Nonlinear wave equations in infinite waveguides

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Abstract: We present sharp decay rates as time tends to infinity for solutions to linear Klein-Gordon and wave equations in domains with infinite boundaries like infinite waveguides, as well as the global well-posedness and the asymptotics for small data for the solutions to the associated nonlinear initial-boundary value problems.

1 Introduction

We consider fully nonlinear Klein-Gordon equations

$$u_{tt} - \Delta u + mu = f(u, u_t, \nabla u, \nabla u_t, \nabla^2 u), \tag{1.1}$$

for a function $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \Omega \subset \mathbb{R}^n$, $m > 0$ being a constant, as well as wave equations where $m = 0$, with initial conditions

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \tag{1.2}$$

and Dirichlet type boundary conditions

$$u(t,\cdot) = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

The domain Ω has an infinite smooth boundary, hence is neither bounded nor an exterior domain (domain with bounded complement), but represents infinite cylinders (waveguides) or domains like strips in $\mathbb{R}^2$ or domains between two parallel planes in $\mathbb{R}^3$. More generally, Ω is assumed to satisfy

$$\Omega = \mathbb{R}^l \times B, \quad B \subset \mathbb{R}^{n-l} \text{ bounded}, \tag{1.4}$$

where $1 \le l \le n-1$. Typical examples are:

$n = 2$, $l = 1$: Infinite strip, see Figure 1.1,
$n = 3$, $l = 2$: Domain between two planes, see Figure 1.2,
$n = 3$, $l = 1$: Infinite cylinder, see Figure 1.3.

Figure 1.1

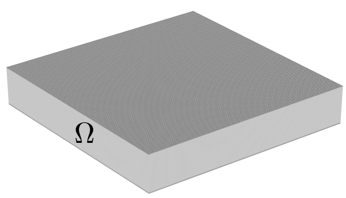

Figure 1.2

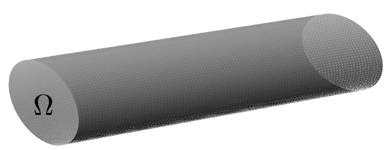

Figure 1.3

The aim of the paper is to prove sharp L^p–L^q-decay rates for the associated linearized problem

[0] AMS subject classification: 35 L 70

($f \equiv 0$ or f at most depending on t and x), and to present a global existence theorem for small data to the initial-boundary value problem (1.1)–(1.3).

There has been done a lot of work on the Cauchy problem, i.e. for $\Omega = \mathbb{R}^n$, see for example [7, 8, 12, 21, 22], or [20] for a partial survey, also the case of exterior domains has been dealt with, see for example [6, 23]. It seems that the case of infinite waveguides as given in (1.4) has not yet been treated, neither for the question of decay rates to the linearized problem nor for the question of global existence of solutions to the nonlinear problem.

The results that we are going to prove show an interesting phenomenon concerning the spreading of waves under presence of a boundary as given above. For example, the decay of solutions to the linear wave equation ($m = 0$, $f = f(t, x)$) in whole of $\mathbb{R}^3$ (Cauchy problem) is the same as for the region between two planes in $\mathbb{R}^3$, while it is weaker for infinite cylinders. More generally, a trapping of rays through the infinite boundaries affects the decay rates only if the dimension of the bounded part B is large enough.

As well known for Klein-Gordon or wave equations in exterior domains, in particular for the Cauchy problem, the decay rates for solutions to the linearized problem determine where to prove a general global existence theorem for small data for the nonlinear system, cp. [20] for a survey and a method for the Cauchy problem, or [23] for wave equations in general exterior domains.

For the *linear* Cauchy problem, $\Omega = \mathbb{R}^n$, there are different ways to get L^p-L^q-decay rates for the solutions, for example using representation formulae like the Kirchhoff formula in $\mathbb{R}^3$, or using the Fourier transform, see e.g. [18, 28, 20]. Already for the linear problem in exterior domains, these methods are not applicable, instead, the following ingredients were needed in [23] to get the decay of solutions: information on the linear Cauchy problem, the so-called local energy decay of solutions, and combinations with cut-off functions that allow to take advantage of the knowledge on the Cauchy problem.

From these short descriptions, it is obvious that one has to argue differently in our situation because there is an infinite boundary where the techniques of comparison to the Cauchy problem lead to difficulties as well as the methods to obtain the local energy decay, cp. [27]. We shall use a partial eigenfunction expansion in the bounded direction and information on the growth of eigenvalues of elliptic operators in general domains.

Another difficulty arises in the discussion of right-hand sides $f = f(t, x) \neq 0$ in contrast to the situation of the Cauchy problem, and in contrast to the situation in exterior domains. There are boundary conditions to be respected, but also cut-off techniques do not seem to work. Here, a transformation working with the inverse of the Laplacian will lead to success.

For the fully *nonlinear* system we shall then be able to follow the strategy for exterior domains as given in [23].

Remarks: 1. We remark that for the Cauchy problem it was possible to exploit the invariances of the wave operator to obtain further types of decay estimates which in turn lead to improved results for the nonlinear system, cp. [9, 10, 11]. Also for exterior domains with star-shaped complement there were recently extensions of these results, see [6], not including the infinite boundaries considered here.

2. Our results can be extended to the case of wave operators with $-\Delta$ being replaced by a

more general elliptic operator having variable coefficients

$$A = -\partial_i(a_{ij}(x))\partial_j$$

for example in the case that a_{ij} admits a decomposition like

$$a_{ij}(x) = a_{ij}(x', x'') = a'_{ij}(x') + a''_{ij}(x''),$$

where $x' \in \mathbb{R}^l$, $x'' \in \mathbb{R}^{n-l}$. Of course, further properties of the coefficients are needed to assure the applicability of the method, in particular the corresponding results on the Cauchy problem are needed. We do not pursue this here.

The paper is organized as follows: In Section 2 we shall provide the essential notation, estimates on elliptic operators that we need and information on the linear Cauchy problem. The decay estimates for the linearized case with $f = 0$ are presented in Section 3, together with a comment on Neumann boundary conditions, while in Section 4 the corresponding case with general $f = f(t, x)$ is discussed. In Section 5, the optimality of the decay rates given for the linearized situation is proved. Finally, in Section 6, the result on global existence of small, smooth solutions to the fully nonlinear system is presented.

2 Elliptic estimates and the linear Cauchy problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ fixed, be of the type

$$\Omega = \mathbb{R}^l \times B, \quad B \subset \mathbb{R}^{n-l} \text{ bounded},$$

where $1 \leq l \leq n-1$ is fixed, and ∂B is assumed to be smooth.

For $x \in \Omega$ we write

$$x = (x', x'') \quad \text{with } x' \in \mathbb{R}^l, \quad x'' \in B,$$

analogously

$$\Delta = \sum_{j=1}^{n} \partial_j^2, \qquad \Delta' = \sum_{j=1}^{l} \partial_j^2, \qquad \Delta'' = \sum_{j=l+1}^{n} \partial_j^2, \qquad \partial_j = \frac{\partial}{\partial x_j}.$$

Let

$$A : D(A) \subset L^2(\Omega) \to L^2(\Omega),$$
$$D(A) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad A\varphi := -\Delta\varphi,$$

where we use standard notations for Sobolev spaces (cp. [1]), similarly

$$A' : D(A') \subset L^2(\mathbb{R}^l) \to L^2(\mathbb{R}^l),$$

$$D(A') := W^{2,2}(\mathbb{R}^l), \quad A'\varphi := -\Delta'\varphi,$$

$$A'' : D(A'') \subset L^2(B) \to L^2(B),$$

$$D(A'') := W^{2,2}(B) \cap W_0^{1,2}(B), \quad A''\varphi := -\Delta''\varphi.$$

The operators A, A', A'' are selfadjoint, A'' is positive definite with compact inverse, having a complete orthogonal set $(w_j)_{j\in\mathbb{N}}$ of eigenfunctions corresponding to positive eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ being arranged such that

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \to \infty, \quad \text{as } j \to \infty.$$

The spectrum of A' resp. A is purely continuous and consists of

$$\sigma(A') = [0,\infty), \quad \sigma(A) = [\lambda_1,\infty),$$

cp. e.g. [14].
For $A_1 \in \{A, A', A''\}$ we have

$$\|\varphi\|^2_{W^{1,\,2}} = \langle A_1\varphi, \varphi\rangle_{L^2} + \|\varphi\|^2_{L^2}, \tag{2.1}$$

where $\|\cdot\|_{W^{1,\,2}}$ denotes the norm in $W^{1,\,2}(\Omega)$ (resp. in $W^{1,\,2}(\mathbb{R}^l), W^{1,\,2}(B)$) and $\langle\cdot,\cdot\rangle_{L^2}$ denotes the inner product in $L^2(\Omega)$ (resp. in $L^2(\mathbb{R}^l)$, $L^2(B)$) with associated norm $\|\cdot\|_{L^2}$.
For $\varphi \in D(A^{1/2})$ we have

$$\|\varphi\| \le \frac{1}{\sqrt{\lambda_1}}\,\|A^{1/2}\varphi\|$$

and from (2.1) we conclude

$$\|\varphi\|_{1,\,2} \le c\|A^{1/2}\varphi\|,$$

where c will denote (various) positive constants in this paper. By elliptic regularity theory we get for $\varphi \in D(A^{j/2})$, $j \in \mathbb{N}$,

$$\|\varphi\|_{j,2} \le c(j)\|A^{j/2}\varphi\| \tag{2.2}$$

In L^p-spaces with $p > 1$ we have

Lemma 2.1 *(L^p-regularity) Let $j \in \mathbb{N}$, $\varphi \in D(A)$, $p > 1$, $\varphi \in L^p(\Omega)$, $A\varphi \in W^{j,p}(\Omega)$. Then it holds*

$$\|\varphi\|_{W^{2+j,\,p}(\Omega)} \le c\|A\varphi\|_{W^{j,\,p}(\Omega)},$$

where c is a positive constant at most depending on j and p.

