zero through the falling factorial.

We get

$$\sum_{q=2}^{k+1} (n+1)^{\underline{q}} \frac{1}{q} \left\{ k \atop q-1 \right\} = (n+1) \sum_{q=2}^{k+1} n^{\underline{q-1}} \frac{1}{q} \left\{ k \atop q-1 \right\}$$

or alternatively

$$\boxed{\sum_{r=0}^{n} r^k = (n+1) \sum_{q=1}^{k} n^{\underline{q}} \frac{1}{q+1} \left\{ k \atop q \right\}.}$$

In this way we get e.g. with $k = 4$

$$(n+1) \times \left[n^{\underline{1}} \frac{1}{2} \left\{ 4 \atop 1 \right\} + n^{\underline{2}} \frac{1}{3} \left\{ 4 \atop 2 \right\} + n^{\underline{3}} \frac{1}{4} \left\{ 4 \atop 3 \right\} + n^{\underline{4}} \frac{1}{5} \left\{ 4 \atop 4 \right\} \right]$$

The Stirling numbers may be evaluated by inspectiona as before and we find

$$\sum_{r=0}^{n} r^4 = (n+1) \times \left[\frac{1}{2} n^{\underline{1}} + \frac{7}{3} n^{\underline{2}} + \frac{3}{2} n^{\underline{3}} + \frac{1}{5} n^{\underline{4}} \right].$$

This was math.stackexchange.com problem 4095795.

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