Resonance phenomena for water waves in channels of arbitrary cross-section

M. D. Groves*
Institut für angewandte Mathematik
Universität Hannover
Welfengarten 1
30167 Hannover
Germany

P. H. Lesky
Mathematisches Institut A
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
Germany

Abstract

This article studies the evolutionary problem for linear gravity waves on the surface of water in a uniform, symmetric channel which is excited by an antisymmetric pressure force of frequency ω at the free surface. It is shown that there is a countably infinite set of frequencies $\{\omega_0, \omega_1, \ldots\}$ which give rise to resonance phenomena: the amplitude of the wave motion grows like $t^{1/2}$ as $t \to \infty$ in a sense which is precisely specified. Under pressure forcing at any other frequency the solution obeys the principle of limiting amplitude. These results are obtained by combining methods developed for problems in acoustic waveguides with regularity theory for elliptic boundary-value problems in nonsmooth domains.

MOS subject classification: 35B40, 35C10, 76B15

*Supported by a Research Fellowship from the Alexander von Humboldt Foundation. Permanent address: Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK.

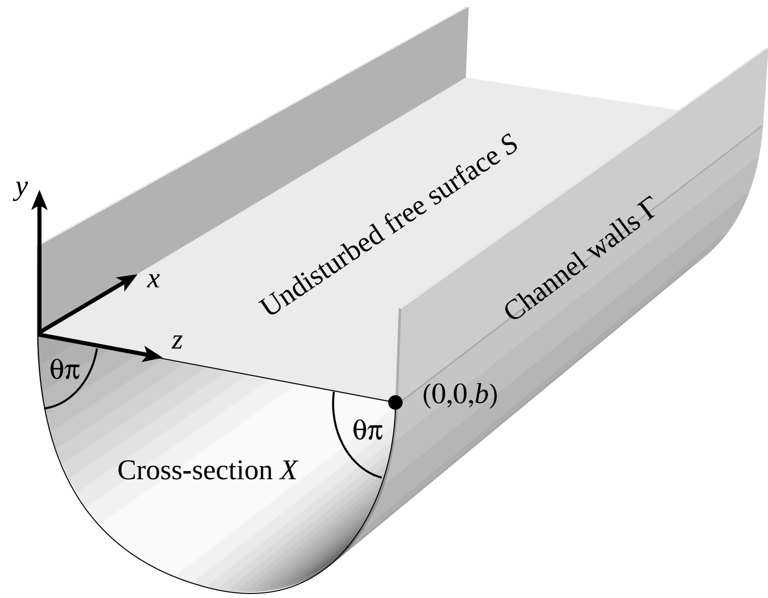

Figure 1: Sketch of the channel geometry

1 Introduction

1.1 The hydrodynamic problem

This paper treats an evolution problem for linear gravity waves on the surface of water in a uniform horizontal channel of bounded cross-section $X \subset \mathbb{R}^2$ (see Figure 1). In its undisturbed state the fluid occupies a cylindrical domain $D = \mathbb{R} \times X$ bounded by the undisturbed free surface

$$S = \{(x, 0, z) : (x, z) \in \mathbb{R} \times (0, b)\}$$

and the piecewise smooth channel walls Γ formed by the smooth surfaces $\Gamma_1, \ldots, \Gamma_N$. The cross-section X is supposed to be symmetric about the centre line $L = \{(y, b/2) : y \in \mathbb{R}\} \cap X$; its boundary ∂X consists of a finite number of smooth curves which coincide with straight lines near their ends. Let us denote the angle enclosed by the corners at $(0, 0)$ and $(0, b)$ by $\theta\pi$ and suppose that the angles at the other corners (all in $\partial X \cap \Gamma$) are not greater than $\theta^\star\pi$. Attention is focused upon *antisymmetric motions* that are driven by an antisymmetric pressure force $p(x, z)\mathrm{e}^{-\mathrm{i}\omega t}$ at the free surface

$$S_\eta = \{(x, y, z) : (x, z) \in \mathbb{R} \times (0, b), y = \eta(x, z, t)\}.$$

The restriction to antisymmetric wave motions in symmetric channels simplifies the subsequent analysis by excluding the constant solutions of a certain boundary-value problem (equations (12)–(14) below); the general case requires additional considerations which will be discussed elsewhere.

The mathematical problem is to find solutions $\phi(x, y, z, t)$ of Laplace's equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad \text{in } D \tag{1}$$

subject to the boundary conditions

$$\partial_n \phi = 0 \qquad \text{on } \Gamma, \tag{2}$$

$$
\begin{aligned}
|\nabla\phi| &\to 0 && \text{as } x \to \pm\infty, && (3)\\
\eta_t &= \phi_y && \text{on } S, && (4)\\
\phi_t + \mathrm{g}\eta &= -p\mathrm{e}^{-\mathrm{i}\omega t} && \text{on } S && (5)
\end{aligned}
$$

(Stoker [10, §§1, 2.1]) and suitable initial conditions (see below). Here ϕ is supposed to be antisymmetric about $\mathbb{R} \times L$ and g is the acceleration due to gravity.

This problem can be formulated in terms of a *Dirichlet-Neumann* operator G which is formally defined as follows. For given $\Phi(t, x, z)$ with $(x, 0, z) \in S$, let ϕ be the unique solution of (1) with boundary conditions (2), (3) and

$$\phi = \Phi \qquad \text{on } S \tag{6}$$

and define $G\Phi = \phi_y|_S$. Equations (4), (5) may then be replaced by

$$\Phi_{tt} + A\Phi = f\mathrm{e}^{-\mathrm{i}\omega t}, \tag{7}$$

where $f = \mathrm{i}\omega p$ and $A = gG$. Specifying initial conditions

$$\Phi(0, x, z) = \Phi_0(x, z), \qquad \Phi_t(0, x, z) = \Phi_1(x, z), \tag{8}$$

observe that the evolutionary problem (7), (8) completely determines the solution of (1)–(5): at each time $t \geq 0$ the velocity potential ϕ is uniquely determined as the solution of (1) subject to the boundary conditions (2), (3), (6).

The operator G is defined rigorously in Section 2 in terms of a weak formulation of the elliptic boundary-value problem (1)–(3), (6), where it is shown to be a self-adjoint, positive, unbounded operator on $\check{L}^2(S)$, the closed subspace of $L^2(S)$ consisting of those functions which are antisymmetric about $\mathbb{R} \times \{b/2\}$. The same is therefore true of A and clearly $\mathcal{D}(A^s) = \mathcal{D}(G^s)$ for each $s \geq 0$. These observations indicate that it is appropriate to study the following weak formulation of (7), (8): find $\Phi \in C^2([0, \infty), \check{L}^2(S))$ such that $\Phi(t) \in \mathcal{D}(A)$ for $t \geq 0$ and

$$\Phi_{tt}(t) + A\Phi(t) = f\mathrm{e}^{-\mathrm{i}\omega t}, \qquad t \geq 0, \tag{9}$$

$$\Phi(0) = \Phi_0, \qquad \Phi_t(0) = \Phi_1 \tag{10}$$

for $f \in \check{L}^2(S)$, $\Phi_0 \in \mathcal{D}(A)$, $\Phi_1 \in \mathcal{D}(A^{1/2})$. This problem has a unique solution given by

$$\Phi(t) = \int_0^\infty \cos(\sqrt{\lambda}t)\,\mathrm{d}(P_\lambda\Phi_0) + \int_0^\infty \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,\mathrm{d}(P_\lambda\Phi_1) + \int_0^\infty \chi(\lambda, t)\,\mathrm{d}(P_\lambda f), \qquad t \geq 0, \tag{11}$$

where $\{P_\lambda\}_{\lambda\in(-\infty,\infty)}$ is the spectral family of A and

$$\chi(\lambda, t) = \begin{cases} \dfrac{1}{\lambda - \omega^2}\left(\mathrm{e}^{-\mathrm{i}\omega t} - \cos(\sqrt{\lambda}t) + \dfrac{\mathrm{i}\omega}{\sqrt{\lambda}}\sin(\sqrt{\lambda}t)\right), & \lambda \neq \omega^2, \\ \dfrac{\mathrm{i}}{2\omega}\left(t\mathrm{e}^{-\mathrm{i}\omega t} - \dfrac{1}{\omega}\sin(\omega t)\right), & \lambda = \omega^2 \end{cases}$$

(e.g. see Mikhlin [8, pp. 487–491]). The theory in the present paper is concerned with the behaviour of the function (11) as $t \to \infty$ for functions $f, \Phi_0, \Phi_1 \in \check{C}_0^\infty(S)$, where $\check{C}_0^\infty(S)$

denotes the subset of $C_0^\infty(S)$ consisting of those functions which are antisymmetric about $\mathbb{R} \times \{b/2\}$.

One of the major difficulties encountered in the following theory is the lack of smoothness of the boundary ∂D: the edges $\mathbb{R} \times (0,0)$ and $\mathbb{R} \times (0,b)$ where Γ meets S are an essential feature of the above linear water-wave problem. This difficulty becomes apparent right at the start of the analysis: one has to establish the inclusion $\check{C}_0^\infty(S) \subset \mathcal{D}(A) = \mathcal{D}(G)$ in order to use (11), and standard regularity theory is not available to guarantee its validity. Similar problems arise in a number of the proofs presented below. These difficulties are overcome using regularity theory due to Grisvard [3, 4], which was originally developed for elliptic boundary-value problems in two-dimensional polygonal and three-dimensional prismatic domains. Since ∂X coincides with a polygon near its corners one may apply Grisvard's regularity theory to obtain conditions upon θ and $\theta^\star$ which give results analogous to those found in standard elliptic regularity theory for smooth domains. *In this paper the angles $\theta\pi$ and $\theta^\star\pi$ are always supposed to be respectively acute and convex.* This hypothesis guarantees the validity of all the results presented here when $\Phi_0 = \Phi_1 = 0$; the stronger hypothesis that $\theta < 1/4$, $\theta^\star < 1/2$ is necessary in the general case. (It is shown in Appendix A that the results below are also valid for a channel with rectangular cross-section.) Under these conditions one finds that $\check{C}_0^\infty(S) \subset \mathcal{D}(A)$ and $\Phi(t) \in C(\bar{S})$ for each $t \geq 0$, so that pointwise estimates upon the behaviour of $\Phi(t)(x,z)$ as $t \to \infty$ are possible.

The spectral family $\{P_\lambda\}_{\lambda \in (-\infty,\infty)}$ of A is studied in detail in Section 4, where it is shown to depend crucially upon a family $\{G_k\}_{k \in (-\infty,\infty)}$ of Dirichlet-Neumann operators for certain elliptic boundary-value problems in the cross-section X (see Section 3). The conclusions of this study are then used to examine the behaviour of (11) as $t \to \infty$. The results depend upon whether ω^2/g coincides with one of the eigenvalues $\lambda_0, \lambda_1, \ldots$ of the boundary-value problem

$$\psi_{yy} + \psi_{zz} = 0 \qquad \text{in } X, \tag{12}$$

$$\partial_n \psi = 0 \qquad \text{on } \bar{X} \cap \Gamma, \tag{13}$$

$$\psi_y = \lambda\psi \qquad \text{on } \bar{X} \cap S, \tag{14}$$

in which ψ is antisymmetric about L. The theory in Section 5 shows that the *principle of limiting amplitude* holds whenever $\omega^2/g \notin \{\lambda_0, \lambda_1, \ldots\}$, that is, there exists a function $V_\omega \in C(\bar{S})$ such that

$$\Phi(t)(x,z) = V_\omega(x,z)\mathrm{e}^{-\mathrm{i}\omega t} + o(1) \qquad \text{as } t \to \infty$$

uniformly in every compact subset of $\bar{S}$. On the other hand it is shown in Section 6 that a *resonance phenomenon* occurs whenever $\omega^2/g \in \{\lambda_0, \lambda_1, \ldots\}$: there exist functions $V, V_\omega \in C(\bar{S})$ such that

$$\Phi(t)(x,z) = \sqrt{t}\,\mathrm{e}^{-\mathrm{i}\omega t}V(x,z) + V_\omega(x,z)\mathrm{e}^{-\mathrm{i}\omega t} + o(1) \qquad \text{as } t \to \infty$$

uniformly in every compact subset of $\bar{S}$. Similar results may be obtained for η using equation (5) and the representation (11) of Φ; this matter is discussed in Appendix B.

The first example of resonance phenomena of the above kind was presented by Werner [11], who studied a problem similar to (9), (10) in which S is replaced by a smooth cylindrical waveguide and A is replaced by a self-adjoint extension of the Laplacian. Another connection between problems in acoustic waveguides and linear water waves in uniform channels was discussed by Groves [5], who looked for *trapped modes* in uniform, symmetric channels containing symmetric obstacles, that is, antisymmetric eigenvectors of a Dirichlet-Neumann operator for a perturbed channel. His work was inspired by that of Evans, Levitin & Vassiliev [2], who obtained similar results for antisymmetric eigenvectors of the Neumann Laplacian in a two-dimensional strip containing a symmetric obstacle.