Proof: The assertion is proved for bounded Ω in [26, Theorem 9.1]. In our case Ω has bounded cross-section. Hence Poincaré's estimate holds:

$$\|\varphi\|_{W^{0,\,p}(\Omega)} \le c\,\|\nabla\varphi\|_{W^{0,\,p}(\Omega)} \qquad \text{for } \varphi \in W^{1,p}(\Omega).$$

By this the proof of [26, Theorem 9.1] extends to our case.

Q.E.D.

Corollary 2.2 *Let $j \in \mathbb{N}$, $\varphi \in D(A^j)$, $p > 1$ and φ, $A\varphi$, $\ldots$, $A^j\varphi \in L^p(\Omega)$. Then we have*

$$\|\varphi\|_{W^{2j,\,p}(\Omega)} \le c\|A^j\varphi\|_{L^p(\Omega)},$$

c at most depending on j and p.

Lemma 2.3 *The eigenvalues $(\lambda_j)_j$ of A'' satisfy*

$$\lambda_j \geq c j^{\frac{2}{n-l}},$$

where $c > 0$ is independent of j (but depends on B).

PROOF: This kind of estimate goes back to early work of Weyl, we use [2, Theorem 14.6] yielding

$$N(\lambda) = c\lambda^{\frac{n-l}{2}} + o\left(\lambda^{\frac{n-l}{2}}\right), \quad \text{as } \lambda \to \infty,$$

where $N(\lambda)$ denotes the number of eigenvalues λ_j satisfying $\lambda_j \leq \lambda$, and $c = c(B) > 0$. Hence we obtain

$$N(\lambda) \leq c\lambda^{\frac{n-l}{2}} \quad \text{for } \lambda \geq \lambda_1.$$

Since

$$N(\lambda_j) \geq j$$

this proves the assertion.

Q.E.D.

We shall need the decay estimates for the solutions to Klein-Gordon equations in $\mathbb{R}^l$ with explicit description on the dependence on the mass term.

Lemma 2.4 *(Decay for Klein-Gordon in $\mathbb{R}^l$) Let*

$$K_1 := \left[\frac{l+3}{2}\right] \quad \left(= \max\left\{j \in \mathbb{N} \;\middle|\; j \leq \frac{l+3}{2}\right\}\right),$$

and let $M \geq M_0 > 0$, $\quad v_0 \in W^{K_1,1}(\mathbb{R}^l)$, $v_1 \in W^{K_1-1,1}(\mathbb{R}^l)$.
Then the unique solution $v \in \bigcap_{j=0}^{1} C^j([0,\infty), W^{1-j,2}(\mathbb{R}^l))$ to

$$\begin{aligned} v_{tt} - (\Delta' - M)v &= 0 \qquad \text{in } [0,\infty) \times \mathbb{R}^l, \\ v(0,\cdot) = v_0, \quad v_t(0,\cdot) &= v_1 \qquad \text{in } \mathbb{R}^l \end{aligned}$$

satisfies for $t \geq \frac{1}{\sqrt{M}}$:

$$\|v(t)\|_{L^\infty(\mathbb{R}^l)} \leq \frac{c}{t^{l/2}} \left(M^{l/4} \|v_0\|_{W^{K_1,1}(\mathbb{R}^l)} + M^{\frac{l-2}{4}} \|v_1\|_{W^{K_1-1,1}(\mathbb{R}^l)} \right),$$

$c > 0$ being a constant depending at most on M_0.

PROOF:
The unique existence is standard, and it is sufficient to prove the estimate for $v_0, v_1 \in C_0^\infty(\mathbb{R}^l)$. Let

$$\tilde{v}(t,x') := v\left(\frac{t}{\sqrt{M}}, \frac{x'}{\sqrt{M}}\right) \quad \text{for } t \geq 0,\ x' \in \mathbb{R}^l.$$

Then $\tilde{v}$ satisfies

$$\tilde{v}_{tt} - (\Delta' - 1)\tilde{v} = 0,$$

$$\tilde{v}(0,x') = v_0\left(\frac{x'}{\sqrt{M}}\right) =: \tilde{v}_0, \quad v_t(0,x') = \frac{1}{\sqrt{M}}\, v_1\left(\frac{x'}{\sqrt{M}}\right) =: \tilde{v}_1.$$

By well-known estimates for Klein-Gordon equations, cp. e.g. ([20, (11.169)] or [28]), we have for $t \geq 1$:

$$|\tilde{v}(t,x')| \leq \frac{c}{t^{l/2}} \left(\|\tilde{v}_0\|_{W^{K_1,1}(\mathbb{R}^l)} + \|\tilde{v}_1\|_{W^{K_1-1,1}(\mathbb{R}^l)} \right). \tag{2.3}$$

A substitution $y' = \frac{x'}{\sqrt{M}}, \quad dx' = (\sqrt{M})^l dy'$, yields

$$\|\tilde{v}_0\|_{L^1(\mathbb{R}^l)} = M^{l/2} \|v_0\|_{L^1(\mathbb{R}^l)}$$

and, since

$$\partial_j \tilde{v}_0(x') = \frac{1}{\sqrt{M}}\, v_0\left(\frac{x'}{\sqrt{M}}\right), \quad 1 \leq j \leq l,$$

we obtain

$$\|\tilde{v}_0\|_{W^{K_1,1}(\mathbb{R}^l)} \leq c M^{l/2} \|v_0\|_{W^{K_1,1}(\mathbb{R}^l)}, \tag{2.4}$$

$$\|\tilde{v}_1\|_{W^{K_1-1,1}(\mathbb{R}^l)} \leq c M^{\frac{l-1}{2}} \|v_1\|_{W^{K_1,1}(\mathbb{R}^l)}, \tag{2.5}$$

where $c = c(M_0)$. Combining (2.3)–(2.5) we get

$$\begin{aligned} |v(t,x')| &= |\tilde{v}(\sqrt{M}t, \sqrt{M}x')| \\ &\leq \frac{c}{(\sqrt{M}\,t)^{l/2}} \left(M^{l/2} \|v_0\|_{W^{K_1,1}(\mathbb{R}^l)} + M^{\frac{l-1}{2}} \|v_1\|_{W^{K_1-1,1}(\mathbb{R}^l)} \right). \end{aligned}$$

Q.E.D.

Remark: The consideration of the optimality in Section 5 below will show that the estimate in Lemma 2.4 is optimal with respect to the decay rate (power in t) and to the power in M.

Corollary 2.5 *In addition to the assumptions in Lemma 2.4 let $v_0 \in W^{K_1+1,1}(\mathbb{R}^l)$, $v_1 \in W^{K_1,1}(\mathbb{R}^l)$. Then we have for $t \geq \frac{1}{\sqrt{M}}$:*

$$\begin{aligned} \|\nabla' v(t)\|_{L^\infty(\mathbb{R}^l)} &\leq \frac{c}{t^{l/2}} \left(M^{l/4} \|v_0\|_{W^{K_1+1,1}(\mathbb{R}^l)} + M^{\frac{l-2}{4}} \|v_1\|_{W^{K_1,1}(\mathbb{R}^l)} \right), \\ \|v_t(t)\|_{L^\infty(\mathbb{R}^l)} &\leq \frac{c}{t^{l/2}} \left(M^{\frac{l+2}{4}} \|v_0\|_{W^{K_1+1,1}(\mathbb{R}^l)} + M^{\frac{l}{4}} \|v_1\|_{W^{K_1,1}(\mathbb{R}^l)} \right), \end{aligned}$$

where $c > 0$ depends at most on M_0.

PROOF:

(i) $w_j := \partial_j v$, $j \in \{1, \ldots, l\}$, satisfies the same Klein-Gordon equation as v, now with initial data

$$w_j(0) = \partial_j v_0, \quad w_{j,t}(0) = \partial_j v_1.$$

Applying Lemma 2.4 yields the first claimed estimate for $\nabla' v = \begin{pmatrix} \partial_1 v \\ \vdots \\ \partial_l v \end{pmatrix}$.

(ii) $w_0 := v_t$ also satisfies the same differential equation, but with initial data

$$w_0(0) = v_1, \quad w_{0,t}(0) = (\Delta' - M)v_0.$$

Hence Lemma 2.4 implies

$$\begin{aligned} |w_0(t,x')| &\le \frac{c}{t^{l/2}}\left(M^{l/4}\|v_1\|_{W^{K_1,1}(\mathbb{R}^l)} + M^{\frac{l-2}{4}}\|(\Delta' - M)v_0\|_{W^{K_1-1}}\right), \\ &\le \frac{c}{t^{l/2}}\left(M^{l/4}\|v_1\|_{W^{K_1,1}(\mathbb{R}^l)} + M^{\frac{l-2}{4}}\|v_0\|_{W^{K_1+1,1}(\mathbb{R}^l)} + M^{\frac{l+2}{4}}\|v_0\|_{W^{K_1-1,1}(\mathbb{R}^l)}\right) \end{aligned}$$

proving the second claimed estimate.

Q.E.D.