1.2 Sobolev spaces and trace theorems

Let us now collect together some trace theorems for Sobolev spaces of functions defined on the nonsmooth domains D and X described above. Lemmata 1, 2 below are direct applications of respectively Theorems 1.5.1.3 and 1.5.2.1 in reference [4].

Lemma 1 *The mapping*

$$u \mapsto u|_{\partial X},$$

which is defined for $u \in C^{0,1}(\bar{X})$, the space of functions which are Lipschitz continuous on $\bar{X}$, has a unique extension to a continuous operator from $H^1(X)$ onto $H^{1/2}(\partial X)$.

Lemma 2

(i) The mapping

$$u \mapsto u|_{\bar{X}\cap S},$$

which is defined for $u \in C^{\infty}(\bar{X})$, has a unique extension to a continuous operator from $H^m(X)$ onto $H^{m-1/2}(\bar{X} \cap S)$, $m \in \mathbb{N}$.

(ii) The mapping

$$u \mapsto \partial_n u|_{\bar{X}\cap S},$$

which is defined for $u \in C^{\infty}(\bar{X})$, has a unique extension to a continuous operator from $H^m(X)$ onto $H^{m-3/2}(\bar{X} \cap S)$, $m = 2, 3, \ldots$.

(iii) Results (i) and (ii) also hold when $\bar{X} \cap S$ is replaced by one of $\bar{X} \cap \Gamma_1, \ldots, \bar{X} \cap \Gamma_N$.

The theory below uses the closed subspaces $\check{H}^m(X)$ of $H^m(X)$, $m \in \mathbb{N}$, $\check{L}^2(\bar{X} \cap S)$ of $L^2(\bar{X} \cap S)$ and $\check{H}^{m+1/2}(\bar{X} \cap S)$ of $H^{m+1/2}(\bar{X} \cap S)$, $m \in \mathbb{N}$ consisting of those functions which are antisymmetric about respectively L and $(0, b)$. Propositions 3 and 4 state some elementary properties of these spaces which are needed in the sequel.

Proposition 3

(i) The trace map $H^m(X) \to H^{m-1/2}(\bar{X} \cap S)$ defined in Lemma 2(i) induces a continuous, surjective trace map $\check{H}^m(X) \to \check{H}^{m-1/2}(\bar{X} \cap S)$, $m \in \mathbb{N}$.

(ii) The trace map $H^m(X) \to H^{m-3/2}(\bar{X} \cap S)$ defined in Lemma 2(ii) induces a continuous, surjective trace map $\check{H}^m(X) \to \check{H}^{m-3/2}(\bar{X} \cap S)$, $m = 2, 3, \ldots$.

Proof The existence and continuity of the induced trace maps is obvious; it remains only to check that they are surjective. Take $u \in \check{H}^{m-1/2}(\bar{X} \cap S)$ and let $\phi \in H^m(X)$ be a function such that $\phi|_{\bar{X}\cap S} = u$ in $H^{m-1/2}(\bar{X} \cap S)$. The function $\psi \in H^m(X)$ defined by

$$\psi(y,z) = \frac{1}{2}(\phi(y,z) - \phi(y, b-z))$$

belongs to $\check{H}^m(X)$ and satisfies $\psi|_{\bar{X}\cap S} = u$ in $H^{m-1/2}(\bar{X} \cap S)$ and hence in $\check{H}^{m-1/2}(\bar{X} \cap S)$. The surjectivity of the other induced trace map is proved in a similar fashion. □

Proposition 4 *The Dirichlet norm $\|u\|_D := \|\nabla u\|_0$ is equivalent to the usual norm $\|u\|_1 = [\|u\|_0^2 + \|\nabla u\|_0^2]^{1/2}$ on $\check{H}^1(X)$, where ∇ denotes the two-dimensional gradient.*

Proof Each $u \in \check{H}^1(X)$ admits a decomposition $u = u^{(1)} + u^{(2)}$, $u^{(1)} \in \hat{H}^1(X^-)$, $u^{(2)} \in \hat{H}^1(X^+)$, where

$$\begin{aligned} X^- &= \{(y,z) \in X : z < b/2\}, \\ X^+ &= \{(y,z) \in X : z > b/2\} \end{aligned}$$

and $\hat{H}^1(X^-)$ denotes the closure in $H^1(X^-)$ of the set of functions in $H^1(X^-) \cap C^\infty(X^-)$ that vanish on an open set containing L; the space $\hat{H}^1(X^+)$ is defined in a similar fashion. A simple argument based upon the fact that X^- and X^+ are of finite width in z shows that the Dirichlet norm is equivalent to the usual norm on these spaces, and the stated result is a direct consequence of this observation. □

The following version of Green's theorem is taken from Grisvard [4, Lemma 1.5.3.7].

Lemma 5 (Green's theorem) *For each $u_1 \in H^2(X)$, $u_2 \in H^1(X)$ one has that*

$$\int_X u_2 \Delta u_1 + \int_X \nabla u_2 \nabla u_1 = \int_{\bar{X}\cap S} u_2\, \partial_n u_1 + \sum_{i=1}^N \int_{\bar{X}\cap\Gamma_i} u_2\, \partial_n u_1,$$

in which ∇ and Δ denote respectively the two-dimensional gradient and Laplacian.

Define

$$H_\Delta(X) := \{u \in H^1(X) : \Delta u \in L^2(X)\},$$

$$\|u\|_{H_\Delta(X)} := \left[\|u\|^2_{H^1(X)} + \|\Delta u\|^2_{L^2(X)}\right]^{1/2},$$

where Δ denotes the two-dimensional distributional Laplacian in X, and set

$$\tilde{H}^{1/2}(\bar{X} \cap S) := \{u \in H^{1/2}(\bar{X} \cap S) : \tilde{u} \in H^{1/2}(\partial X)\},$$

$$\|u\|_{\tilde{H}^{1/2}} := \|\tilde{u}\|_{H^{1/2}(\partial X)},$$

where $\tilde{u}$ denotes the extension of u by zero. The next result states a relationship between these spaces; its proof was given by Grisvard [4, Theorem 1.5.3.10].

Lemma 6 *Let $(\tilde{H}^{1/2}(\bar{X}\cap S))^*$ denote the dual of $\tilde{H}^{1/2}(\bar{X}\cap S)$ with respect to the inner product in $L_2(\bar{X}\cap S)$. The mapping*

$$u \mapsto \partial_n u|_{\bar{X}\cap S},$$

which is defined for $u \in C^\infty(\bar{X})$, has a unique extension to a continuous operator from $H_\Delta(X)$ into $(\tilde{H}^{1/2}(\bar{X}\cap S))^*$ which is given by the Green's formula

$$\langle \partial_n u|_{\bar{X}\cap S}, v\rangle = \int_X w\Delta u + \int_X \nabla w \nabla u, \qquad v \in \tilde{H}^{1/2}(\bar{X}\cap S).$$

Here $\langle\cdot\,,\cdot\rangle$ denotes the duality between $(\tilde{H}^{1/2}(\bar{X}\cap S))^*$ and $\tilde{H}^{1/2}(\bar{X}\cap S)$ and w is an element of $H^1(X)$ such that $w|_{\bar{X}\cap S} = v$ in $H^{1/2}(\bar{X}\cap S)$ and $w|_{\partial X\setminus S} = 0$ in $H^{1/2}(\bar{X}\cap\Gamma_1), \ldots, H^{1/2}(\bar{X}\cap\Gamma_N)$ (the existence of w is assured by Lemma 1).

The corresponding result holds when $\bar{X}\cap S$ is replaced by one of $\bar{X}\cap\Gamma_1, \ldots, \bar{X}\cap\Gamma_N$.

Finally, note that the above results with the obvious modifications also hold for functions defined on D (see Grisvard [3]).

2 The Dirichlet-Neumann operator

Let us begin by studying the boundary-value problem

$$\begin{aligned} \phi_{xx} + \phi_{yy} + \phi_{zz} &= 0 \qquad \text{in } D, & (15)\\ \partial_n\phi &= 0 \qquad \text{on } \Gamma, & (16)\\ \phi &= u \qquad \text{on } S, & (17)\\ \phi &\to 0 \qquad \text{as } x \to \pm\infty, & (18) \end{aligned}$$

in which u is a function which belongs to $\check{H}^{1/2}(S)$. A *weak solution* of this boundary-value problem is a function $\phi \in \check{H}^1(D)$ which satisfies $\phi = u$ in $\check{H}^{1/2}(S)$ and

$$\int_D (\phi_x\xi_x + \phi_y\xi_y + \phi_z\xi_z)\,\mathrm{d}(x,y,z) = 0$$

for each $\xi \in \bar{H}^1(D) := \{\xi \in \check{H}^1(D) : \xi = 0 \text{ in } \check{H}^{1/2}(S)\}$.

Proposition 7 *The boundary-value problem (15)–(18) has a unique weak solution $\phi \in \check{H}^1(D)$ for each $u \in \check{H}^{1/2}(S)$.*

Proof Using Proposition 3(i), one finds that there is a function $\chi \in \check{H}^1(D)$ such that $\chi|_S = u$ in $\check{H}^{1/2}(S)$. Since the Dirichlet norm is equivalent to the usual norm on $\bar{H}^1(D)$ the Lax-Milgram lemma asserts that there is a unique function $\zeta \in \bar{H}^1(D)$ such that

$$\int_D (\zeta_x\xi_x + \zeta_y\xi_y + \zeta_z\xi_z)\,\mathrm{d}(x,y,z) = -\int_D (\chi_x\xi_x + \chi_y\xi_y + \chi_z\xi_z)\,\mathrm{d}(x,y,z)$$

for all $\xi \in \bar{H}^1(D)$. The function $\phi \in \check{H}^1(D)$ defined by $\phi = \chi + \zeta$ is therefore a weak solution of (15)–(18). Finally, suppose that ϕ_2 is also a weak solution of this boundary-value problem, so that $\phi_1 - \phi_2 \in \bar{H}^1(D)$. Using the above definition of a weak solution with $\phi = \phi_1$, $\xi = \phi_1 - \phi_2$ and $\phi = \phi_2$, $\xi = \phi_1 - \phi_2$, one finds that $\|\phi_1 - \phi_2\|_{\bar{H}^1(D)} = 0$, so that $\phi_1 = \phi_2$. □

Observe that the weak solution ϕ to (15)–(18) satisfies (15) in $\mathcal{D}'(D)$, (16) in $(\tilde{H}^{1/2}(\Gamma_1))^*$, ..., $(\tilde{H}^{1/2}(\Gamma_N))^*$ (see Lemma 6) and (17) in $\check{H}^{1/2}(S)$ (see Proposition 3(i)). Moreover, its trace $\phi_y|_S$ always exists as an element of $(\tilde{H}^{1/2}(S))^*$. The object of central interest in the present paper, the Dirichlet-Neumann operator for the above boundary-value problem, is defined using this observation.

Definition 8 *The Dirichlet-Neumann operator $G : \mathcal{D}(G) \subset \check{L}^2(S) \to \check{L}^2(S)$ for the elliptic boundary-value problem (15)–(18) is defined by*

$$Gu := \phi_y|_S, \qquad u \in \mathcal{D}(G) := \{u \in \check{H}^{1/2}(S) : \phi_y|_S \in \check{L}^2(S)\}.$$

Lemma 9 *The Dirichlet-Neumann operator G has a positive, self-adjoint, continuous inverse $B : \check{L}^2(S) \to \check{L}^2(S)$.*

Proof The first step is to show that the boundary-value problem

$$\begin{aligned} \phi_{xx} + \phi_{yy} + \phi_{zz} &= 0 \qquad \text{in } D, &(19)\\ \partial_n \phi &= 0 \qquad \text{on } \Gamma, &(20)\\ \phi_y &= u \qquad \text{on } S, &(21)\\ \phi &\to 0 \qquad \text{as } x \to \pm\infty &(22) \end{aligned}$$

has a unique weak solution $\phi \in \check{H}^1(D)$ for each $u \in \check{L}^2(S)$, that is, a function $\phi \in \check{H}^1(D)$ such that

$$\int_D (\phi_x \xi_x + \phi_y \xi_y + \phi_z \xi_z)\, \mathrm{d}(x,y,z) = \langle u, I_1 T \xi \rangle_{\check{L}^2(S)} \tag{23}$$

for all $\xi \in \check{H}^1(D)$. Here T and I denote respectively the trace operator $\check{H}^1(D) \to \check{H}^{1/2}(S)$ and the embedding $\check{H}^{1/2}(S) \to \check{L}^2(S)$, both of which are continuous. The existence and uniqueness of the weak solution ϕ are immediate consequences of the Lax-Milgram lemma; note that ϕ satisfies (19) in $\mathcal{D}'(D)$, (20) in $(\tilde{H}^{1/2}(\Gamma_1))^*$, ..., $(\tilde{H}^{1/2}(\Gamma_N))^*$ and (21) in $(\tilde{H}^{1/2}(S))^*$. Defining $B : \check{L}^2(S) \to \check{L}^2(S)$ by $Bu = I_1 T\phi$ and comparing the weak formulations of the boundary-value problems (15)–(18) and (19)–(22), one finds that $B = G^{-1}$.