3 Linear L^p–L^q-estimates for zero forces

In this section we study the linearized equations

$$u_{tt} - \Delta u + mu = 0 \qquad \text{in } [0,\infty) \times \Omega, \tag{3.1}$$

$$u = 0 \qquad \text{in } [0,\infty) \times \partial\Omega, \tag{3.2}$$

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 \qquad \text{in } \Omega, \tag{3.3}$$

where $m \ge 0$ is a constant. The L^1-L^∞-decay of solutions to (3.1)–(3.3) is described as follows:

Theorem 3.1 *Let*

$$K_2 := \left[\frac{n}{2}\right] + \left[\frac{n-l}{2}\right] + 3, \quad K_3 := \left[\frac{l+3}{2}\right] + \left[\frac{n-l+1}{2}\right],$$

and

$$u_0 \in D(A^{K_2/2}) \cap W^{K_2+K_3+1,1}(\Omega), \tag{3.4}$$

$$u_1 \in D(A^{(K_2-1)/2}) \cap W^{K_2+K_3-1,1}(\Omega). \tag{3.5}$$

Then the unique solution u to (3.1)–(3.3) satisfies

$$\|(u(t), u_t(t), \nabla u(t))\|_{L^\infty(\Omega)} \le \frac{c}{(1+t)^{l/2}}\left(\|u_0\|_{W^{K_2+K_3+1,1}(\Omega)} + \|u_1\|_{W^{K_2+K_3-1,1}(\Omega)}\right),$$

where the positive constant c depends at most on m.

Remarks:

1. Of course, c also may depend on the *fixed* n and B.
2. In Section 5 we shall demonstrate that these decay rates are sharp.

3. For the case of the wave equation, i.e. for $m=0$, one notices an interesting behavior for the case $l=n-1$, where the decay is the same as that for the Cauchy problem — no loss in the rate of decay. This holds e.g. for the infinite strip in $\mathbb{R}^2$ ($n=2$, $l=1$; see Figure 1.1 on page 1) or the region between two parallel planes in $\mathbb{R}^3$ ($n=3$, $l=2$, see Figure 1.2). In general, the decay becomes weaker as the number of bounded dimensions increases (while n is fixed).

Proof of Theorem 3.1:
<u>Case 1</u>: $0 \le t \le 1$:
Let $K_4 := [\frac{n}{2}]+2$. Observe that $u_0 \in D(A^{K_4/2})$, $u_1 \in D(A^{(K_4-1)/2})$ since $K_4 \le K_2$. Let $(P_\lambda)_{\lambda\in\mathbb{R}}$ denote the spectral family of A, then

$$u(t) = \int_{\lambda_1}^{\infty} \cos(\sqrt{\lambda+m}\,t)dP_\lambda u_0 + \int_{\lambda_1}^{\infty} \frac{\sin(\sqrt{\lambda+m}\,t)}{\sqrt{\lambda+m}} dP_\lambda u_1.$$

Therefore we get for $t \ge 0$, $x \in \Omega$

$$\begin{aligned}
|u(t,x)| + |\nabla u(t,x)| &\le c\|u(t,\cdot)\|_{W^{K_4,2}(\Omega)} \le c\|A^{K_4/2}u(t,\cdot)\|_{L^2(\Omega)} \\
&\le c\left(\left(\int_{\lambda_1}^{\infty} \lambda^{K_4}\cos^2(\sqrt{\lambda+m}\,t)d\|P_\lambda u_0\|^2_{L^2(\Omega)}\right)^{1/2}\right. \\
&\quad \left. + \left(\int_{\lambda_1}^{\infty} \frac{\lambda^{K_4}}{\lambda+m}\sin^2(\sqrt{\lambda+m}\,t)d\|P_\lambda u_1\|^2\right)^{1/2}\right) \\
&\le c\left(\|A^{K_4/2}u_0\|_{L^2(\Omega)} + A^{(K_4-1)/2}u_1\|_{L^2(\Omega)}\right) \\
&\le c\left(\|u_0\|_{W^{K_4,\,2}(\Omega)} + \|u_1\|_{W^{K_4-1,\,2}(\Omega)}\right) \\
&\le c\left(\|u_0\|_{W^{n+2,\,1}(\Omega)} + \|u_1\|_{W^{n+1,\,1}(\Omega)}\right),
\end{aligned}$$

the latter since $n+2-K_4 \ge \frac{n}{2} \ge 0$. In particular it follows for $0 \le t \le 1$, $x \in \Omega$:

$$|u(t,x)| + |\nabla u(t,x)| \le \frac{c}{(1+t)^{l/2}}\left(\|u_0\|_{W^{n+2,\,1}(\Omega)} + \|u_1\|_{W^{n+1,\,1}(\Omega)}\right). \tag{3.6}$$

Analogously, we get for $0 \le t \le 1$, $x \in \Omega$:

$$|u_t(t,x)| \le \frac{c}{(1+t)^{l/2}}\left(\|u_0\|_{W^{n+2,\,1}(\Omega)} + \|u_1\|_{W^{n+1,\,1}(\Omega)}\right). \tag{3.7}$$

<u>Case 2</u>: $t \ge 1$:

For $j \in \mathbb{N}$ let

$$v_j(t,x') := \langle u(t,x',\cdot), w_j\rangle_{L^2(B)}$$

where $(w_j)_j$ still denotes the eigenfunctions of A'' in $L^2(B)$.
If

$$v_{j,0}(x') := \langle u_0(x',\cdot), w_j\rangle_{L^2(B)}, \quad v_{j,1}(x') := \langle u_1(x',\cdot), w_j\rangle_{L^2(B)}$$

and

$$M := m + \lambda_j$$

then v_j satisfies

$$\begin{aligned} v_{j,tt} - (\Delta' - M)v_j = 0 \quad &\text{in } [0,\infty| \times \mathbb{R}^l, \\ v_j(0,\cdot) = v_{j,0}, \quad v_{j,t}(0,\cdot) = v_{j,1} &\text{ in } \mathbb{R}^l, \end{aligned}$$

and we shall be able to use the information on the decay for solutions to Klein-Gordon equations given in Lemma 2.4.
Let

$$K_5 := \left[\frac{n-l}{2}\right] + 2$$

and $\nabla'' = \begin{pmatrix} \partial_{l+1} \\ \vdots \\ \partial_n \end{pmatrix}$. Then we conclude, using (2.2),

$$\begin{aligned} |u(t,x',x'')|^2 + |\nabla'' u(t,x',x'')|^2 &\le c\|u(t,x',\cdot)\|^2_{W^{K_5,2}(B)} \\ &\le c\|(A'')^{K_5/2}u(t,x',\cdot)\|^2_{L^2(B)}, \end{aligned}$$

hence we obtain, using Lemma 2.4,

$$\begin{aligned} |u(t,x',x'')|^2 + |\nabla'' u(t,x',x'')|^2 &\le c\sum_{j=1}^{\infty} \lambda_j^{K_5}|v_j(t,x')|^2 \\ &\le \frac{c}{t^l}\sum_{j=1}^{\infty} \lambda_j^{K_5}((m+\lambda_j)^{l/2}\|\langle u_0(x'),\cdot),w_j\rangle_{L^2(B)}\|^2_{W^{K_1,1}(\mathbb{R}^l)} \\ &\quad +(m+\lambda_j)^{\frac{l-2}{2}}\|\langle u_1(x',\cdot),w_j\rangle_{L^2(B)}\|^2_{W^{K_1-1,1}(\mathbb{R}^l)}) \\ &\le \frac{c}{t^l}\sum_{j=1}^{\infty} \lambda^{K_5+l/2}\|\langle u_0,w_j\rangle_{L^2(B)}\|^2_{W^{K_1,1}(\mathbb{R}^l)} \\ &\quad +\lambda_j^{K_5+l/2-1}\|\langle u_1,w_j\rangle_{L^2(B)}\|_{W^{K_1-1,1}(\mathbb{R}^l)}. \end{aligned}$$

Observing

$$\begin{aligned} \langle u_0,w_j\rangle_{L^2(B)} &= \frac{1}{\lambda_j^{K_2/2}}\langle (A'')^{K_2/2}u_0,w_j\rangle_{L^2(B)}, \\ \langle u_1,w_j\rangle_{L^2(B)} &= \frac{1}{\lambda_j^{(K_2-1)/2}}\langle (A'')^{(K_2-1)/2}u_1,w_j\rangle_{L^2(B)}, \end{aligned}$$

we conclude

$$\begin{aligned} |u(t)(t,x',x'')|^2 + |\nabla'' u(t,x,x',x'')|^2 &\le \frac{c}{t^l}\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{K_2-K_5-l/2}}\Big(\|\langle (A'')^{K_2/2}u_0,w_j\rangle_{L^2(B)}\|^2_{W^{K_1,1}(\mathbb{R}^l)} \\ &\quad + \|(A'')^{(K_2-1)/2}u_1,w_j\rangle\|_{W^{K_1-1,1}(\mathbb{R}^l)}\Big). \end{aligned} \tag{3.8}$$

By Lemma 2.3 we know

$$\lambda_j^{K_2-K_5-l/2} \geq cj^{(K_2-K_5-l/2)\frac{2}{n-l}},$$

and, since

$$(K_2-K_5-l/2)\frac{2}{n-l} > 1$$

by the choice of K_2, K_5, the series $\sum\limits_{j=1}^{\infty} \frac{1}{\lambda_j^{K_2-K_5-l/2}}$ converges. Moreover,

$$\begin{aligned}
|\langle (A'')^{K_2/2} u_0, w_j\rangle_{L^2(B)}| &\leq \|(A'')^{K_2/2} u_0\|_{L^2(B)} \\
&\leq \|u_0\|_{W^{K_2,2}(B)} \\
&\leq \|u_0\|_{W^{K_2+[\frac{n-l+1}{2}],1}(B)},
\end{aligned}$$

similarly

$$|\langle (A'')^{(K_2-1)/2} u_1, w_j\rangle_{L^2(B)}| \leq \|u_1\|_{W^{K_2+[\frac{n-l+1}{2}]-1,1}(B)}.$$

Hence we obtain from (3.8), observing the definition of K_3, that

$$|u(t,x',x'')| + (\nabla'' u(t,x',x'')| \leq \frac{c}{t^{l/2}} \left(\|u_0\|_{W^{K_2+K_3,1}(\Omega)} + \|u_1\|_{W^{K_2+K_3-2,1}(\Omega)} \right). \tag{3.9}$$

Analogously we get

$$\begin{aligned}
|\nabla' u(t,x',x'')|^2 &\leq c\|(A'')^{(K_5-1)/2} \nabla' u(t,x',\cdot)\|^2_{L^2(B)} \\
&= c\sum_{j=1}^{\infty} \lambda_j^{K_5-1} |\nabla' \langle u(t,x',\cdot), w_j\rangle_{L^2(B)}|^2 \\
&\leq \frac{c}{t^l} \sum_{j=1}^{\infty} \lambda_j^{K_5-1} \Big((m+\lambda_j)^{\lambda_j})^{l/2} \|\langle u_0, w_j\rangle_{L^2(B)}\|^2_{W^{K_2+1,1}(\mathbb{R}^l)} \\
&\quad + (m+\lambda_j)^{(l-2)/2} \|\langle u_1, w_j\rangle_{L^2(B)}\|^2_{W^{K_1,1}(\mathbb{R}^l)} \Big) \\
&\leq \frac{c}{t^l} \left(\|u_0\|^2_{W^{K_2+K_3,1}(\Omega)} + \|u_1\|^2_{W^{K_2+K_3-2,1}(\Omega)} \right).
\end{aligned} \tag{3.10}$$

Finally,

$$\begin{aligned}
|u_t|(t,x',x'')|^2 &\leq \|u_t(t,x',\cdot)\|_{W^{K_5-1,2}(B)} \\
&\leq \frac{c}{t^l} \left(\|u_0\|^2_{W^{K_2+K_3+1,1}(\Omega)} + \|u_1\|^2_{W^{K_2+K_3-1,1}(\Omega)} \right).
\end{aligned} \tag{3.11}$$

Combining (3.7), (3.9)–(3.11) completes the proof.

Q.E.D.

Having proved the main estimate on L^1-L^∞-delay, we now provide the usual L^2-L^2-estimate and the consequences by interpolation.

Theorem 3.2 *Let $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$. Then the unique solution u to (3.1)–(3.3) satisfies for $t \geq 0$*

$$\|u(t)\|^2_{W^{1,2}(\Omega)} + \|u_t(t)\|_{L^2(\Omega)} \leq c\left(\|u_0\|^2_{W^{1,2}(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\right),$$

where c depends at most on m (and B).

Proof: The classical energy identity, obtained from (3.1)–(3.3) by multiplication of (3.1) with u_t, gives

$$\|u_t(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + m\|u(t)\|^2_{L^2(\Omega)} = \|u_1\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)} + m\|u_0\|^2_{L^2(\Omega)},$$

which proves the assertion using Poincaré's estimate

$$\|u(t)\| \leq c\|\nabla u(t)\|.$$

Q.E.D.

By interpolation we shall conclude the desired L^p–L^q-decay:

Theorem 3.3 *Let the assumptions of Theorem 3.1 be satisfied, let u be the unique solution to (3.1)–(3.3), and let $2 \leq q \leq \infty$, $\quad 1/p + 1/q = 1$.*

(i) If $u_0 = 0$, then u satisfies for $t \geq 0$:

$$\|(u(t), u_t(t), \nabla u(t))\|_{L^q(\Omega)} \leq \frac{c}{(1+t)^{(1-\frac{2}{q})^{l/2}}}\|u_1\|_{W^{N_p,p}(\Omega)},$$

where

$$N_p := \begin{cases} (1-\frac{2}{q})(K_2+K_3-1) & \text{if } \ q \in \{2,\infty\} \\ [(1-\frac{2}{q})(K_2+K_3)] & \text{if } \ 2 < q < \infty, \end{cases}$$

and c depends at most on q and m.

(ii) If additionally

$$u_1 \in D(A^{K_2-1)/2}) \cap W^{K_2+K_3,1}(\Omega) \tag{3.12}$$

then u satisfies for $t \geq 0$:

$$\|(u(t), u_t(t), \nabla u(t))\|_{L^q(\Omega)} \leq \frac{c}{(1+t)^{(1-\frac{2}{q})^{l/2}}}\|(u_0, u_1, \nabla u_1)\|_{W^{\tilde{N}_p,p}(\Omega)},$$

where

$$\tilde{N}_p := \begin{cases} (1-\frac{2}{q})(K_2+K_3) & \text{if } \ q \in \{2,\infty\}, \\ [(1-\frac{2}{q})(K_2+K_3)]+1 & \text{if } \ 2 < q < \infty, \end{cases}$$

and c depends at most on q and m.

PROOF: Using Theorem 3.1 and Theorem 3.2 the assertion follows by interpolation, cp. [4].

Q.E.D.

We remark that we can also treat other boundary conditions replacing (3.2), e.g. Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{in} \quad [0,\infty) \times \partial\Omega, \tag{3.13}$$

where $\nu = \nu(x)$ denotes the exterior normal in $x \in \partial\Omega$. There are, of course, eigenfunctions of $-\Delta''$, say w_0 corresponding to the eigenvalue $\lambda_0 = 0$. Looking at the proof of Theorem 3.1 above, it is obvious that

$$v_0(t,x') := \langle u(t,x',\cdot) w_0 \rangle_{L^2(B)}$$

only satisfies a wave equation, not a Klein-Gordon equation:

$$v_{0,tt}(t,x') - \Delta' v_0(t,x') = 0$$

and hence only provides a decay in L^∞ of order $t^{-\frac{l-1}{2}}$ (instead of $t^{-l/2}$). Indeed, if $\tilde{u} = \tilde{u}(t,x')$ is a solution to

$$\begin{aligned} \tilde{u}_{tt} - \Delta\tilde{u} &= 0 && \text{in } [0,\infty) \times \mathbb{R}^l, \\ \tilde{u}(0,x') &= \varphi(x') && \text{in } \mathbb{R}^l, \\ \tilde{u}_t(0,x') &= \psi(x') && \text{in } \mathbb{R}^l, \end{aligned}$$

with $\varphi, \psi \neq 0$, then

$$u(t,x',x'') := \tilde{u}(t,x') w_0(x'')$$

solves the linear wave equation ((3.1) with $m = 0$) in Ω and satisfies the Neumann boundary conditions (3.13), together with nonvanishing data. The decay in L^∞ is only of order $t^{-\frac{l-1}{2}}$. On the other hand, if the initial data u_0 and u_1 in (3.3) are orthogonal to the eigenfunctions w_0,

$$\langle u_0(x',\cdot), w_0 \rangle = 0 = \langle u_1(x',\cdot), w_0 \rangle, \qquad x' \in \mathbb{R}^l,$$

then the better decay of order $t^{-l/2}$ follows again.

4 Linear L^p–L^q-estimates, general case

Now we study the general linearized case

$$u_{tt} - \Delta u + mu = f \qquad \text{in } [0,\infty) \times \Omega, \tag{4.1}$$

$$u = 0 \qquad \text{in } [0,\infty) \times \Omega, \tag{4.2}$$

$$u(0,\cdot) = u_0, \qquad u_t(0,\cdot) = u_1 \qquad \text{in } \Omega, \tag{4.3}$$

where $f = f(t,x)$.
One could think of using a variation of constants formula (Duhamel's principle) to express the solution to the nonhomogeneous problem (4.1)–(4.3) with the help of solutions to the homogeneous problem (3.1)–(3.3). But this would require a set of boundary conditions for $f = f(t,\cdot)$ for fixed t (cp. the conditions on u_0, u_1 in Theorem 3.1) which are in general not satisfied in

particular for the application to the nonlinear problem (1.1)–(1.3) (cp. [19] for special nonlinearities). To overcome this difficulty we use the following idea.
Let $B := A + m$ on $D(B) := D(A)$ and

$$u_{tt} + Bu = f,$$
$$u(0) = u_0, \;\; u_t(0) = u_1.$$

Then

$$v := B^{-1} u_{tt}$$

satisfies

$$v_{tt} + Bv = B^{-1} f_{tt} =: g$$
$$v(0) = B^{-1} u_{tt}(0), \quad v_t(0) = B^{-1} u_{ttt}(0).$$

that is, v satisfies the same wave or Klein-Gordon equation but with a right-hand side g that belongs to $D(B)$, hence satisfies boundary conditions. This way, a higher regularity of f (in t) is needed and replaces the in general missing boundary conditions. The estimate for u will then be obtained from

$$u = A^{-1} f - v.$$

For this purpose let $T \in (0, \infty]$ and $K \in \mathbb{N}_0$ be arbitrary, but fixed. We look at the solution u to

$$u_{tt}(t) + Bu(t) \;=\; f(t), \qquad t \geq 0, \tag{4.4}$$
$$u(t) \in D(B), \quad t \geq 0, \tag{4.5}$$
$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{4.6}$$
$$u \in \bigcap_{j=2K}^{2K+2} C^j([0,T],\; W^{2K+2-j,2}(\Omega)). \tag{4.7}$$

Let

$$u_j := \begin{cases} \sum\limits_{k=0}^{\frac{j}{2}-1} (\Delta - m)^k f^{(j-2-2k)}(0) + (\Delta - m)^{\frac{j}{2}} u_0, & j \geq 2 \text{ even}, \\[2ex] \sum\limits_{k=0}^{\frac{j-1}{2}-1} (\Delta - m)^k f^{(j-2-2k)}(0) + (\Delta - m)^{\frac{j-1}{2}} u_1, & j \geq 3 \text{ odd}, \end{cases} \tag{4.8}$$

where $j = 2, 3, \ldots, 2K+2$ and $f^{(m)} := \left(\frac{\partial}{\partial t}\right)^m f, \;\; m \in \mathbb{N}_0$.
Then it holds

$$\left(\frac{d}{dt}\right)^j u(0) \equiv d_t^j u(0) = u_j, \quad j = 0, 1, \ldots, 2K+2.$$