Take u_1, $u_2 \in \check{L}^2(S)$ and let ϕ_1, $\phi_2 \in \check{H}^1(D)$ be the corresponding weak solutions of (19)–(22). Equation (23) shows that

$$\int_D (\phi_{1x}\phi_{2x} + \phi_{1y}\phi_{2y} + \phi_{1z}\phi_{2z})\, \mathrm{d}(x,y,z) = \langle u_1, Bu_2 \rangle_{\check{L}^2(S)} \tag{24}$$

and that

$$\begin{aligned} \|\phi_1\|^2_{\check{H}^1(D)} &= \langle u_1, I_1 T \phi_1 \rangle_{\check{L}^2(S)} \\ &\leq \|T\| \|I_1\| \|u_1\|_{\check{L}^2(S)} \|\phi_1\|_{\check{H}^1(X)}, \end{aligned} \tag{25}$$

where the inequality follows from the Cauchy-Schwarz inequality and the continuity of T and I_1. Equation (24) asserts that B is symmetric and positive on $\check{L}^2(S)$, and since $\mathcal{D}(B) = \check{L}^2(S)$ it is self-adjoint. Equation (25) shows that the linear mapping $K : \check{L}^2(S) \to \check{H}^1(D)$ defined by $Ku_1 = \phi_1$ is continuous. Because K, T and I_1 are continuous, the relation $B = I_1 T K$ implies that B is also continuous. □

Corollary 10 *The Dirichlet-Neumann operator $G : \mathcal{D}(G) \subset \check{L}^2(S) \to \check{L}^2(S)$ is positive and self-adjoint.*

Standard results in elliptic regularity theory do not apply to the boundary-value problems (15)–(18) and (19)–(22) because of the edges in the boundary of D. Regularity properties of weak solutions to Poisson's equation in a prismatic domain $\mathbb{R} \times \Omega$ were studied by Grisvard [3], and an inspection of his method shows that it requires only that $\partial\Omega$ is piecewise smooth and coincides with a polygon near its corners. Grisvard's theory therefore applies to the boundary-value problems under consideration here and gives the following regularity results.

Lemma 11

(i) The weak solution of the boundary-value problem (15)–(18) belongs to $\check{H}^{m+2}(D)$ whenever $u \in \check{H}^{m+3/2}(S)$ and $\theta < 1/2(m+1)$, $\theta^\star < 1/(m+1)$ for some $m \in \mathbb{N}_0$.

(ii) The weak solution of the boundary-value problem (19)–(22) belongs to $\check{H}^{m+2}(D)$ whenever $u \in \check{H}^{m+1/2}(S)$ and $\theta, \theta^\star < 1/(m+1)$ for some $m \in \mathbb{N}_0$.

In the next set of results Lemma 11 is applied to the operators G, B and $A = \mathrm{g}G$.

Lemma 12 *Suppose that $\theta < 1/2(m+1)$, $\theta^\star < 1/(m+1)$ for some $m \in \mathbb{N}_0$. The operator G defines an isomorphism from $\check{H}^{m+3/2}(S)$ to $\check{H}^{m+1/2}(S)$ whose inverse is given by B.*

Proof For each $u \in \check{H}^{m+3/2}(S)$ the weak solution ϕ of (15)–(18) belongs to $\check{H}^{m+2}(D)$ (see Lemma 11(i)); its trace $\phi_y|_S$ therefore belongs to $\check{H}^{m+1/2}(S)$ (Proposition 3(ii)). Thus G maps $\check{H}^{m+3/2}(S)$ injectively into $\check{H}^{m+1/2}(S)$. Repeating this argument for the boundary-value problem (19)–(22), one finds that B maps $\check{H}^{m+1/2}(S)$ into $\check{H}^{m+3/2}(S)$, whereupon $G[\check{H}^{m+3/2}(S)] = \check{H}^{m+1/2}(S)$.

Define $K : \check{H}^{m+3/2}(S) \to \check{H}^{m+2}(D)$ by the formula $Ku = \phi$ and suppose that $u_n \to u$ in $\check{H}^{m+3/2}(S)$, $Ku_n \to \phi$ in $\check{H}^{m+2}(S)$; the continuity of the trace operator asserts that $\phi = Ku$. The operator K is therefore closed and hence continuous (by virtue of the closed graph theorem). The continuity of G as an operator from $\check{H}^{m+3/2}(S)$ onto $\check{H}^{m+1/2}(S)$ follows from that of K and that of the trace operator. □

Corollary 13 *For each $m \in \mathbb{N}$ the inclusion $\check{H}^{m+1/2}(S) \subset \mathcal{D}(G^m)$, and hence $\check{C}_0^\infty(S) \subset \mathcal{D}(G^m)$, holds whenever $\theta < 1/2m$, $\theta^\star < 1/m$.*

Proposition 14 *The set $\mathcal{D}(A^2)$ is included in $C(\bar{S})$ and there exists a constant $C > 0$ such that*

$$|u(x,z)| \leq C\|A^2 u\|$$

for each $u \in \mathcal{D}(A^2)$ and each $(x,z) \in \bar{S}$.

Proof It was shown in the proof of Lemma 9 that B is a continuous operator from $\check{L}^2(S)$ into $\check{H}^{1/2}(S)$, and Lemma 12 states that B is a continuous operator from $\check{H}^{1/2}(S)$ onto $\check{H}^{\frac{3}{2}}(S)$, so that

$$\|B^2 w\|_{\check{H}^{\frac{3}{2}}(S)} \leq c_1 \|Bw\|_{\check{H}^{1/2}(S)} \leq c_2 \|w\|_{\check{L}^2(S)}$$

for every $w \in \check{L}^2(S)$. It follows that each $u \in \mathcal{D}(G^2)$ belongs to $\check{H}^{\frac{3}{2}}(S)$ with

$$\|u\|_{\check{H}^{\frac{3}{2}}(S)} \leq c_2 \|G^2 u\|, \tag{26}$$

and the continuity of the embedding $\check{H}^{\frac{3}{2}}(S) \subset C_B(S)$ shows that

$$|u(x,z)| \leq \sup_{(x,z)\in\bar{S}} |u(x,z)| \leq c_3 \|u\|_{\check{H}^{\frac{3}{2}}(S)}. \tag{27}$$

The stated result is obtained from (26), (27) and the fact that $A = gG$. □

Let us conclude this section with a result which will later prove useful in estimating spectral integrals defined in terms of the spectral family $\{P_\lambda\}_{\lambda\in(-\infty,\infty)}$ of A. Its proof combines the functional calculus for A with the above regularity results.

Lemma 15 *Take $u \in \check{L}^2(S)$ and α, β with $0 \leq \alpha < \beta \leq \infty$, let $f : (\alpha, \beta) \to \mathbb{C}$ be a continuous, bounded function and define $v_1 \in \check{L}^2(S)$ by*

$$v_1 = \int_\alpha^\beta f(\lambda)\, \mathrm{d}(P_\lambda u).$$

Suppose that $u \in \mathcal{D}(A^s)$ for some $s \geq 0$ and that $\lambda^{2-s}|f(\lambda)|$ is bounded on (α, β). The function v_1 belongs to $\mathcal{D}(A^2)$ and satisfies

$$|v_1(x,z)| \leq C \sup_{\lambda\in(\alpha,\beta)} |\lambda^{2-s} f(\lambda)|\, \|(P_\beta - P_\alpha)A^s u\|$$

for each $(x,z) \in \bar{S}$.

Proof Note that v_1 lies in the domain of A^2 if

$$\int_\alpha^\beta \lambda^4 |f(\lambda)|^2\, \mathrm{d}(\|P_\lambda u\|^2) < \infty \tag{28}$$

and that

$$\int_\alpha^\beta \lambda^{2s}\, \mathrm{d}(\|P_\lambda u\|^2) < \infty$$

because $u \in \mathcal{D}(A^s)$. Condition (28) is therefore satisfied if $\lambda^{2-s}|f(\lambda)|$ is bounded on (α, β), and Proposition 14 shows that

$$\begin{aligned} |v_1(x,z)|^2 &\leq C^2 \int_\alpha^\beta \lambda^4 |f(\lambda)|^2\, d(\|P_\lambda u\|^2) \\ &\leq C^2 \sup_{\lambda\in(\alpha,\beta)} |\lambda^{2-s} f(\lambda)|^2 \int_\alpha^\beta \lambda^{2s}\, d(\|P_\lambda u\|^2) \\ &= C^2 \sup_{\lambda\in(\alpha,\beta)} |\lambda^{2-s} f(\lambda)|^2 \|(P_\beta - P_\alpha)A^s u\|^2 \end{aligned}$$

for each $(x,z) \in \bar{S}$. □

3 Boundary-value problems in the cross-section

This section is devoted to a study of the Dirichlet-Neumann operator for the elliptic boundary-value problem

$$\begin{aligned}
\psi_{yy} + \psi_{zz} &= k^2\psi && \text{in } X, && (29)\\
\partial_n \psi &= 0 && \text{on } \bar{X} \cap \Gamma, && (30)\\
\psi &= f && \text{on } \bar{X} \cap S, && (31)
\end{aligned}$$

in which k is a real number and f is a function which belongs to $\check{H}^{1/2}(0,b)$. A *weak solution* of this boundary-value problem is a function $\psi \in \check{H}^1(X)$ which satisfies $\psi = f$ in $\check{H}^{1/2}(\bar{X} \cap S)$ and

$$\int_X (\psi_y \xi_y + \psi_z \xi_z + k^2 \psi \xi)\, \mathrm{d}(y,z) = 0$$

for each $\xi \in \bar{H}^1(X) := \{\xi \in \check{H}^1(X) : \xi = 0 \text{ in } \check{H}^{1/2}(\bar{X} \cap S)\}$. The following existence result is proved in the same way as Proposition 7.

Proposition 16 *The boundary-value problem (29)–(31) has a unique weak solution $\psi \in \check{H}^1(X)$ for each $f \in \check{H}^{1/2}(0,b)$ and each $k \in \mathbb{R}$.*

The trace $\psi_y|_{\bar{X}\cap S}$ of the weak solution to (29)–(31) always exists in $(\tilde{H}^{1/2}(\bar{X} \cap S))^*$, and the Dirichlet-Neumann operator for the boundary-value problem (29)–(31) is defined using this observation.

Definition 17 *The Dirichlet-Neumann operator $G_k : \mathcal{D}(G_k) \subset \check{L}^2(0,b) \to \check{L}^2(0,b)$ for the boundary-value problem (29)–(31) is defined by*

$$G_k f := \psi_y|_{\bar{X}\cap S}, \qquad f \in \mathcal{D}(G_k) := \{f \in \check{H}^{1/2}(0,b) : \psi_y|_{\bar{X}\cap S} \in \check{L}^2(0,b)\}.$$

Lemma 18 *The Dirichlet-Neumann operator G_k has a positive, self-adjoint, compact inverse $B_k : \check{L}^2(0,b) \to \check{L}^2(0,b)$. The family $\{B_k\}_{k\in(-\infty,\infty)}$ is a holomorphic family of type (A) in the terminology of Kato [6].*

Proof Consider the elliptic boundary-value problem

$$\begin{aligned}
\psi_{yy} + \psi_{zz} &= k^2\psi && \text{in } X, && (32)\\
\partial_n \psi &= 0 && \text{on } \bar{X} \cap \Gamma, && (33)\\
\psi_y &= f && \text{on } \bar{X} \cap S, && (34)
\end{aligned}$$

in which $f \in \check{L}^2(0,b)$ and $k \in \mathbb{R}$. Using the fact that the Dirichlet norm is equivalent to the usual norm on $\check{H}^1(X)$, one finds that

$$\int_X (\xi_y^2 + \xi_z^2 + k^2\xi^2)\, \mathrm{d}(y,z) \geq \int_X (\xi_y^2 + \xi_z^2)\, \mathrm{d}(y,z) \geq c_4 \|\xi\|^2_{\check{H}^1(X)}, \qquad \xi \in \check{H}^1(X),$$

where c_4 is independent of k, and for $k \in (-\kappa, \kappa)$, $\kappa > 0$ one has that

$$\int_X (\xi_y^2 + \xi_z^2 + k^2\xi^2)\, \mathrm{d}(y,z) \leq \max(1,\kappa^2) \|\xi\|^2_{\check{H}^1(X)}, \qquad \xi \in \check{H}^1(X).$$

Recall that the trace operator $U : \check{H}^1(X) \to \check{H}^{1/2}(0,b)$ is continuous and that the embedding $I_2 : \check{H}^{1/2}(0,b) \to \check{L}^2(0,b)$ is compact. It therefore follows from the Lax-Milgram lemma that the boundary-value problem (32)–(34) has a unique weak solution, that is, there is a unique function $\psi \in \check{H}^1(X)$ such that

$$\int_X (\psi_y \xi_y + \psi_z \xi_z + k^2 \psi \xi)\, \mathrm{d}(y,z) = \langle f, I_2 U \xi \rangle_{\check{L}^2(0,b)} \tag{35}$$

for all $\xi \in \check{H}^1(X)$; this function depends holomorphically upon $k \in (-\kappa, \kappa)$ (e.g. see Sanchez Hubert & Sanchez Palencia [9, p. 184]).