Observing that B^{-1} exists since $\sigma(B) = [\lambda_1 + m, \infty)$ with $\lambda_1 > 0, \;\; m \geq 0$, we have

Theorem 4.1 *Let $f \in C^{2K}([0,T],\, L^2(\Omega))$. Then u is the solution to (4.4)–(4.7) if and only if*

$$v := u + \sum_{j=0}^{K-1} (-B)^{-(j+1)} f^{(2j)} \tag{4.9}$$

is the solution to

$$v_{tt} + Bv = (-B)^{-K} f^{(2K)}, \quad t \geq 0, \tag{4.10}$$

$$v(t) \in D(B), \quad t \geq 0, \tag{4.11}$$

$$v(0) = (-B)^{-K} u_{2K}, \quad v_t(0) = (-B)^{-K} u_{2K+1}, \tag{4.12}$$

$$v \in \bigcap_{j=0}^{2} C^j([0,T], W^{2-j,2}(\Omega)). \tag{4.13}$$

PROOF:

(i) Let v solve (4.10)–(4.13), and let u implicitly be defined by (4.9). Then we get

$$\begin{aligned}
(d_t^2 + B)u &= (d_t^2 + B)v - \sum_{j=0}^{K-1} (d_t^2 + B)(-B)^{-(j+1)} f^{(2j)} \\
&= (-B)^{-K} f^{(2K)} - \sum_{j=0}^{K-1} (-B)^{-(j+1)} f^{(2j+2)} + \sum_{j=0}^{K-1} (-B)^{-j} f^{(2j)} \\
&= f, \\
u(0) &= v(0) - \sum_{j=0}^{K-1} (-B)^{-(j+1)} f^{(2j)}(0) \\
&= (-B)^{-K} \left(\sum_{k=0}^{K-1} (-B)^k f^{(2K-2-2k)}(0) + (-B)^K u_0 \right) \\
&\quad - \sum_{j=0}^{K-1} (-B)^{-(j+1)} f^{(2j)}(0) \\
&= u_0,
\end{aligned}$$

and analogously

$$u_t(0) = u_1.$$

Hence u satisfies (4.4)–(4.7).

(ii) Let u solve (4.4)–(4.7). Let

$$\tilde{v} := (-B)^{-K} d_t^{2K} u.$$

Then

$$\begin{aligned}
(d_t^2 + B)\tilde{v} &= -(B)^{-K} d_t^{2K} (d_t^2 + B)u \\
&= (-B)^{-K} f^{(2K)},
\end{aligned}$$

and it follows that $\tilde{v}$ solves (4.10) –(4.13). According to step (i), a solution $\tilde{u}$ to (4.4)–(4.7) is defined by (4.9), which equals u by uniqueness, hence $\tilde{v} = v$.

Q.E.D.

Before stating the general result on the L^p–L^q-decay, we formulate the compatibility conditions for the data. We say that (u_0, u_1, f) satisfies the *compatibility condition of order 2K* if the u_j, defined for $k \geq 2$ in (4.8), satisfy

$$\left.\begin{array}{c} u_j \in W_0^{1,2}(\Omega) \cap W^{2K+2-j,2}(\Omega) \cap W^{2K+2-j,1}(\Omega), \;\; j = 0, 1, \ldots, 2K+1, \\ u_{2K+2} \in L^2(\Omega). \end{array}\right\} \tag{4.14}$$

Theorem 4.2 *Let*

$$K \geq \frac{K_2 + K_3 - 1}{2} = \frac{1}{2}\left(\left[\frac{n}{2}\right] + \left[\frac{l+1}{2}\right] + n - l\right),$$

$$f \in \bigcap_{j=0}^{2K} C^j([0,T], W^{2K-j,2}(\Omega) \cap W^{2K-j,1}(\Omega)) \cap C^{2K+1}([0,T], L^2(\Omega)),$$

and (u_0, u_1, f) satisfy the compatibility conditions (4.14) of order 2K. Let $2 \leq q < \infty$ and $1/p + 1/q = 1$. Then the unique solution u to (4.1)–(4.3) satisfies

$$\begin{aligned} &\|(u(t), u_t(t), \nabla u(t))\|_{L^q(\Omega)} \\ &\leq \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \left(\|(u_0, u_1, \nabla u_0)\|_{W^{2K+1,p}(\Omega)} + \sum_{j=0}^{2K-1} \|f^{(j)}(0)\|_{W^{2K-j,p}(\Omega)} \right) \\ &\quad + c \int_0^t \frac{1}{(1+t-\tau)^{(1-\frac{2}{q})\frac{l}{2}}} \|f^{(2K)}(\tau)\|_{L^p(\Omega)} d\tau + c \sum_{j=0}^{2K-1} \|f^{(j)}(t)\|_{W^{2K-1-j,p}(\Omega)}, \end{aligned}$$

where the constant $c > 0$ depends at most on m.

PROOF:

(i) According to Theorem 4.1, the function v defined by (4.9) solves (4.10)–(4.13). Observe that

$$u_{2K} \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) = D(B), \quad u_{2K+1} \in W_0^{1,2}(\Omega) = D(B^{1/2}),$$

and

$$\begin{aligned} v_0 &:= (-B)^{-K} u_{2K} \in D(B^{K+1}) \subset W^{2K+2,2}(\Omega), \\ v_1 &:= (-B)^{-K} u_{2K+1} \in D(B^{K+1,2}) \subset W^{2K+1,2}(\Omega), \\ g(t) &:= (-B)^{-K} f^{(2K)}(t) \in D(B^K) \subset W^{2K,2}(\Omega), \;\; t \geq 0. \end{aligned}$$

Observing that we can replace, for $p > 1$, the condition (3.12) in Theorem 3.3 by

$$u_1 \in D(A^{(K_2-1/2)} \cap W^{\tilde{N}_p, p}(\Omega),$$

we conclude for the solution $\hat{v}$ to

$$\hat{v}_{tt} + B\hat{v} = 0, \quad \hat{v}(0) = v_0, \quad \hat{v}_t(0) = v_1$$

from Theorem 3.3 (ii) that

$$\|(\hat{v}(t), \hat{v}_t(t), \nabla \hat{v}(t)\|_{L^q(\Omega)} \leq \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \|(v_0, v_1, \nabla v_0)\|_{W^{\tilde{N}_p, p}(\Omega)} \tag{4.15}$$

holds.

The solution $\check{v}$ to

$$\check{v}_{tt} + B\check{v} = g, \quad \check{v}(0) = 0, \quad \check{v}_t(0) = 0$$

is given by

$$\check{v}(t) = \int_0^t v^\tau(t-\tau)d\tau,$$

where v^τ solves, for fixed $0 \le \tau \le t$,

$$v^\tau_{tt} + Bv^\tau = 0, \quad v^\tau(0) = 0, \quad v_\tau(0) = g(\tau).$$

Using Theorem 3.3 (i) we conclude

$$\|(\check{v}(t), \check{v}_t(t), \nabla\check{v}(t))\|_{L^q(\Omega)} \le c\int_0^t \frac{1}{(1+t-\tau)^{(1-\frac{2}{q})\frac{l}{2}}}\|g(\tau)\|_{W^{N_p,p}(\Omega)}d\tau. \tag{4.16}$$

Since $v = \hat{v} + \check{v}$, we get from (4.15), (4.16)

$$\begin{aligned}\|(v(t), v_t, \nabla v(t)\|_{L^q(\Omega)} &\le \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}}\|(v_0, v_1, \nabla v_0)\|_{W^{\tilde{N}_p,p}(\Omega)} \\ &\quad + c\int_0^t \frac{1}{(1+t-\tau)^{(1-\frac{2}{q})\frac{l}{2}}}\|g(\tau)\|_{W^{\tilde{N}_p,p}(\Omega)}dt. \end{aligned}\tag{4.17}$$

(ii) Since $p > 1$ we can use Corollary 2.2 to conclude, using $N_p \le K_2 + K_3 - 1 = 2K$,

$$\|g(\tau)\|_{W^{N_p,p}(\Omega)} \le c\|f^{(2K)}\|_{L^p(\Omega)}.$$

Also we have $\tilde{N}_p \le 2K+1$ and get

$$\|(v_0, v_1, \nabla v_0)\|_{W^{\tilde{N}_p,p}(\Omega)} \le c(\|u_{2K}\|_{W^{2,p}(\Omega)} + \|u_{2K+1}\|_{W^{1,p}(\Omega)}).$$

Hence (4.17) turns into

$$\begin{aligned}\|(v(t), v_t(t), \nabla v(t)\|_{L^q(\Omega)} &\le \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}}\left(\|u_{2K}\|_{W^{2,p}(\Omega)} + \|u_{2K+1}\|_{W^{1,p}(\Omega)}\right) \\ &\quad + c\int_0^t \frac{1}{(1+t-\tau)^{(1-\frac{2}{q})\frac{l}{2}}}\|f^{(2K)}(\tau)\|_{L^p(\Omega)}d\tau. \end{aligned}\tag{4.18}$$

(iii) By (4.9) we have

$$\begin{aligned}\|(u(t), u_t(t), \nabla u(t))\|_{L^q(\Omega)} &\le \|(v(t), v_t(t), \nabla v(t)\|_{L^q(\Omega)} \\ &\quad + \sum_{j=0}^{K-1}(\|B^{-(j+1)}f^{(2j)}(t)\|_{L^q(\Omega)} + \|B^{-(j+1)}f^{(2j+1)}(t)\|_{L^q(\Omega)} \\ &\quad + \|\nabla B^{-(j+1)}f^{(2j)}(t)\|_{L^q(\Omega)}. \end{aligned}\tag{4.19}$$