Defining $B_k : \check{L}^2(0,b) \to \check{L}^2(0,b)$ by $B_k f = I_2 U \psi$ and comparing the weak formulations of the the boundary-value problems (29)–(31) and (32)–(34), one finds that $B_k = G_k^{-1}$. The function $B_k f$ depends holomorphically upon k since I_2 and U are continuous and ψ depends holomorphically upon k. The family $\{B_k\}_{k\in(-\infty,\infty)}$ is therefore a holomorphic family of type (A) in the terminology of Kato [6, p. 375]: the domain $\check{L}^2(0,b)$ of B_k does not depend upon k and $B_k f$ is a holomorphic function of k for each $f \in \check{L}^2(0,b)$.

The facts that B_k is self-adjoint, positive and compact are obtained using the method explained in the proof of Lemma 9. □

Corollary 19 *The Dirichlet-Neumann operator $G_k : \mathcal{D}(G_k) \subset \check{L}^2(0,b) \to \check{L}^2(0,b)$ is positive and self-adjoint.*

Let us now turn to a regularity result for the boundary-value problems (29)–(31) and (32)–(34). Regularity properties of weak solutions to Poisson's equation in a polygonal domain were studied by Grisvard [3, 4]; his findings remain valid when Ω is replaced by X, whose boundary coincides with a polygon near its corners. Lemma 20 below is obtained directly from Grisvard's results.

Lemma 20

(i) The weak solution $\psi(k)$ of (29)–(31) lies in $\check{H}^{m+2}(X)$ whenever $f \in \check{H}^{m+3/2}(0,b)$ and $\theta < 1/2(m+1)$, $\theta^\star < 1/(m+1)$ for some $m \in \mathbb{N}_0$.

(ii) The weak solution $\psi(k)$ of (32)–(34) lies in $\check{H}^{m+2}(X)$ whenever $f \in \check{H}^{m+1/2}(0,b)$ and $\theta, \theta^\star < 1/(m+1)$ for some $m \in \mathbb{N}_0$.

Because $G_k : \mathcal{D}(G_k) \subset \check{L}^2(0,b) \to \check{L}^2(0,b)$ is positive, self-adjoint and has a compact inverse, its spectrum consists entirely of eigenvalues $\lambda_n(k)$, $n \in \mathbb{N}_0$ of finite multiplicity which satisfy $0 < \lambda_0(k) \le \lambda_1(k) \le \lambda_2(k) \le \ldots$ with $\lambda_n(k) \to \infty$ as $n \to \infty$. The corresponding eigenvectors $f_n(k)$, $n \in \mathbb{N}_0$ can be chosen so that $\{f_n(k)\}_{n=0}^\infty$ forms an orthonormal basis for $\check{L}^2(0,b)$ and that the elements of the sequences $\{f_n(k)\} \subset \check{L}^2(0,b)$, $\{\lambda_n(k)\} \subset \mathbb{R}$ depend holomorphically upon k (since $\{B_k\}_{k\in(-\infty,\infty)}$ is a holomorphic family of type (A)). The proof of Lemma 18 shows that the same is true of $\psi_n(k) \in \check{H}^1(X)$, the weak solution of (32)–(34) with $f = \lambda_n(k) f_n(k)$.

Proposition 21 *For each $n \in \mathbb{N}_0$ the mappings $k \mapsto \psi_n(k)$, $k \mapsto f_n(k)$ are holomorphic into respectively $\check{H}^2(X)$ and $C[0,b]$.*

Proof Using Lemma 20(ii), one finds that $\psi_h(k) \in \check{H}^2(X)$ (in fact under the weakened hypothesis that $\theta, \theta^\star < 1$), and the argument used in the proof of Lemma 12 shows that $\psi_n(k) \in \check{H}^2(X)$ depends continuously upon $\lambda_n(k) f_n(k) \in \check{H}^{1/2}(0,b)$; it therefore also depends continuously upon $\lambda_n(k) f_n(k) \in \check{L}^2(0,b)$, which is a holomorphic function of k. It follows that $\psi_n : k \to \check{H}^2(X)$ is holomorphic. Since $f_n(k) = \psi_n(k)|_{\bar{X}\cap S} \subset \check{H}^{\frac{3}{2}}(0,b) \subset C[0,b]$, one concludes from the continuity of the trace operator $\check{H}^2(X) \to \check{H}^{\frac{3}{2}}(0,b)$ and the embedding $\check{H}^{\frac{3}{2}}(0,b) \to C[0,b]$ that $f_n : \mathbb{R} \to C[0,b]$ is also holomorphic. □

Corollary 22 *For each $n \in \mathbb{N}_0$ and $k' \in \mathbb{R}$ one has that*

$$f_n(k)(z) \;=\; f_n(k')(z) + O(|k - k'|)$$

as $k \to k'$ uniformly with respect to $z \in [0,b]$.

The eigenvalue $\lambda_n(k)$ depends only upon k^2 since the same is true of G_k; in particular it is an even function of k. The following result establishes the asymptotic behaviour of $\lambda_n(k)$ as $k \downarrow 0$ and $k \to \infty$.

Proposition 23 *For each $n \in \mathbb{N}_0$ the eigenvalue $\lambda_n(k)$ is a strictly increasing, unbounded function of $k \geq 0$ and has the expansion*

$$\lambda_n(k) = \lambda_n(0) + k^2 \int_X |\psi_n(0)|^2 \,\mathrm{d}(y,z) + O(k^4)$$

as $k \downarrow 0$.

Proof Observe that

$$\langle f_n(k), f_n'(k)\rangle_{\check{L}^2(0,b)} = \frac{1}{2}\frac{d}{dk}\|f_n^2(k)\|_{\check{L}^2(0,b)} = \frac{1}{2}\frac{d}{dk}(1) = 0, \tag{36}$$

in which the prime denotes differentiation with respect to k, and that

$$\begin{aligned} \lambda_n(k) &= \langle G_k f_n(k), f_n(k)\rangle_{\check{L}^2(0,b)} \\ &= \int_X [\psi_{ny}^2(k) + \psi_{nz}^2(k) + k^2\psi_n^2(k)] \,\mathrm{d}(y,z), \end{aligned} \tag{37}$$

in which the second line follows by Green's theorem (Lemma 5 is valid because $\psi \in \check{H}^2(X)$). Differentiating (37) with respect to k, one finds that

$$\begin{aligned} \lambda_n'(k) &= 2k \int_X \psi_n^2(k) \,\mathrm{d}(y,z) \\ &\quad + 2\int_X [\psi_{ny}(k)\psi_{ny}'(k) + \psi_{nz}(k)\psi_{nz}'(k) + k^2\psi_n(k)\psi_n'(k)] \,\mathrm{d}(y,z) \\ &= 2k \int_X \psi_n^2(k) \,\mathrm{d}(y,z) + 2\langle G_k f_n(k), f_n'(k)\rangle_{\check{L}^2(0,b)} \\ &= 2k \int_X \psi_n^2(k) \,\mathrm{d}(y,z) + 2\lambda_n(k)\langle f_n(k), f_n'(k)\rangle_{\check{L}^2(0,b)} \\ &= 2k \int_X \psi_n^2(k) \,\mathrm{d}(y,z); \end{aligned} \tag{38}$$

here Green's theorem and equation (36) have been used. Equation (38) shows that $\lambda_n'(k) > 0$ for $k > 0$ and $\lambda_n(0) = 0$, so that $\lambda_n(k)$ is a strictly increasing function of $k \geq 0$. Differentiation of (38) yields

$$\lambda_n''(k) = 2\int_X \psi_n^2(k)\,\mathrm{d}(y,z) + 2k\frac{d}{dk}\int_X \psi_n^2(k)\,\mathrm{d}(y,z),$$

whence

$$\lambda_n''(0) = 2\int_X \psi_n^2(0)\,\mathrm{d}(y,z).$$

The stated estimate as $k \to 0$ follows from the observation that $\lambda_n(k)$ depends only upon k^2.

Finally, suppose that $\lambda_n(k)$ is a bounded function of k. Because the Dirichlet norm is equivalent to the usual norm on $\check{H}^1(X)$, equation (37) shows that $\|\psi_n(k)\|_{\check{H}^1(X)}$ is bounded independently of k and that $\psi_n(k) \to 0$ in $\check{L}^2(X)$ as $k \to \infty$. It follows that there is a sequence $\{k_j\}_{j=0}^{\infty}$ of real numbers with $k_j \to \infty$ as $j \to \infty$ for which $\{\psi_{k_j}\}$ converges weakly in $\check{H}^1(X)$. Denoting the weak limit by ψ, note that $\{\psi_{k_j}\}$ converges strongly to ψ in $\check{L}^2(X)$, and by the uniqueness of limits one finds that $\psi = 0$. It follows from the relation $f_n(k_j) = I_2 U\psi_n(k_j)$, the continuity of the trace operator $U : \check{H}^1(X) \to \check{H}^{1/2}(0,b)$ and the compactness of the embedding $I_2 : \check{H}^{1/2}(0,b) \to \check{L}^2(0,b)$ that $\{f_n(k_j)\}$ converges strongly to zero in $\check{L}^2(0,b)$, which contradicts the fact that $\|f_n(k_j)\|_{\check{L}^2(0,b)} = 1$ for each $n \in \mathbb{N}_0$. □

4 The spectral family of A

The next theorem shows that A is unitarily equivalent to a multiplication operator (cf. Davies [1, §2.5]) and establishes a formula for its spectral family.

Theorem 24 *The operator $\Upsilon : \check{L}^2(S) \to \oplus_{n=0}^{\infty} L^2(-\infty,\infty)$ defined by $\Upsilon(u) = \{a_n^u\}$, where*

$$a_n^u(k) = \lim_{R\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-R}^{R}\int_0^b u(x,z)f_n(k)(z)\mathrm{e}^{-ikx}\,\mathrm{d}(x,z), \qquad n \in \mathbb{N}_0 \tag{39}$$

is a unitary transformation whose inverse is given by the formula

$$u(x,z) = \lim_{\lambda\to\infty}\frac{1}{\sqrt{2\pi}}\int\sum_{(n,k)\in N_\lambda} a_n^u(k)f_n(k)(z)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \tag{40}$$

Here $N_\lambda = \{(n,k) \in \mathbb{N}_0 \times \mathbb{R} : \mathrm{g}\lambda_n(k) < \lambda\}$ and the limits in (39), (40) exist in the strong sense in respectively $L^2(-\infty,\infty)$ and $\check{L}^2(S)$.

The spectral family $\{P_\lambda\}_{\lambda\in(-\infty,\infty)}$ of A is given by the formula

$$(P_\lambda u)(x,z) = \frac{1}{\sqrt{2\pi}}\int\sum_{(n,k)\in N_\lambda} a_n^u(k)f_n(k)(z)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \qquad u \in \check{L}^2(S). \tag{41}$$

In particular, one has that

$$(Au)(x,z) = \lim_{\lambda\to\infty}\frac{1}{\sqrt{2\pi}}\int\sum_{(n,k)\in N_\lambda} \mathrm{g}\lambda_n(k)a_n^u(k)f_n(k)(z)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \tag{42}$$

with

$$\mathcal{D}(A) = \left\{ u \in \check{L}^2(S) : \lim_{\lambda\to\infty} \int \sum_{(n,k)\in N_\lambda} \lambda_n(k) a_n^u(k) f_n(k)(z) \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k \in \check{L}^2(S) \right\}; \tag{43}$$

it follows that $\sigma(A) = \sigma_{\mathrm{c}}(A) = [\mathrm{g}\lambda_0(0), \infty)$.

Proof The first part of the theorem is proved in the same way as the Fourier expansion theorem for functions in $L^2(\mathbb{R}^n)$. Straightforward calculations show that P_λ is bounded, symmetric and idempotent for each $\lambda \in \mathbb{R}$ and that $\|P_\lambda u\|$ is monotone increasing with $P_\lambda u \to u$ as $\lambda \to \infty$. Hence $\{P_\lambda\}_{\lambda\in(-\infty,\infty)}$ is the spectral family of a positive, self-adjoint operator $\tilde{A}$ which is defined by (42), (43) and $\sigma(\tilde{A}) = \sigma_{\mathrm{c}}(\tilde{A}) = [g\lambda_0(0), \infty)$. It remains to confirm that $\tilde{A} = A$.

Take $u \in \check{C}_0^\infty(S) \subset \mathcal{D}(A)$ (see Corollary 13) and let ϕ be the weak solution of (15)–(18). Recall that $\phi \in \check{H}^2(D)$ and that $\psi_n(k)$, the weak solution of (32)–(34) with $f = \lambda_n(k) f_n(k)$, belongs to $\check{H}^2(X)$. Writing $\chi_n^k(x, y, z) = \psi_n(k)(y, z)\mathrm{e}^{-\mathrm{i}kx}$, one finds that ϕ, $\chi_n^k \in \check{H}^2(D_R)$ for each $R \in \mathbb{R}$, where $D_R = \{(x, y, z) \in D : |x| < R\}$. Since $\Delta\phi$ and $\Delta\chi_n^k$ both vanish in D_R, it follows from Green's theorem (see Lemma 5) that

$$\int_{\partial D_R} \chi_n^k\, \partial_n \phi = \int_{\partial D_R} \phi\, \partial_n \chi_n^k,$$

where the integration is taken piecewise over the smooth components of ∂D_R. The contributions to these integrals from $\partial D_R \cap \Gamma$ and $\partial D_R \cap S$ are identically zero while the contributions from $\{-R\} \times X$, $\{R\} \times X$ vanish as $R \to \infty$, so that

$$\begin{aligned}
a_n^{Au}(k) &= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \int_0^b (Au)(x, z) f_n(k)(z) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}(x, z) \\
&= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \int_0^b \mathrm{g}\phi_y(x, 0, z) \chi_n^k(x, 0, z)\, \mathrm{d}(x, z) \\
&= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \int_0^b \mathrm{g}\phi(x, 0, z) \chi_{ny}^k(x, 0, z)\, \mathrm{d}(x, z) \\
&= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \int_0^b \mathrm{g}u(x, z) \lambda_n(k) f_n(k) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}(x, z) \\
&= \mathrm{g}\lambda_n(k) a_n^u(k),
\end{aligned}$$

in which the limits are taken in the strong sense in $L^2(-\infty, \infty)$. This calculation shows that $\check{C}_0^\infty(S) \subset \mathcal{D}(\tilde{A})$ and that $Au = \tilde{A}u$ for each $u \in \check{C}_0^\infty(S)$. Because $\check{C}_0^\infty(S)$ is a dense subset of both $\mathcal{D}(A)$ and $\mathcal{D}(\tilde{A})$ (in the $\check{L}^2(S)$-metric) and A, $\tilde{A}$ define bijections from respectively $\mathcal{D}(A)$, $\mathcal{D}(\tilde{A})$ onto $\check{L}^2(S)$, the continuous operators A^{-1} and $\tilde{A}^{-1}$ agree on the dense subset $A[\check{C}_0^\infty(S)] = \tilde{A}[\check{C}_0^\infty(S)]$ of $\check{L}^2(S)$. One concludes that $A^{-1} = \tilde{A}^{-1}$ and hence $A = \tilde{A}$. □

The remainder of this section is dedicated to a study of the family $\{P_\lambda u\}_{\lambda\in(-\infty,\infty)} \subset \check{L}^2(S)$ for a fixed function $u \in \check{C}_0^\infty(S)$. Using Corollary 13, one finds that $P_\lambda u \in C(\bar{S})$ for each $\lambda \in (-\infty, \infty)$. The formula (39) for a_n^u also simplifies; one has that

$$a_n^u(k) = \frac{1}{\sqrt{2\pi}} \int_S u(x, z) f_n(k)(z) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}(x, z), \qquad k \in \mathbb{R}, \tag{44}$$

and elementary calculations show that $a_n^u \in C^\infty(-\infty,\infty)$. Recall that $\mathrm{g}\lambda_n : [0,\infty) \to \mathbb{R}$ is holomorphic, increasing and has range $[\mathrm{g}\lambda_n(0),\infty)$; it has an inverse $k_n : [\mathrm{g}\lambda_n(0),\infty) \to [0,\infty)$ which is continuous on $[\mathrm{g}\lambda_n(0),\infty)$ and smooth on $(\mathrm{g}\lambda_n(0),\infty)$. It follows from (41) that

$$(P_\lambda u)(x,z) = \begin{cases} \displaystyle\sum_{n=0}^{J(\lambda)} \int_{-k_n(\lambda)}^{k_n(\lambda)} \frac{1}{\sqrt{2\pi}} a_n^u(k) f_n(k)(z) \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, & \lambda \geq g\lambda_0(0), \\ 0, & \lambda < \mathrm{g}\lambda_0(0), \end{cases} \tag{45}$$

where $J(\lambda) = \max\{n \in \mathbb{N}_0 : \mathrm{g}\lambda_n(0) \leq \lambda\}$. Let us now take the derivative with respect to λ. Using (44) and the fact that $f_j(k) = f_j(-k)$ for each $j \in \mathbb{N}_0$ and each $k \in \mathbb{R}$, one fi nds that

$$\frac{d}{d\lambda}(P_\lambda u)(x,z) = (Q_\lambda^{J(\lambda)} u)(x,z), \tag{46}$$

for $\lambda \notin \{\mathrm{g}\lambda_0(0), \mathrm{g}\lambda_1(0), \ldots\}$, where

$$(Q_\lambda^m u)(x,z) = \sum_{j=0}^m g_j^u(\lambda)(x,z)$$

and

$$g_j^u(\lambda)(x,z) = \frac{1}{\pi}\frac{dk_j(\lambda)}{d\lambda} f_j(k_j(\lambda))(z) \int_S u(x',z') f_j(k_j(\lambda))(z') \cos(k_j(\lambda)(x'-x))\,\mathrm{d}(x',z').$$

Theorem 25 *For each $u \in \check{C}_0^\infty(S)$ one has that*

$$\begin{aligned}
&\frac{d}{d\lambda}(P_\lambda u)(x,z) \\
&= \begin{cases} (Q_{\lambda'}^{J(\lambda')} u)(x,z) + O_M(|\lambda-\lambda'|) & \text{as } \lambda \to \lambda', \\ (Q_{\mathrm{g}\lambda_m(0)}^{m-1} u)(x,z) + O_M(\mathrm{g}\lambda_m(0) - \lambda) & \text{as } \lambda \uparrow \mathrm{g}\lambda_m(0), \\ (Q_{\mathrm{g}\lambda_m(0)}^{m-1} u)(x,z) \\ \qquad + \dfrac{1}{\sqrt{\lambda - \mathrm{g}\lambda_m(0)}}(\tilde{P}^m u)(x,z) + O_M\left(\sqrt{\lambda - \mathrm{g}\lambda_m(0)}\right) & \text{as } \lambda \downarrow \mathrm{g}\lambda_m(0) \end{cases}
\end{aligned} \tag{47}$$

for $m \in \mathbb{N}_0$ and $\lambda' \notin \{\mathrm{g}\lambda_0(0), \mathrm{g}\lambda_1(0), \ldots\}$. Here

$$(\tilde{P}^m u)(x,z) = \sum_{\{j : \lambda_j(0) = \lambda_m(0)\}} h_j^u(x,z),$$

in which

$$h_j^u(x,z) = \frac{\sqrt{\mathrm{g}}}{2\pi \|\psi_j(0)\|_{\check{L}^2(X)}} f_j(0)(z) \int_S u(x',z') f_j(0)(z')\,\mathrm{d}(x',z')$$

and the subscript M indicates that the estimate holds uniformly in each compact subset M of $\bar{S}$.

Proof Take $j \in \mathbb{N}_0$, $\lambda' > g\lambda_j(0)$ and note that Corollary 22 implies that

$$f_j(k_j(\lambda))(z) = f_j(k_j(\lambda'))(z) + O(|k_j(\lambda) - k_j(\lambda')|) \tag{48}$$

as $\lambda \to \lambda'$ uniformly with respect to $z \in [0, b]$. Choose $a > 0$ such that $\operatorname{supp} u \in (-a, a)$ and observe that

$$\begin{aligned} &\left|\int_S u(x',z') f_j(k_j(\lambda'))(z') \left(\cos(k_j(\lambda)(x'-x)) - \cos(k_j(\lambda')(x'-x))\right) \mathrm{d}(x',z')\right|^2 \\ &\leq \|u\|^2_{\check{L}^2(S)} \|f_j(k_j(\lambda'))\|^2_{\check{L}^2(0,b)} \int_{-a}^{a} |\cos(k_j(\lambda)(x'-x)) - \cos(k_j(\lambda')(x'-x))|^2 \,\mathrm{d}x' \\ &= O_M(|k_j(\lambda) - k_j(\lambda')|^2) \end{aligned} \tag{49}$$

and

$$\begin{aligned} &\left|\int_S u(x',z') \left(f_j(k_j(\lambda'))(z') - f_j(k_j(\lambda))(z')\right) \cos(k_j(\lambda)(x'-x)) \,\mathrm{d}(x',z')\right|^2 \\ &\leq 2a\|u\|^2_{\check{L}^2(S)} \|f_j(k_j(\lambda')) - f_j(k_j(\lambda))\|^2_{\check{L}^2(0,b)} \\ &= O_M(|k_j(\lambda) - k_j(\lambda')|^2) \end{aligned} \tag{50}$$

as $\lambda \to \lambda'$. Combining (48)–(50) and using the fact that $k_j \in C^\infty(\mathrm{g}\lambda_j(0), \infty)$, one finds that

$$g_j^u(\lambda)(x,z) = g_j^u(\lambda')(x,z) + O_M(|\lambda - \lambda'|) \tag{51}$$

as $\lambda \to \lambda'$. The first two estimates in (47) are direct consequences of (51).

The function $\psi_j : \mathbb{R} \to \check{L}^2(X)$ is holomorphic because $\psi_j : \mathbb{R} \to \check{H}^1(X)$ is holomorphic and $\check{H}^1(X)$ is continuously embedded in $\check{L}^2(X)$. It clearly depends only upon k^2, so that

$$\|\psi_j(k)\|_{\check{L}^2(X)} = \|\psi_j(0)\|_{\check{L}^2(X)} + O(k^2) \tag{52}$$

as $k \downarrow 0$. Proposition 23 shows that

$$k_j(\lambda) = \frac{\sqrt{\lambda - \mathrm{g}\lambda_j(0)}}{\sqrt{\mathrm{g}}\|\psi_j(0)\|_{\check{L}^2(X)}} + O((\lambda - \mathrm{g}\lambda_j(0))^{\frac{3}{2}}) \tag{53}$$

as $\lambda \downarrow \mathrm{g}\lambda_j(0)$, and using the formula

$$\frac{d}{d\lambda} k_j(\lambda) = \frac{1}{g\lambda_j'(k_j(\lambda))},$$

equation (38) and estimates (52) and (53), one finds that

$$\frac{d}{d\lambda} k_j(\lambda) = \frac{\sqrt{g}}{2\|\psi_j(0)\|_{\check{L}^2(X)}\sqrt{\lambda - \mathrm{g}\lambda_j(0)}} + O\left(\sqrt{\lambda - \mathrm{g}\lambda_j(0)}\right) \tag{54}$$

as $\lambda \downarrow \mathrm{g}\lambda_j(0)$. Observe that

$$\begin{aligned} f_j(k_j(\lambda))(z) &= f_j(0)(z) + O(k_j(\lambda)^2) \\ &= f_j(0)(z) + O(\lambda - \mathrm{g}\lambda_j(0)) \end{aligned} \tag{55}$$

as $\lambda \downarrow \mathrm{g}\lambda_j(0)$ uniformly with respect to $z \in [0,b]$ (see Corollary 22) and that, by (53),

$$\begin{aligned}
&\left|\int_S u(x',z') f_j(0)(z')(\cos(k_j(\lambda)(x'-x)) - 1)\,\mathrm{d}(x',z')\right|^2 \\
&\qquad \le \|u\|^2_{\check{L}^2(S)} \|f_j(k_j(\lambda'))\|^2_{\check{L}^2(S)} \int_{-a}^{a} |\cos(k_j(\lambda)(x'-x)) - 1|^2 \,\mathrm{d}x' \\
&\qquad = O_M(k_j^4(\lambda)) \\
&\qquad = O_M(|\lambda - \mathrm{g}\lambda_j(0)|^2)
\end{aligned} \tag{56}$$

as $\lambda \downarrow \mathrm{g}\lambda_j(0)$. Equations (54)–(56) show that

$$g_j^u(\lambda)(x,z) = \frac{1}{\sqrt{\lambda - \mathrm{g}\lambda_j(0)}} h_j^u(x,z) + O_M\left(\sqrt{\lambda - \mathrm{g}\lambda_j(0)}\right) \tag{57}$$

as $\lambda \downarrow \mathrm{g}\lambda_j(0)$, and the third estimate in (47) is an immediate consequence of (51), (57). □

The final result in this section is repeatedly used in Sections 5 and 6 below to estimate certain spectral integrals.