Since $n \leq 2K$ we conclude

$$\begin{aligned}
\|B^{-(j+1)} f^{(2j)}(t)\|_{W^{1,q}(\Omega)} &\leq c\|B^{-(j+1)} f^{(2j)}(t)\|_{W^{K+1,p}(\Omega)} \\
&\leq c\|f^{(2j)}(t)\|_{W^{2K-2j-1,p}(\Omega)}
\end{aligned} \tag{4.20}$$

and, analogously,

$$\|B^{-(j+1)} f^{(2j+1)}(t)\|_{L^q(\Omega)} \leq c\|f^{(2j+1)}(t)\|_{W^{2K-2j-2,p}(\Omega)}. \tag{4.21}$$

Combining (4.19)–(4.21) we obtain

$$\|(u(t), u_t(t), \nabla u(t)\|_{L^q(\Omega)} \leq \|(v(t), v_t(t), \nabla v(t))\|_{L^q(\Omega)} + \sum_{j=0}^{2K-1} \|f^{(j)}(t)\|_{W^{2K-1-j,p}}. \tag{4.22}$$

(iv) Finally, from the definition (4.14), we conclude

$$\|u_{2K}\|_{W^{2,p}(\Omega)} \leq \sum_{j=0}^{K-1} \|f^{2j}(0)\|_{W^{2K-2j,p}(\Omega)} + \|u_0\|_{W^{2K+2,p}(\Omega)}, \tag{4.23}$$

$$\|u_{2K+1}\| \leq \sum_{j=1}^{K} \|f^{(2j-1)}(0)\|_{W^{2K-(2j-1),p}(\Omega)} + \|u_0\|_{W^{2K+1,p}(\Omega)}. \tag{4.24}$$

Combining (4.18), (4.22)–(4.24) the assertion of the Theorem follows.

Q.E.D.

5 Optimality

The following example will show that the decay rates proved in the previous sections are optimal. We look for a solution to the linear problem

$$\begin{aligned}
u_{tt}(t,x) - \Delta\, u(t,x) + m\, u(t,x) &= 0 && \text{in } [0,\infty) \times \Omega, \\
u(t,x) &= 0 && \text{in } [0,\infty) \times \partial\Omega, \\
u(0,x) = u_0(x), \quad u_t(0,x) &= u_1(x) && \text{in } \Omega,
\end{aligned} \tag{5.1}$$

($m = \text{const.} \geq 0$) which has, as $t \to \infty$, exactly the L^∞-decay $O\Big((\frac{1}{t})^l\Big)$. For this purpose let $(w_j)_{j\in\mathbb{N}}$ denote again the orthormal system of eigenfunctions to the eigenvalues λ_j of $-\Delta''$ in B. Then we shall prove

Theorem 5.1 *Let $m = \text{const.} \geq 0$, $v_0 \in C_0^\infty(\mathbb{R}^l)$ and*

$$\begin{aligned}
u_0(x) &= u_0(x', x'') := v_0(x')\, w_1(x'') \qquad \textit{for } x \in \Omega, \\
u_1 &:= 0.
\end{aligned}$$

Then we have for the solution u to (5.1)

$$u(t, x', x'') = \frac{(\lambda_1 + m)^{l/4}\, \mathrm{i}^{l/2}}{(2\pi)^{l/2}} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda_1 + m}\, t}}{t^{l/2}} \int_{\mathbb{R}^l} v_0(x')\, dx'\, w_1(x'') + O\left(\frac{1}{t^{(l+1)/2}}\right) \quad \textit{as } t \to \infty \tag{5.2}$$

for any fixed $x = (x', x'') \in \Omega$.

The proof of this theorem will use the following representation of solutions to Klein-Gordon equations in $\mathbb{R}^l$:

Theorem 5.2 *Let $v_0 \in C_0^\infty(\mathbb{R}^l)$, $M = \text{const.} > 0$, and let $v \in C^2([0,\infty) \times \mathbb{R}^l)$ be the solution to*

$$\begin{aligned} v_{tt}(t,x') - \Delta' v(t,x') + M\, v(t,x') &= 0 \quad \text{in } [0,\infty) \times \mathbb{R}^l, \\ v(0,x') = v_0(x'), \quad v_t(0,x') &= 0 \quad \text{in } \mathbb{R}^l. \end{aligned} \tag{5.3}$$

Then we have

$$v(t,x') = \frac{M^{l/4}\, \mathrm{i}^{l/2}}{(2\pi)^{l/2}} \frac{\mathrm{e}^{\mathrm{i}\sqrt{M}\,t}}{t^{l/2}} \int_{\mathbb{R}^l} v_0(x')\, dx' + O\Big(\frac{1}{t^{(l+1)/2}}\Big) \quad \text{as } t \to \infty \tag{5.4}$$

for any fixed $x' \in \mathbb{R}^l$.

Proof of Theorem 5.1: Let

$$u(t,x) = u(t,x',x'') := v(t,x')\, w_1(x'') \qquad \text{for } x \in \overline{\Omega}.$$

u is the solution to (5.1) with the given data if and only if v is the solution to (5.3) with $M := m + \lambda_1$. Now we get (5.1) from (5.4).

Q.E.D.

Remark: The proof of the optimality of the well-known decay rates for solutions to Klein-Gordon or wave equations seems to be not explicitly stated in the literature. In a paper of Sideris, for example, the optimality for the wave equation is contained in the considerations on the blow-up there, see Lemma 6 and the remarks at the end in [25]. Although the result on the optimality for solutions to Klein-Gordon equations is folklore, we present a proof here, simultaneously giving a representation and an asymptotic expansion.

Proof of Theorem 5.2: Let $\{\tilde{P}_\lambda\}$ denote the spectral family of the selfadjoint Operator $-\Delta'$ in $L^2(\mathbb{R}^l)$. For the spectral family $\{P_\lambda\}$ of the selfadjoint operator $-\Delta' + M$ in $L^2(\mathbb{R}^l)$, we have $P_\lambda = \tilde{P}_{\lambda - M}$,

$$P_\lambda = 0 \qquad \text{for } \lambda \le M$$

and

$$\begin{aligned} \frac{dP_\lambda v_0}{d\lambda} &= \frac{d\tilde{P}_{\lambda-M} v_0}{d\lambda}(x') \\ &= \frac{1}{2(2\pi)^{l/2}} |\lambda - M|^{\frac{l-2}{4}} \int_{\mathbb{R}^l} \frac{v_0(y')}{|x'-y'|^{\frac{l}{2}-1}} J_{\frac{l}{2}-1}(|x'-y'|\sqrt{\lambda - M})\, dy' \qquad \text{for } \lambda > M \end{aligned}$$

(see e.g. [15], $J_{\frac{l}{2}-1}$ = Bessel function).

The solution to (5.3) is given by

$$v(t,x') = \int_M^\infty \cos\Big(\sqrt{\lambda}\,t\Big) \frac{dP_\lambda v_0}{d\lambda}(x')\, d\lambda \qquad \text{for } t \ge 0,\ x' \in \mathbb{R}^l. \tag{5.5}$$

For fixed $x' \in \mathbb{R}^l$ we write

$$\frac{dP_\lambda v_0}{d\lambda}(x') = (\lambda - M)^{\frac{l}{2}-1} \varphi(\lambda) \tag{5.6}$$

with

$$\varphi(\lambda) \;:=\; \frac{1}{2(2\pi)^{l/2}} \int_{\mathbb{R}^l} u_0(y') \frac{J_{\frac{l}{2}-1}(|x'-y'|\sqrt{\lambda - M})}{(|x'-y'|\sqrt{\lambda-M})^{\frac{l}{2}-1}}\, dy' \tag{5.7}$$

It should be observed that

$$\tilde{J}_l(z) \;:=\; \frac{J_{\frac{l}{2}-1}(z)}{z^{\frac{l}{2}-1}} \;=\; \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j+\frac{l}{2}-1}\, j!\, \Gamma\left(\frac{l}{2}+j\right)}\, z^{2j} \tag{5.8}$$

is an entire function (see [17]). In particular, φ is analytic on $(-\infty,\infty)$, and we have

$$\varphi(M) = \frac{1}{2(2\pi)^{l/2}} \frac{1}{2^{\frac{l}{2}-1}\Gamma\left(\frac{l}{2}\right)} \int_{\mathbb{R}^2} u_0(y')\, dy'. \tag{5.9}$$

The claim (5.4) now follows from (5.5), (5.6), (5.9) and the following two lemmata.

Q.E.D.

Lemma 5.3 *Let $g \in C_0^\infty(\mathbb{R}^l)$. For fixed $x \in \Omega$ and arbitrary $j,k \in \mathbb{N}$ it holds*

$$\frac{d^j P_\lambda f}{d\lambda^j}(x) \;=\; O\left(\lambda^{-k}\right) \;\text{ as } \lambda \to \infty. \tag{5.10}$$

Lemma 5.4 *Let $l \in \mathbb{N}$, $L := [\frac{l+2}{2}] := \max\{j \in \mathbb{N} : j \le \frac{l+2}{2}\}$. Moreover, let $M > 0$ and $\varphi \in C^L([M,\infty))$ with $\varphi(M) \neq 0$, such that the function Φ defined by*

$$\Phi(\lambda) \;:=\; (\lambda - M)^{\frac{l}{2}-1}\varphi(\lambda) \qquad \text{for } \lambda \in [M,\infty)$$

satisfies

$$\frac{d^j \Phi}{d\lambda^j}(\lambda) \;=\; O\left(\lambda^{-1-\varepsilon-j/2}\right) \;\text{ as } \lambda \to \infty \qquad \forall j \in \{0,1,\ldots,L\}$$

with fixed $\varepsilon > 0$. Then it follows

$$\int_M^\infty e^{i\sqrt{\lambda}t}\Phi(\lambda)\, d\lambda \;=\; \Gamma\left(\frac{l}{2}\right)\left(2\sqrt{M}\right)^{l/2} i^{l/2} \frac{e^{i\sqrt{M}t}}{t^{l/2}}\, \varphi(M) + O\left(\frac{1}{t^{(l+1)/2}}\right) \tag{5.11}$$

as $t \to \infty$.