Lemma 26 *The equation*

$$\left[\int_\alpha^\beta \phi(\lambda)\,\mathrm{d}(P_\lambda u)\right](x,z) = \sum_{j=0}^{J(\beta)} \int_{\max(\alpha,\mathrm{g}\lambda_j(0))}^{\beta} \phi(\lambda) g_j^u(\lambda)(x,z)\,\mathrm{d}\lambda, \qquad (x,z) \in \bar{S},$$

where $J(\beta)$ is defined below equation (45), holds for each $u \in \check{C}_0^\infty(S)$ and each bounded function $\phi \in C(\alpha,\beta)$ with $0 \le \alpha < \beta < \infty$.

Proof First note that the function defined by the spectral integral on the left-hand side of the above equation belongs to $C_B(\bar{S})$ (this result follows directly from Lemma 15 and Proposition 14, since (α,β) is a finite interval). Take $v \in \check{C}_0^\infty(S)$ and observe that

$$\begin{aligned}
\left\langle \int_\alpha^\beta \phi(\lambda)\,\mathrm{d}(P_\lambda u), v \right\rangle &= \int_\alpha^\beta \phi(\lambda)\,\mathrm{d}(\langle P_\lambda u, v\rangle) \\
&= \int_\alpha^\beta \phi(\lambda) \frac{d}{d\lambda}(\langle P_\lambda u, v\rangle)\,\mathrm{d}\lambda \\
&= \int_\alpha^\beta \phi(\lambda) \left\langle \frac{d}{d\lambda} P_\lambda u, v \right\rangle \mathrm{d}\lambda \\
&= \left\langle \int_\alpha^\beta \phi(\lambda) \frac{d}{d\lambda} P_\lambda u\,\mathrm{d}\lambda, v \right\rangle,
\end{aligned} \tag{58}$$

where $\langle \cdot\,,\cdot \rangle$ denotes the $\check{L}^2(S)$-inner product. Here the functional calculus and the facts that $\frac{d}{d\lambda} P_\lambda u$ is continuous on $(\mathrm{g}\lambda_0(0),\infty) \setminus \{\mathrm{g}\lambda_0(0), \mathrm{g}\lambda_1(0), \dots\}$ and integrable near $\mathrm{g}\lambda_0(0)$, $\mathrm{g}\lambda_1(0)$, ... have been used. Lemma 26 now follows from (46) and (58). □

5 The principles of limiting absorption and limiting amplitude

According to the functional calculus the resolvent $R_\mu : \check{L}^2(S) \to \check{L}^2(S)$ of G exists for $\mu \in \mathbb{C} \setminus [\lambda_0(0), \infty)$ and is given by

$$R_\mu u = \int_{\lambda_0(0)}^{\infty} \frac{1}{\mu - \lambda} \, \mathrm{d}(P_\lambda u), \qquad u \in \check{L}^2(S).$$

Notice that by definition $R_\mu u \in \mathcal{D}(G^2)$ for each $u \in \mathcal{D}(G)$; in particular Proposition 14 implies that $R_\mu u \in C_{\mathrm{B}}(\bar{S})$ whenever $u \in \check{C}_0^\infty(S)$. The following theorem examines the behaviour of $R_{\omega^2+\mathrm{i}\tau} u$ as $\tau \downarrow 0$ for $\omega^2 \notin \{\mathrm{g}\lambda_0(0), \mathrm{g}\lambda_1(0), \ldots\}$ and $u \in \check{C}_0^\infty(S)$.

Theorem 27 (The principle of limiting absorption) *For each fixed $\omega^2 \notin \{\mathrm{g}\lambda_0(0), \mathrm{g}\lambda_1(0), \ldots\}$ and $u \in \check{C}_0^\infty(S)$ the function $R_{\omega^2+\mathrm{i}\tau} u$ converges uniformly in each compact subset of $\bar{S}$ as $\tau \downarrow 0$. The limit function $R_{\omega^2+\mathrm{i}0} u$ is given by the formula*

$$(R_{\omega^2+\mathrm{i}0} u)(x,z) = \lim_{\delta \downarrow 0} \left[\int_{V_\delta} \frac{1}{\lambda - \omega^2} \, \mathrm{d}(P_\lambda u) \right](x,z) + \mathrm{i}\pi (Q_{\omega^2}^{J(\omega^2)} u)(x,z), \qquad (x,z) \in M,$$

where $V_\delta = (\mathrm{g}\lambda_0(0), \infty) \setminus (\omega^2 - \delta, \omega^2 + \delta)$ and $Q_\lambda^m u$ is defined below equation (46).

Proof Let δ be a positive real number such that $\sigma(\mathrm{g}G_0) \cap (\omega^2 - \delta, \omega^2 + \delta)$ is empty and observe that

$$\lambda \left| \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} - \frac{1}{\lambda - \omega^2} \right| \le \frac{\lambda \tau}{|\lambda - \omega^2|^2} = \frac{\tau}{\lambda - \omega^2} \left(1 + \frac{\omega^2}{\lambda - \omega^2} \right) \le \tau \left(\frac{1}{\delta} + \frac{\omega^2}{\delta^2} \right)$$

for $|\lambda - \omega^2| \ge \delta$; Lemma 15 therefore asserts that

$$\left| \left[\int_{\omega^2+\delta}^{\infty} \left(\frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} - \frac{1}{\lambda - \omega^2} \right) d(P_\lambda u) \right](x,z) \right| \le \tau \left(\frac{1}{\delta} + \frac{\omega^2}{\delta^2} \right) \|(I - P_{\omega^2+\delta}) A u\|,$$

whence

$$\left[\int_{\omega^2+\delta}^{\infty} \frac{1}{\lambda - \omega^2 - i\tau} \, d(P_\lambda u) \right](x,z) = \left[\int_{\omega^2+\delta}^{\infty} \frac{1}{\lambda - \omega^2} \, d(P_\lambda u) \right](x,z) + o(1)$$

as $\tau \downarrow 0$ uniformly with respect to $(x,z) \in \bar{S}$. The same result holds when the limits of integration are replaced by $\mathrm{g}\lambda_0(0)$ and $\omega^2 - \delta$.

Let us now investigate the behaviour of

$$\left[\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} \, \mathrm{d}(P_\lambda u) \right](x,z) = \sum_{j=0}^{J(\omega^2)} \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} g_j^u(\lambda)(x,z) \, \mathrm{d}\lambda$$

(see Lemma 26) as $\tau \downarrow 0$. Let M be a fixed compact subset of $\bar{S}$ and take $j \in \mathbb{N}_0$ such that $\mathrm{g}\lambda_j(0) < \omega^2$. It follows from equation (51) that

$$\left| \int_{\omega^2-\delta}^{\omega^2+\delta} \left(\frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} - \frac{1}{\lambda - \omega^2} \right) \left(g_j^u(\lambda)(x,z) - g_j^u(\omega^2)(x,z) \right) \mathrm{d}\lambda \right|$$
$$\le \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{2}{|\lambda - \omega^2|} c_5 |\lambda - \omega^2| \, d\lambda \;=\; 4\delta c_5 \qquad (59)$$

for $(x,z)\in M$. Since

$$\lim_{\tau\to 0}\int_{\omega^2-\delta}^{\omega^2+\delta}\frac{1}{\lambda-\omega^2-\mathrm{i}\tau}\,\mathrm{d}\lambda=\mathrm{i}\pi, \tag{60}$$

one concludes from (59) and (60) that

$$\int_{\omega^2-\delta}^{\omega^2+\delta}\frac{1}{\lambda-\omega^2-\mathrm{i}\tau}g_j^u(\lambda)(x,z)\,\mathrm{d}\lambda=\mathrm{i}\pi g_j^u(\omega^2)(x,z)+w(x,z,\delta)+o_M(1)$$

as $\tau\downarrow 0$. (Here, and in the remainder of this article, the symbols $w(x,z,\delta)$ and $w(x,z,t,\delta)$ denote quantities which tend to zero uniformly in respectively M and $M\times[0,\infty)$ as $\delta\downarrow 0$.)

The above calculations show that $R_{\omega^2+\mathrm{i}\tau}u$ converges uniformly in each compact subset M of $\bar{S}$ as $\tau\to 0$ and that

$$\begin{aligned}\lim_{\tau\downarrow 0}(R_{\omega^2+\mathrm{i}\tau}u)(x,z) &= \left[\int_{V_\delta}\frac{1}{\lambda-\omega^2}\,\mathrm{d}(P_\lambda u)\right](x,z)\\ &\quad+\mathrm{i}\pi\sum_{j=0}^{J(\omega^2)}g_j^u(\omega^2)(x,z)+w(x,z,\delta),\qquad (x,z)\in M\end{aligned} \tag{61}$$

for each $\delta>0$. Observe that the left-hand side of (61) does not depend upon δ; one may therefore take the limit $\delta\downarrow 0$ to obtain the stated formula for $R_{\omega^2+\mathrm{i}0}u$. □

Let us now return to the solution (11) of (9), (10), which may be rewritten as

$$\begin{aligned}\Phi(t) &= \int_{\mathrm{g}\lambda_0(0)}^{\infty}\cos(\sqrt{\lambda}t)\,\mathrm{d}(P_\lambda\Phi_0)+\int_{\mathrm{g}\lambda_0(0)}^{\infty}\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,\mathrm{d}(P_\lambda\Phi_1)\\ &\quad+\int_{\mathrm{g}\lambda_0(0)}^{\infty}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}(1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t})}{\lambda-\omega^2}-\frac{\mathrm{i}\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}(\sqrt{\lambda}+\omega)}\right)\mathrm{d}(P_\lambda f).\end{aligned} \tag{62}$$

Theorem 28 below examines the behaviour of $\Phi(t)$ as $t\to\infty$.

Theorem 28 (The principle of limiting amplitude) *Suppose that* $\omega^2\notin\{\mathrm{g}\lambda_0(0),\mathrm{g}\lambda_1(0),\ldots\}$ *and that* $\theta<1/4$, $\theta^\star<1/2$ *if* Φ_0 *and* Φ_1 *are not both zero. The function* $\Phi(t)$ *has the asymptotic behaviour*

$$\Phi(t)(x,z)=\mathrm{e}^{-\mathrm{i}\omega t}(R_{\omega^2+\mathrm{i}0}f)(x,z)+o_M(1)$$

as $t\to\infty$.

Proof Here M is again supposed to be a fixed compact subset of $\bar{S}$ and $\epsilon,\delta>0$ are fixed numbers chosen such that $\sigma(\mathrm{g}G_0)\cap(\omega^2-\delta,\omega^2+\delta)$ is empty.

Define

$$\begin{aligned}I_1(x,z,t) &= \left[-\mathrm{i}\int_{\mathrm{g}\lambda_0(0)}^{\infty}\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}(\sqrt{\lambda}+\omega)}\,\mathrm{d}(P_\lambda f)\right](x,z),\\ I_2(x,z,t) &= \left[\int_{\mathrm{g}\lambda_0(0)}^{\infty}\cos(\sqrt{\lambda}t)\,\mathrm{d}(P_\lambda\Phi_0)\right](x,z),\\ I_3(x,z,t) &= \left[\int_{\mathrm{g}\lambda_0(0)}^{\infty}\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\,\mathrm{d}(P_\lambda\Phi_1)\right](x,z).\end{aligned}$$

Applying Lemma 15 with $s = 1$, one finds that there exists a real number $a > 0$ such that

$$I_1(x,z,t) = \left[\int_{\mathrm{g}\lambda_0(0)}^{a} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}(\sqrt{\lambda}+\omega)}\,\mathrm{d}(P_\lambda f)\right](x,z) + v(x,z,t), \tag{63}$$

where $|v(x,z,t)| < \epsilon$ for $(x,z) \in \bar{S}$, $t \geq 0$. The interval $[\lambda_0(0), a/g]$ contains only a finite number $\lambda_0(0)$, $\lambda_1(0)$, …, $\lambda_s(0)$ of eigenvalues of G_0. Define $U_\delta = \cup_{j=0}^{\infty}(\mathrm{g}\lambda_j(0) - \delta, \mathrm{g}\lambda_j(0) + \delta) \cap [\mathrm{g}\lambda_0(0), a]$; it follows from Lemma 15 and the continuity of P_λ that

$$\left|\left[\int_{U_\delta} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}(\sqrt{\lambda}+\omega)}\,\mathrm{d}(P_\lambda f)\right](x,z)\right| < \epsilon \tag{64}$$

for all $x \in \bar{S}$, $t \geq 0$ and sufficiently small δ. The methods used in the proof of Theorem 25 show that $\frac{d^2}{d\lambda^2}(P_\lambda f)(x,z)$ exists and is continuous and bounded for $\lambda \in [\mathrm{g}\lambda_0(0), a] \setminus U_\delta$, $(x,z) \in M$; one may therefore integrate by parts to find that

$$\int_{(\lambda_0(0),a)\setminus U_\delta} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}(\sqrt{\lambda}+\omega)}\,\frac{d}{d\lambda}(P_\lambda f)(x,z)\,\mathrm{d}\lambda = O_M\left(\frac{1}{t}\right) \tag{65}$$

as $t \to \infty$. Equations (63)–(65) imply that $I_1(x,z,t) = o_M(1)$ as $t \to \infty$. This argument also shows that $I_2(x,z,t)$ and $I_3(x,z,t)$ are $o_M(1)$ as $t \to \infty$; note, however, that in order to obtain the analogue of formula (63) one has to apply Lemma 15 with $s = 2$, and the assumption $\theta < 1/4$ is therefore necessary.