Proof of Lemma 5.3: The series expansion (5.8) implies

$$-\Delta'_{y'}\, \tilde{J}_l\left(|x'-y'|\sqrt{\lambda-M}\right) \;=\; (\lambda - M)\, \tilde{J}_l\left(|x'-y'|\sqrt{\lambda-M}\right),$$

hence (cp. (5.7) and (5.8))

$$\begin{aligned}
\varphi(\lambda) &= \frac{1}{2(2\pi)^{l/2}} \int_{\mathbb{R}^l} f(y')\, \tilde{J}_l\left(|x'-y'|\sqrt{\lambda-M}\right)\, dy' \\
&= \frac{1}{2(2\pi)^{l/2}} \int_{\mathbb{R}^l} f(y')\, \frac{-\Delta'_{y'}}{\lambda - M}\, \tilde{J}_l\left(|x'-y'|\sqrt{\lambda-M}\right)\, dy' \\
&= -\frac{1}{\lambda - M}\, \frac{1}{2(2\pi)^{l/2}} \int_{\mathbb{R}^l} (\Delta' f)(y')\, \tilde{J}_l\left(|x'-y'|\sqrt{\lambda-M}\right)\, dy',
\end{aligned}$$

and, with (5.6),

$$\frac{dP_\lambda f}{d\lambda} = -\frac{1}{\lambda - M}\,\frac{dP_\lambda(\Delta' f)}{d\lambda}$$

Iteration yields

$$\frac{dP_\lambda f}{d\lambda} = \frac{(-1)^k}{(\lambda - M)^k}\,\frac{dP_\lambda({\Delta'}^k f)}{d\lambda} \qquad \text{for } k \in \mathbb{N}. \tag{5.12}$$

According to [17, p. 139] we have

$$|\tilde{J}_l(\xi)| = \left|\frac{J_{\frac{l}{2}+1}(\xi)}{\xi^{\frac{l}{2}+1}}\right| = O\Big(\frac{1}{\xi^{(l+3)/2}}\Big) \qquad \text{as } \xi \to \infty.$$

Using the holomorphy of $\tilde{J}_l$ the boundedness of $\tilde{J}_l$ on $[0,\infty)$ follows. Therefore, we obtain from (5.6) and (5.7)

$$\left|\frac{dP_\lambda({\Delta'}^k f)}{d\lambda}(x)\right| \le (\lambda - M)^{\frac{l}{2}-1}\,\frac{c}{2(2\pi)^{l/2}}\int_{\mathbb{R}^l}\left|{\Delta'}^k f(y')\right| dy'$$

for $x' \in \mathbb{R}^l$, $k \in \mathbb{N}$, since f has compact support. With (5.12) the validity of (5.10) is proved for the case $j = 1$. For $j \ge 2$ we differentiate (5.12) with respect to λ:

$$\frac{d^2 P_\lambda f}{d\lambda^2} = \frac{(-1)^k(-k)}{(\lambda - M)^{k+1}}\,\frac{dP_\lambda({\Delta'}^k f)}{d\lambda} + \frac{(-1)^k}{(\lambda - M)^k}\,\frac{d^2 P_\lambda({\Delta'}^k f)}{d\lambda^2} \qquad \text{for } k \in \mathbb{N}$$

and similarly for higher derivatives $\frac{d^j P_\lambda f}{d\lambda^j}$. Moreover, from (5.8),

$$\frac{d}{d\lambda}\Big((\lambda - M)^{\frac{l}{2}-1}\tilde{J}_l\Big(|x' - y'|\sqrt{\lambda - M}\Big)\Big) = \frac{1}{2}\,(\lambda - M)^{\frac{l-2}{2}-1}\,\tilde{J}_{l-2}\Big(|x' - y'|\sqrt{\lambda - M}\Big).$$

Here one should notice that those terms (5.8) are defined to be zero for which the argument of the Gamma function is a negative integer. Hence (5.10) follows also for the case $j \ge 2$ with the same arguments as in the case $j = 1$.

Q.E.D.

Proof of Lemma 5.4: By partial integration we get

$$\begin{aligned}
\int e^{i\sqrt{\lambda}\,t}\Phi(\lambda)\,d\lambda &= \int \frac{e^{i\sqrt{\lambda}\,t}}{\sqrt{\lambda}}\sqrt{\lambda}\,\Phi(\lambda)\,d\lambda \\
&= -\frac{2i}{t}\,e^{i\sqrt{\lambda}\,t}\sqrt{\lambda}\,\Phi(\lambda) + \frac{i}{t}\int e^{i\sqrt{\lambda}\,t}\Big(2\sqrt{\lambda}\,\Phi'(\lambda) + \frac{\Phi(\lambda)}{\sqrt{\lambda}}\Big)\,d\lambda
\end{aligned} \tag{5.13}$$

for $\lambda > M$.

1. The case $l = 2$: One hase $\Phi = \varphi$ and, with (5.13),

$$\begin{aligned}
\int_M^\infty e^{i\sqrt{\lambda}\,t}\,\varphi(\lambda)\,d\lambda &= \frac{2i}{t}\,e^{i\sqrt{M}\,t}\sqrt{M}\,\varphi(M) + \frac{i}{t}\int_M^\infty e^{i\sqrt{\lambda}\,t}\Big(2\sqrt{\lambda}\,\varphi'(\lambda) + \frac{\varphi(\lambda)}{\sqrt{\lambda}}\Big)\,d\lambda \\
&= \frac{2i}{t}\,e^{i\sqrt{M}\,t}\sqrt{M}\,\varphi(M) + \frac{i}{t}\Big(-\frac{2i}{t}\,e^{i\sqrt{\lambda}\,t}(2\lambda\,\varphi'(\lambda) + \varphi(\lambda))\Big|_{\lambda=M}^{\infty} \\
&\quad + \frac{i}{t}\int_M^\infty e^{i\sqrt{\lambda}\,t}(4\lambda\varphi''(\lambda) + 6\varphi'(\lambda))\,d\lambda\Big) \\
&= \frac{2i}{t}\,e^{i\sqrt{M}\,t}\sqrt{M}\,\varphi(M) + O\Big(\frac{1}{t^2}\Big) \qquad \text{as } t \to \infty.
\end{aligned}$$

This proves (5.11) for $l = 2$.

2. The case $l = 1$: Applying (5.13) yields

$$I_1(t) \;:=\; \int_{M+1}^{\infty} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}\,\Phi(\lambda)\,d\lambda \;=\; O\Big(\frac{1}{t}\Big) \qquad \text{as } t \to \infty. \tag{5.14}$$

We write

$$\begin{aligned}\int_M^{M+1} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}\Phi(\lambda)\,d\lambda \;&=\; \int_M^{M+1} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}}{\sqrt{\lambda - M}}\,\varphi(\lambda)\,d\lambda \\ &=\; \frac{\varphi(M)}{\sqrt{2\sqrt{M}}}\int_M^{M+1} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}}{\sqrt{\sqrt{\lambda}-\sqrt{M}}}\,d\lambda + \int_M^{M+1} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}\,\psi(\lambda)\,d\lambda \end{aligned} \tag{5.15}$$

with

$$\psi(\lambda) \;:=\; \frac{\varphi(\lambda)-\varphi(M)}{\sqrt{\lambda - M}} + \varphi(M)\left(\frac{1}{\sqrt{\lambda-M}} - \frac{1}{\sqrt{2\sqrt{M}}\sqrt{\sqrt{\lambda}-\sqrt{M}}}\right)$$

and

$$\lim_{\lambda\downarrow 0}\psi(\lambda) \;=\; 0, \quad \lim_{\lambda\downarrow 0}\sqrt{\lambda - M}\psi'(\lambda) \text{ exists}(= \frac{1}{2}\,\varphi'(M) - \frac{5}{32M}\,\varphi(M)).$$

Therefore we conclude using (5.13) (with ψ replacing Φ)

$$\int_M^{M+1} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}\,\psi(\lambda)\,d\lambda \;=\; O\Big(\frac{1}{t}\Big) \qquad \text{as } t \to \infty \tag{5.16}$$

With $\mu := \left(\sqrt{\lambda} - \sqrt{M}\right) t$ and $\delta := \sqrt{M+1} - \sqrt{M}$ we obtain

$$\begin{aligned}\int_M^{M+1} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}}{\sqrt{\sqrt{\lambda}-\sqrt{M}}}\,d\lambda \;&=\; \mathrm{e}^{\mathrm{i}\sqrt{M}\,t}\int_0^{\delta t} \frac{\mathrm{e}^{\mathrm{i}\mu}}{\sqrt{\mu/t}}\,\frac{2}{t}\left(\frac{\mu}{t} + \sqrt{M}\right)d\mu \\ &=\; \mathrm{e}^{\mathrm{i}\sqrt{M}\,t}\left(\frac{\sqrt{M}}{\sqrt{t}}J_2(t) + \frac{1}{t^{3/2}}J_3(t)\right),\end{aligned}$$

where

$$\begin{aligned} J_2(t) \;&:=\; 2\int_0^{\delta t} \frac{\mathrm{e}^{\mathrm{i}\mu}}{\sqrt{\mu}}\,d\mu \\ &=\; 2\int_0^{\infty} \frac{\mathrm{e}^{\mathrm{i}\mu}}{\sqrt{\mu}}\,d\mu - 2\int_{\delta t}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\mu}}{\sqrt{\mu}}\,d\mu \\ &=\; \sqrt{2\pi}(1+\mathrm{i}) + O\Big(\frac{1}{\sqrt{t}}\Big) \qquad \text{as } t \to \infty \end{aligned}$$

(cp. [3]) and

$$J_3(t) \;:=\; 2\int_0^{\delta t} \mathrm{e}^{\mathrm{i}\mu}\sqrt{\mu}\,d\mu \;=\; \frac{2}{\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\mu}\sqrt{\mu}\,\Big|_{\mu=0}^{\delta t} - \frac{1}{\mathrm{i}}\int_0^{\delta t} \frac{\mathrm{e}^{\mathrm{i}\mu}}{\sqrt{\mu}}\,d\mu \;=\; O\left(\sqrt{t}\right) \qquad \text{as } t \to \infty.$$

This proves

$$\int_M^{M+1} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\,t}}{\sqrt{\sqrt{\lambda}-\sqrt{M}}}\,d\lambda \;=\; \frac{\sqrt{2\pi}\,(1+\mathrm{i})\,\sqrt{M}}{\sqrt{t}}\,\mathrm{e}^{\mathrm{i}\sqrt{M}\,t} + O\Big(\frac{1}{t}\Big) \qquad \text{as } t \to \infty.$$

and, in combination with (5.13), (5.14) and (5.15), the claim (5.11) for $l = 1$.

3. For the case $l \geq 3$ we use induction with respect to l (in two steps each time): (5.13), observing $\Phi(M) = 0$, yields

$$\int_M^\infty e^{i\sqrt{\lambda}t}\Phi(\lambda)\,d\lambda \;=\; \frac{i}{t}\int_M^\infty e^{i\sqrt{\lambda}t}\left(2\sqrt{\lambda}\,\Phi'(\lambda) + \frac{\Phi(\lambda)}{\sqrt{\lambda}}\right)d\lambda.$$

Now we have

$$\tilde{\Phi}(\lambda) \;:=\; 2\sqrt{\lambda}\,\Phi'(\lambda) + \frac{\Phi(\lambda)}{\sqrt{\lambda}} \;=:\; (\lambda - M)^{\frac{l}{2}-2}\tilde{\varphi}(\lambda),$$

where $\tilde{\varphi} \in C^{L-1}([M,\infty))$, $\tilde{\varphi}(M) = 2\sqrt{M}\left(\frac{l}{2} - 1\right)\varphi(M)$ (cp. the assumptions in Lemma 5.4) and

$$\frac{d^j\tilde{\Phi}}{d\lambda^j}(\lambda) \;=\; O\!\left(\lambda^{1-\varepsilon-j/2}\right) \qquad \text{as } \lambda \to \infty,\ j = 0,1,\ldots,L-1.$$

The induction hypothesis (l replaced by $l-2$) yields

$$\begin{aligned}
&\int_M^\infty e^{i\sqrt{\lambda}t}\,\Phi(\lambda)\,d\lambda\\
&= \frac{i}{t}\left(\Gamma\!\left(\frac{l-2}{2}\right)\left(2\sqrt{M}\right)^{(l-2)/2} i^{(l-2)/2}\,\frac{e^{i\sqrt{\lambda}t}}{t^{(l-2)/2}}\,\tilde{\varphi}(M) + O\!\left(\frac{1}{t^{(l-1)/2}}\right)\right)\\
&= \Gamma\!\left(\frac{l}{2}\right)\left(2\sqrt{M}\right)^{l/2} i^{l/2}\,\frac{e^{i\sqrt{\lambda}t}}{t^{l/2}}\,\varphi(M) + O\!\left(\frac{1}{t^{(l+1)/2}}\right)
\end{aligned}$$

as $t \to \infty$.

Q.E.D.

6 Nonlinear equations

We turn to the fully nonlinear system (1.1)–(1.3), i.e. we look for solutions and for the asymptotic behavior of solutions u to

$$u_{tt} - \Delta u + mu = f(u, u_t, \nabla u, \nabla u_t, \nabla^2 u) \qquad \text{in } [0,\infty)\times\Omega \tag{6.1}$$

$$u = 0 \qquad \text{in } [0,\infty)\times\partial\Omega, \tag{6.2}$$

$$u(0,\cdot) = u_0, \qquad u_t(0,\cdot) = u_1 \qquad \text{in } \Omega, \tag{6.3}$$

where $m \geq 0$ again.
In [23] nonlinear wave equations ($m = 0$) were studied in *exterior* domains $\tilde{\Omega}$, i.e. $\mathbb{R}^n \setminus \tilde{\Omega}$ is bounded, which are *non-trapping.*
Remark: In [23] more general nonlinearities were considered like $f_1(t, x, u_t, \nabla u, \nabla u_t, \nabla^2 u) + f_2(t, x)$, but not involving u. This type could also be dealt with here but is just replaced by f as in (6.1) for simplicity. The methods from [23] also apply to the case $m > 0$. In our case we can also treat $f = f(u, \ldots)$ depending on u because of Poincaré's estimate.
The strategy in [23] consists in

(i) having a local existence theorem available from [24],

(ii) proving L^p-L^q-estimates for the linearized system,

(iii) proving a priori estimates for the local solution exploiting (ii).

Part (ii) strongly uses the local energy decay property for non-trapping domains which is not available in our case. But we have already proved the general L^p–L^q-decay result Theorem 4.2 (which replaces the corresponding [23, Theorem 4.1]), and we can proceed as in [23]. Therefore we start formulating the theorem on global existence of solutions for small data which also provides information on the asymptotic behavior as $t \to \infty$, and only give an idea of the proof, referring to [23] for details. Let f be smooth and satisfy

$$f(W) = O(|W|^{\alpha+1}) \quad \text{as } |W| \to 0, \tag{6.4}$$

where

$$\alpha = \alpha(l) := \begin{cases} 3 & \text{if } l = 1, \\ 2 & \text{if } 2 \le l \le 4, \\ 1 & \text{if } l \ge 5. \end{cases} \tag{6.5}$$

Remark: The case $l = 1$, corresponding to $n - 1$ in [23], hence to $n = 2$ there, is not treated in [23].
Let

$$q(l) := 2\alpha(l) + 2 = \begin{cases} 8 & \text{if } l = 1, \\ 6 & \text{if } 2 \le l \le 4, \\ 4 & \text{if } l \ge 5. \end{cases} \tag{6.6}$$

with associated Hölder exponent

$$p(l) = \begin{cases} 8/7 & \text{if } l = 1, \\ 6/5 & \text{if } 2 \le l \le 4, \\ 4/3 & \text{if } l \ge 5. \end{cases} \tag{6.7}$$

Finally, let

$$d(l) := \frac{\alpha(l)}{\alpha(l)+1}\,\frac{l}{2} = \begin{cases} 3/8 & \text{if } l = 1, \\ 2/3 & \text{if } l = 2, \\ 1 & \text{if } l = 3, \\ 4/3 & \text{if } l = 4, \\ l/4 & \text{if } l = 5. \end{cases} \tag{6.8}$$

The number $d(l)$ is the decay rate of the $L^{q(l)}$-norm $\|u(t)\|_{L^{q(l)}}$ for the linearized problem, $\alpha = \alpha(l)$ is determined by the condition

$$\frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right) < \frac{l}{2} \tag{6.9}$$

and then q resp. p by

$$\frac{1}{q} + \frac{1}{p} = 1 \quad \text{and} \quad \frac{\alpha}{q} + \frac{1}{2} = \frac{1}{p}, \tag{6.10}$$

see [20, pp 92, 93] for the general method and motivation behind.
The compatibility condition (4.14) carries over to one for (u_0, u_1, f), where $f^{(m)}(0)$ is now successively expressed in terms of u_0, u_1 by using the differential equation (6.1) (cp also [23]).

Theorem 6.1 *Assume (6.4). Then there are* $K = K(l) \in \mathbb{N}$ *and* $\varepsilon = \varepsilon(l, \Omega) > 0$ *such that if* $u_0 \in W^{2K,2}(\Omega) \cap W^{2K-1,p(l)}(\Omega)$, $u_1 \in W_2^{2K-1,2}(\Omega) \cap W^{2K-2,p(l)}(\Omega)$ *and* (u_0, u_1, f) *satisfies the compatibility condition (4.14) of order 2K, and*

$$\|u_0\|_{W^{2K,\,2}(\Omega)} + \|u_0\|_{W^{2K-1,\,p(l)}(\Omega)} + \|u_1\|_{W^{2K-1,\,2}(\Omega)} + \|u_1\|_{W^{2K-2,\,p(l)}(\Omega)} < \varepsilon,$$

then there exists a unique solution $u \in \bigcap_{j=0}^{2K} C^j\Big([0,\infty), W^{2K-j,2}(\Omega)\Big) \subset C^2([0,\infty) \times \overline{\Omega})$ *to (6.1)–(6.3) satisfying*

$$\sup_{t \geq 0} \Big(\|(u(t), u_t(t), \nabla u(t)\|_{L^2(\Omega)} + (1+t)^{d(l)} \|u(t), u_t(t), \nabla u(t)\|_{L^{q(l)}(\Omega)} \Big) \leq c_1,$$

where the constant $c_1 0$ *depends at most on* l, Ω, m.

The PROOF consists of two steps (see [23] for details).

1. Considering a local solution in time for $0 \leq t \leq T$ for some $T > 0$ in $W^{m,2}(\Omega)$-spaces. This local existence can be taken from [24] or [13, 16].

2. Proving uniform a priori estimates for the local solution using the linear decay described in Theorem 4.2: variation of constants formula, differentiating the equation in t to preserve boundary conditions, using the elliptic estimates for $-\Delta$, cp. [23] for the enormous technical details.

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