Recall that $V_\delta = (\mathrm{g}\lambda_0(0), \infty) \setminus (\omega^2 - \delta, \omega^2 + \delta)$ and define

$$\begin{aligned}
I_4(x,z,t,\delta) &= \left[-\int_{V_\delta} \frac{\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}t}}{\lambda - \omega^2}\,\mathrm{d}(P_\lambda f)\right](x,z),\\
I_5(x,z,\delta) &= \left[\int_{V_\delta} \frac{1}{\lambda - \omega^2}\,\mathrm{d}(P_\lambda f)\right](x,z),\\
I_6(x,z,t,\delta) &= \left[\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1 - \mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda - \omega^2}\,\mathrm{d}(P_\lambda f)\right](x,z);
\end{aligned}$$

the above arguments show that $I_4(x,z,t,\delta) = o_M(1)$ as $t \to \infty$ and equation (61) states that

$$I_5(x,z,\delta) = (R_{\omega^2+\mathrm{i}0} f)(x,z) - \mathrm{i}\pi(Q_{\omega^2}^{J(\omega^2)} f)(x,z) + w(x,z,\delta).$$

Turning to I_6, observe that

$$\left[\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1 - \mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda - \omega^2}\,\mathrm{d}(P_\lambda f)\right](x,z) = \sum_{j=0}^{J(\omega^2)} \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1 - \mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda - \omega^2} g_j^f(\lambda)\,\mathrm{d}\lambda \tag{66}$$

(see Lemma 26), and using (51) one finds that

$$\left|\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1 - \mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda - \omega^2}(g_j^f(\lambda)(x,z) - g_j^f(\omega^2)(x,z))\,\mathrm{d}(x,z)\right| \leq 4c_6\delta, \qquad (x,z) \in M,$$

whence

$$\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} g_j^f(\lambda)(x,z)\,\mathrm{d}\lambda = g_j^f(\omega^2)(x,z) \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\,\mathrm{d}\lambda + w(x,z,t,\delta) \tag{67}$$

for $j \in \mathbb{N}_0$ such that $\mathrm{g}\lambda_j(0) < \omega^2$. A calculation by Lesky [7, equation (5.21)] shows that

$$\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\,\mathrm{d}\lambda = \mathrm{i}\pi + w(t,\delta) + W(t,\delta); \tag{68}$$

here and in the following text the symbol $W(x,z,t,\delta)$ is used to denote a quantity which for each $\delta > 0$ is $o_M(1)$ as $t \to \infty$. It follows from (66)–(68) that

$$I_6(x,z,t,\delta) = \mathrm{i}\pi(Q_{\omega^2}^{J(\omega^2)} f)(x,z) + w(x,z,t,\delta) + W(x,z,t,\delta).$$

Together with formula (62) the above estimates for $I_1, \ldots, I_6$ imply that

$$\Phi(t)(x,z) = \mathrm{e}^{-\mathrm{i}\omega t}(R_{\omega^2+\mathrm{i}0} f)(x,z) + w(x,z,t,\delta) + W(x,z,t,\delta);$$

by definition $|w(x,z,t,\delta)| < \epsilon$ for all $(x,z) \in M$, $t \in [0,\infty)$ and sufficiently small δ and $|W(x,z,t,\delta)| < \epsilon$ for all $(x,z) \in M$, $\delta > 0$ and sufficiently large t. □

6 Resonance phenomena

The behaviour of $R_{\omega^2+\mathrm{i}\tau}$ as $\tau \downarrow 0$ and of $\Phi(t)$ as $t \to \infty$ was examined in detail in Section 5 for $\omega^2 \notin \{\mathrm{g}\lambda_0(0), \mathrm{g}\lambda_1(0), \ldots\}$. Here corresponding results are obtained in the remaining case when $\omega^2 = \mathrm{g}\lambda_m(0)$ for some $m \in \mathbb{N}_0$; without loss of generality one can suppose that $\lambda_{m-1}(0) \neq \lambda_m(0)$ if $\lambda_m(0)$ is a multiple eigenvalue and $m > 0$.

Theorem 29 *Suppose that $\omega^2 = \mathrm{g}\lambda_m(0)$ for some $m \in \mathbb{N}_0$ and that $\lambda_{m-1}(0) \neq \lambda_m(0)$ if $m > 0$. For each $u \in \check{C}_0^\infty(S)$ the function $(R_{\omega^2+\mathrm{i}\tau}u) - (1+\mathrm{i})\pi(2\tau)^{-1/2}\tilde{P}^m u$, where $\tilde{P}^m u$ is defined in Theorem 25, converges uniformly in each compact subset M of $\bar{S}$. The limit $\tilde{R}_{\omega^2+\mathrm{i}0}$ is given by*

$$(\tilde{R}_{\omega^2+\mathrm{i}0}u)(x,z) = \lim_{\delta\downarrow 0}\left[\left[\int_{V_\delta} \frac{1}{\lambda-\omega^2}\,\mathrm{d}(P_\lambda u)\right](x,z) - \frac{2}{\sqrt{\delta}}(\tilde{P}^m u)(x,z)\right] + \mathrm{i}\pi(Q^{m-1}_{\mathrm{g}\lambda_m(0)}u)(x,z), \quad (x,z) \in M.$$

Proof In the following text M is supposed to be a fixed compact subset of $\bar{S}$. Choose $\delta > 0$ such that $\sigma(\mathrm{g}G_0) \cap (\omega^2-\delta, \omega^2+\delta) = \{\omega^2\}$.

It follows from (57) that

$$\left|\int_{\omega^2}^{\omega^2+\delta} \left(\frac{1}{\lambda-\omega^2-\mathrm{i}\tau} - \frac{1}{\lambda-\omega^2}\right)\left(g_j^u(\lambda)(x,z) - \frac{1}{\sqrt{\lambda-\omega^2}} h_j^u(x,z)\right)\,\mathrm{d}\lambda\right| \leq \int_{\omega^2}^{\omega^2+\delta} \frac{2}{|\lambda-\omega^2|} c_7\sqrt{\lambda-\omega^2}\,d\lambda = 4c_7\sqrt{\delta}, \qquad (x,z) \in M, \tag{69}$$

and

$$\left| \int_{\omega^2}^{\omega^2+\delta} \frac{1}{\lambda - \omega^2} \left(g_j(\lambda)(x,z) - \frac{1}{\sqrt{\lambda-\omega^2}} h_j^u(x,z) \right) \mathrm{d}\lambda \right| \le 2c_8\sqrt{\delta}, \tag{70}$$

for each j with $\lambda_j(0) = \lambda_m(0)$, and since

$$\lim_{\tau\downarrow 0} \int_{\omega^2}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau} \frac{1}{\sqrt{\lambda-\omega^2}} \,\mathrm{d}\lambda = \frac{(1+\mathrm{i})\pi}{\sqrt{2\tau}} - \frac{2}{\sqrt{\delta}} \tag{71}$$

(cf. Lesky [7, equation (6.16)]), one concludes from (69)–(71) that

$$\begin{aligned} &\int_{\omega^2}^{\omega^2+\delta} \left(\frac{1}{\lambda-\omega^2-\mathrm{i}\tau} - \frac{1}{\lambda-\omega^2} \right) g_j^u(\lambda)(x,z)\,\mathrm{d}\lambda \\ &\qquad = \frac{(1+\mathrm{i})\pi}{\sqrt{2\tau}} h_j^u(x,z) - \frac{2}{\sqrt{\delta}} h_j^u(x,z) + w(x,z,\delta) + o_M(1) \end{aligned} \tag{72}$$

as $\tau \downarrow 0$ for each j such that $\lambda_j(0) = \lambda_m(0)$.

The argument in the proof of Theorem 27 shows that

$$\left[\int_{V_\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau} \,\mathrm{d}(P_\lambda u) \right](x,z) = \left[\int_{V_\delta} \frac{1}{\lambda-\omega^2} \,\mathrm{d}(P_\lambda u) \right](x,z) + o(1)$$

as $\tau \downarrow 0$ uniformly with respect to $(x,z) \in \bar{S}$ and

$$\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau} g_j^u(\lambda)(x,z)\,\mathrm{d}\lambda = \mathrm{i}\pi g_j^u(\omega^2)(x,z) + w(x,z,\delta) + o_M(1)$$

for $j = 0, 1, \ldots, m-1$ as $\tau \downarrow 0$. These equations, together with the formula

$$\begin{aligned} \left[\int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau} \,\mathrm{d}(P_\lambda u) \right](x,z) &= \sum_{j=0}^{m-1} \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau} g_j^u(\lambda)(x,z)\,\mathrm{d}\lambda \\ &\quad + \sum_{\{j:\lambda_j(0)=\lambda_m(0)\}} \int_{\omega^2}^{\omega^2+\delta} \frac{1}{\lambda-\omega^2-\mathrm{i}\tau} g_j^u(\lambda)(x,z)\,\mathrm{d}\lambda, \end{aligned}$$

which follows from Lemma 26 and the definition of $\tilde{P}^m u$ in Theorem 25, imply that

$$\begin{aligned} &\lim_{\tau\downarrow 0} \left((R_{\omega^2+\mathrm{i}\tau} u)(x,z) - \frac{(1+\mathrm{i})\pi}{\sqrt{2\tau}} (\tilde{P}^m u)(x,z) \right) \\ &= \left[\int_{V_\delta} \frac{1}{\lambda-\omega^2} \,\mathrm{d}(P_\lambda u) \right](x,z) - \frac{2}{\sqrt{\delta}} (\tilde{P}^m u)(x,z) + \mathrm{i}\pi \sum_{j=0}^{m-1} g_j^u(\omega^2)(x,z) + w(x,z,\delta) \end{aligned} \tag{73}$$

for each $\delta > 0$; the convergence as $\tau \downarrow 0$ is uniform in M. Since the left-hand side of this equation does not depend upon δ, the formula for $\tilde{R}_{\omega^2+\mathrm{i}0} u$ follows by taking the limit $\delta \downarrow 0$. □

Theorem 30 *Suppose that* $\omega^2 = \mathrm{g}\lambda_m(0)$ *for some* $m \in \mathbb{N}_0$ *with* $\lambda_{m-1}(0) \neq \lambda_m(0)$ *if* $m > 0$ *and that* $\theta < 1/4$, $\theta^\star < 1/2$ *if* Φ_0 *and* Φ_1 *do not both vanish. The function* $\Phi(t)$ *satisfies*

$$\Phi(t)(x,z) = (1+\mathrm{i}) \left(\frac{\pi t}{\omega} \right)^{1/2} \mathrm{e}^{-\mathrm{i}\omega t} (\tilde{P}^m f)(x,z) + \mathrm{e}^{-\mathrm{i}\omega t} (\tilde{R}_{\omega^2+\mathrm{i}0} f)(x,z) + o_M(1)$$

as $t \to \infty$.

Proof Here M is again supposed to be a fixed compact subset of $\bar{S}$ and ϵ, δ are fixed numbers with $\sigma(\mathrm{g}G_0) \cap (\omega^2 - \delta, \omega^2 + \delta) = \{\omega^2\}$. Define $I_1, \ldots, I_6$ as in the proof of Theorem 28. The arguments given there show that $I_1(x,z,t)$, $I_2(x,z,t)$, $I_3(x,z,t)$, $I_4(x,z,t,\delta)$ are $o_M(1)$ as $t \to \infty$, and equation (73) states that

$$I_5(x,z,\delta) = (\tilde{R}_{\omega^2+\mathrm{i}0}u)(x,z) + \frac{2}{\sqrt{\delta}}(\tilde{P}^m f)(x,z) - \mathrm{i}\pi \sum_{j=0}^{m-1} g_j^f(\omega^2)(x,z) + w(x,z,\delta).$$

Turning to I_6, one finds from Lemma 26 that

$$\begin{aligned} I_6(x,z,t,\delta) &= \sum_{j=0}^{m-1} \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-i(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} g_j^f(\lambda)(x,z)\,\mathrm{d}\lambda \\ &\quad + \sum_{\{j:\lambda_j(0)=\lambda_m(0)\}} \int_{\omega^2}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-i(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} g_j^f(\lambda)(x,z)\,\mathrm{d}\lambda. \end{aligned} \tag{74}$$

It follows from (57) that

$$\begin{aligned} &\left| \int_{\omega^2}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} \left(g_j^f(\lambda)(x,z) - \frac{1}{\sqrt{\lambda-\omega^2}} h_j^f(x,z) \right) \mathrm{d}\lambda \right| \\ &\qquad\qquad \le 2c_9 \int_{\omega^2}^{\omega^2+\delta} \frac{1}{\sqrt{\lambda-\omega^2}}\,\mathrm{d}\lambda = c_9\sqrt{\delta}, \qquad (x,z) \in M, \end{aligned}$$

and since

$$\int_{\omega^2}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{(\lambda-\omega^2)^{\frac{3}{2}}}\,\mathrm{d}\lambda = (1+\mathrm{i})\sqrt{\frac{\pi t}{\omega}} - \frac{\sqrt{2}}{\sqrt{\omega}(\sqrt{\omega^2+\delta}-\omega)^{1/2}} + w(t,\delta) + W(t,\delta)$$

(see Lesky [7, pp. 717–718]), one has that

$$\begin{aligned} &\int_{\omega}^{\omega^2+\delta} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} g_j^f(\lambda)(x,z)\,\mathrm{d}\lambda \\ &\quad = \left[(1+\mathrm{i})\sqrt{\frac{\pi t}{\omega}} - \frac{\sqrt{2}}{\sqrt{\omega}(\sqrt{\omega^2+\delta}-\omega)^{1/2}} \right] h_j^f(x,z) + w(x,z,t,\delta) + W(x,z,t,\delta) \end{aligned} \tag{75}$$

for each j with $\lambda_j(0) = \lambda_m(0)$. Equations (74), (75) and the estimate (67) (which holds for $j = 0, \ldots, m-1$) show that

$$\begin{aligned} I_6(x,z,t,\delta) &= \mathrm{i}\pi \sum_{j=0}^{m-1} g_j^f(\omega^2)(x,z) + \left[(1+\mathrm{i})\sqrt{\frac{\pi t}{\omega}} - \frac{\sqrt{2}}{\sqrt{\omega}(\sqrt{\omega^2+\delta}-\omega)^{1/2}} \right] (\tilde{P}^m f)(x,z) \\ &\quad + w(x,z,t,\delta) + W(x,z,t,\delta). \end{aligned}$$

Formula (62) and the above estimates for $I_1, \ldots, I_6$ imply that

$$\begin{aligned} \Phi(t)(x,z) &= (1+\mathrm{i})\sqrt{\frac{\pi t}{\omega}} \mathrm{e}^{-\mathrm{i}\omega t} (\tilde{P}^m f)(x,z) + \mathrm{e}^{-\mathrm{i}\omega t} (\tilde{R}_{\omega^2+\mathrm{i}0} f)(x,z) \\ &\quad + \left[\frac{2}{\sqrt{\delta}} - \frac{\sqrt{2}}{\sqrt{\omega}(\sqrt{\omega^2+\delta}-\omega)^{1/2}} \right] (\tilde{P}^m f)(x,z) + w(x,z,t,\delta) + W(x,z,t,\delta), \end{aligned}$$

and the stated result follows from

$$\lim_{\delta\downarrow 0}\left[\frac{2}{\sqrt{\delta}}-\frac{\sqrt{2}}{\sqrt{\omega}(\sqrt{\omega^2+\delta}-\omega)^{1/2}}\right]=0$$

and the argument used in the last part of the proof of Theorem 28. □

Appendix A A rectangular channel

The results in Sections 5 and 6 also hold for any f, Φ_0, $\Phi_1 \in \check{C}_0^\infty(S)$ in the special case when X is a rectangle $(-h,0)\times(0,b)$. To verify this fact one has to establish that the weak solution ϕ of (15)–(18) with $u \in \check{C}_0^\infty(S)$ belongs to $\check{H}^2(D)$ and that $\check{C}_0^\infty(S) \subset \mathcal{D}(G^2)$. In fact it is possible to reach much stronger conclusions.

Lemma 31

(i) Take $u \in \check{C}_0^\infty(S)$ and let ϕ be the weak solution of (15)–(18). The function ϕ belongs to $\check{C}^\infty(\bar{D})$.

(ii) The inclusion $\check{C}_0^\infty(S) \subset \mathcal{D}(G^m)$ holds for each $m \in \mathbb{N}$.

Proof Define $\phi_2 \in H^1[\mathbb{R}\times(-h,0)\times(-b,2b)]$ by

$$\phi_2(x,y,z)=\begin{cases}\phi(x,y,-z), & -b<z<0,\\ \phi(x,y,z), & 0\le z\le b,\\ \phi(x,y,2b-z), & b<z<2b.\end{cases}$$

Defining $u_2 \in C_0^\infty(\mathbb{R}\times(-b,2b))$ in the same fashion, observe that ϕ_2 is a weak solution of the elliptic boundary-value problem

$$\begin{aligned} \phi_{xx}+\phi_{yy}+\phi_{zz} &= 0 && \text{in } \mathbb{R}\times(-h,0)\times(-b,2b), && (76)\\ \phi &= u_2 && \text{on } \mathbb{R}\times\{0\}\times(-b,2b), && (77)\\ \phi_y &= 0 && \text{on } \mathbb{R}\times\{-h\}\times(-b,2b), && (78)\\ \phi_z &= 0 && \text{on } (\mathbb{R}\times(-h,0)\times\{-b\})\cup(\mathbb{R}\times(-h,0)\times\{2b\}), && (79)\\ \phi &\to 0 && \text{as } x\to\pm\infty. && (80) \end{aligned}$$

Standard elliptic regularity theory shows that $\phi_2 \in C^\infty(\mathbb{R}\times[-h,0]\times[-b,2b]\setminus E)$, where $E = \{(x,y,z) : x\in\mathbb{R}, (y,z) \neq (-b,0),(2b,0),(-b,-h),(2b,-h)\}$. It follows that $\phi \in \check{C}^\infty(\bar{D})$ (and hence $\phi_2 \in C^\infty(\mathbb{R}\times[-h,0]\times[-b,2b])$), whereby $\check{C}_0^\infty(S) \subset \mathcal{D}(G)$.

Define $u^{(2)} \in \check{C}^\infty(\bar{S})$ by $u^{(2)}(x,z) = (Gu)(x,z) = \phi_y(x,0,z)$ for $(x,z)\in\bar{S}$ and let $\phi^{(2)} \in \check{H}^1(D)$ be the weak solution of (15)–(18) with u replaced by $u^{(2)}$. By construction the function $u^{(2)}$ can be extended to an element $u_2^{(2)} \in C^\infty(\mathbb{R}\times[-b,2b])$ by the procedure explained above and $\phi_2^{(2)}$ is a weak solution of (76)–(80) with u_2 replaced by $u_2^{(2)}$. As before one concludes that $\phi_2 \in \check{C}^\infty(\bar{D})$, and hence $\check{C}_0^\infty(S) \subset \mathcal{D}(G^2)$.

Repeating this argument, one finds that $\check{C}_0^\infty(S) \subset \mathcal{D}(G^m)$ for every $m\in\mathbb{N}$. □

Appendix B Asymptotic behaviour of η

Equation (5) shows that the elevation of the free surface, in the framework of the linearised theory, is given by

$$\eta(t)(x,z) = -\frac{1}{\mathrm{g}}\left(p(x,z)\mathrm{e}^{-\mathrm{i}\omega t} + \Phi_t(t)(x,z)\right).$$

Using (11) with $f = \mathrm{i}\omega p$, one finds that

$$\Phi_t(t) = -\int_0^\infty \sqrt{\lambda}\sin(\sqrt{\lambda}t)\,\mathrm{d}(P_\lambda\Phi_0) + \int_0^\infty \cos(\sqrt{\lambda}t)\,\mathrm{d}(P_\lambda\Phi_1) + \mathrm{i}\omega\int_0^\infty \chi_t(\lambda,t)\,\mathrm{d}(P_\lambda p), \qquad t \geq 0,$$

where

$$\chi_t(\lambda,t) = \begin{cases} \dfrac{-\mathrm{i}\omega}{\lambda-\omega^2}\left(\mathrm{e}^{-\mathrm{i}\omega t} - \cos(\sqrt{\lambda}t) + \dfrac{\mathrm{i}\sqrt{\lambda}}{\omega}\sin(\sqrt{\lambda}t)\right), & \lambda \neq \omega^2, \\ \dfrac{1}{2}\left(t\mathrm{e}^{-\mathrm{i}\omega t} + \dfrac{1}{\omega}\sin(\omega t)\right), & \lambda = \omega^2, \end{cases}$$

which may be rewritten as

$$\chi_t(\lambda,t) = -\mathrm{i}\omega\,\frac{\mathrm{e}^{-\mathrm{i}\omega t}\left(1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}\right)}{\lambda-\omega^2} + \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}+\omega}.$$

The next step is to compare this formula with (62) and inspect the estimates of spectral integrals carried out in Sections 5 and 6. One finds that the same argument applies to the above representation of η. The stronger assumptions $\Phi_0 \in \mathcal{D}(G^{\frac{5}{2}})$, $\Phi_1 \in \mathcal{D}(G^2)$, $p \in \mathcal{D}(G^{\frac{3}{2}})$ are however needed in order to apply Lemma 15 to estimate the relevant spectral integrals, and Corollary 13 shows that $\check{C}_0^\infty(S) \subset \mathcal{D}(G^{\frac{5}{2}})$ for $\theta < 1/6$, $\theta^\star < 1/3$ and $\check{C}_0^\infty(S) \subset \mathcal{D}(G^2), \mathcal{D}(G^{\frac{3}{2}})$ for $\theta < 1/4$, $\theta^\star < 1/2$.

Theorem 32 *Let p, Φ_0, $\Phi_1 \in \check{C}_0^\infty(S)$ and suppose that $\theta < 1/6$, $\theta^\star < 1/3$ in the general case or $\theta < 1/4$, $\theta^\star < 1/2$ if both Φ_0 and Φ_1 vanish.*

(i) If $\omega^2 \notin \{\mathrm{g}\lambda_0(0), \mathrm{g}\lambda_1(0), \ldots\}$, then

$$\eta(t)(x,z) = -\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\mathrm{g}}\left(p(x,z) - \omega^2\left(R_{\omega^2+\mathrm{i}0}p\right)(x,z)\right) + o_M(1)$$

as $t \to \infty$.

(ii) If $\omega^2 = \mathrm{g}\lambda_m(0)$ for some $m \in \mathbb{N}_0$ (with the assumption that $\lambda_{m-1}(0) \neq \lambda_m(0)$ if $m > 0$), then

$$\eta(t)(x,z) = \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\mathrm{g}}\left(\omega^2(1+\mathrm{i})\left(\frac{\pi t}{\omega}\right)^{1/2}\left(\tilde{P}_m p\right)(x,z) + \omega^2\left(\tilde{R}_{\omega^2+\mathrm{i}0}p\right)(x,z) - p(x,z)\right) + o_M(1)$$

as $t \to \infty$.

In particular, Theorem 32 shows how small vibrations in the exterior pressure with suitably chosen frequencies can generate waves whose amplitudes are unbounded as $t \to \infty$. This effect dramatically illustrates the limitations of linear water-wave theory on large time intervals.

Acknowledgement We would like to thank Dr. C. M. Linton, Dr. M. McIver and Dr. P. McIver for many helpful discussions during the preparation of this article.

References

1. Davies, E. B., *Spectral Theory and Differential Operators*. Cambridge: C.U.P. (1995).
2. Evans, D. V., Levitin, M., & Vassiliev, D., ‘Existence theorems for trapped modes’, *J. Fluid Mech.* **261**, 21–31 (1994).
3. Grisvard, P., ‘Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain’, In *Numerical Solution of Partial Differential Equations III* (ed. Hubbard, B.), pages 207–274. London: Academic Press (1975).
4. Grisvard, P., *Elliptic Problems in Nonsmooth Domains*. London: Pitman (1985).
5. Groves, M. D., ‘On the existence of trapped modes in channels of arbitrary cross-section’, *Math. Meth. Appl. Sci.* **20**, 521–545 (1997).
6. Kato, T., *Perturbation Theory for Linear Operators*, 2nd edn. New York: Springer-Verlag (1976).
7. Lesky, P., ‘Resonance phenomena for a class of partial differential equations of higher order in cylindrical waveguides’, *Math. Meth. Appl. Sci.* **11**, 697–723 (1989).
8. Mikhlin, S., *Mathematical Physics, An Advanced Course*. Amsterdam: North-Holland (1970).
9. Sanchez Hubert, J., & Sanchez Palencia, E., *Vibration and Coupling of Continuous Systems: Asymptotic Methods*. Berlin: Springer-Verlag (1989).
10. Stoker, J. J., *Water Waves: The Mathematical Theory with Applications*. New York: Interscience (1957).
11. Werner, P., ‘Resonance phenomena in cylindrical waveguides’, *J. Math. Anal. Appl.* **121**, 215–272 (1987).