Time Asymptotics for the Biharmonic Wave Equation in Exterior Domains

Peter H. Lesky

Mathematisches Institut A

Universitt Stuttgart

Pfaffenwaldring 57
D 70569 Stuttgart

Tel. 0711/685-5557
E-Mail lesky@mathematik.uni-stuttgart.de
Telefax 0711/685-5594

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Math. Inst. A der Universitt Stuttgart, Pfaffenwaldring 57, D 70569 Stuttgart

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Abstract

We consider initial-boundary value problems for the plate equation $u''(t,x)+\Delta^2 u(t,x) = f(x)$ in exterior domains of odd dimension. Especially the boundary condition $\Delta u = \partial\Delta u/\partial\mathbf{n} = 0$ is studied. We prove existence, uniqueness and time asymptotics for the solution. Furthermore, we study the plate equation with Dirichlet boundary condition. We give examples for right-hand sides f having unbounded support such that resonances occur.

1 Introduction

Consider in $\Omega = \mathbb{R}^n$ the initial value problem

$$\left.\begin{aligned} \partial_t^2 u(t,x) + \Delta^2 u(t,x) = f(x) \quad & \text{in } [0,\infty)\times\Omega, \\ u(x,0) = \partial_t u(x,0) = 0 \qquad & \text{in } \Omega \end{aligned}\right\} \tag{1.1}$$

with time independent right-hand side $f \in C_0^\infty(\Omega)$. We restrict our considerations to the case of odd $n \geq 3$. It is well known (compare e.g. [2] and [5]) that (1.1) has a unique solution $u \in \bigcap_{j=0}^{2} C^j([0,\infty), H^{4-2j}(\mathbb{R}^n))$ and that $u \in C^\infty([0,\infty)\times\mathbb{R}^n)$. Furthermore, if $n = 3$, then a resonance of the following form occurs:

$$u(t,x) = \frac{t^{1/2}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(y)\,dy + u_0(x) + o(1) \qquad \text{as } t \to \infty \tag{1.2}$$

uniformly in every bounded subset of $\mathbb{R}^3$. For $n \geq 5$ the principle of limiting amplitude holds:

$$u(t,x) = u_0(x) + o(1) \qquad \text{as } t \to \infty \tag{1.3}$$

uniformly in every bounded subset of $\mathbb{R}^n$, where $u_0 \in C^\infty(\mathbb{R}^n)$ is a solution of the static equation

$$\Delta^2 u_0 = f \qquad \text{in } \mathbb{R}^n. \tag{1.4}$$

We are interested in the question, how the time asymptotics change if Ω is a proper subset of $\mathbb{R}^n$. Then, in addition to (1.1), some boundary condition has to be imposed on u to guarantee uniqueness. The time asymptotics depend essentially on this boundary condition. Let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $\partial\Omega \in C^\infty$. We consider (1.1) combined alternatively with each of the boundary conditions

$$u(t,x) \quad = \quad \frac{\partial u}{\partial \mathbf{n}}(t,x) \quad = \quad 0 \qquad \text{on } [0,\infty)\times\partial\Omega, \tag{1.5}$$

$$\begin{aligned} u(t,x) &= \Delta u(t,x) &= 0 \quad &\text{on } [0,\infty)\times\partial\Omega, &(1.6)\\ \frac{\partial u}{\partial \mathbf{n}}(t,x) &= \frac{\partial \Delta u}{\partial \mathbf{n}}(t,x) &= 0 \quad &\text{on } [0,\infty)\times\partial\Omega, &(1.7)\\ \Delta u(t,x) &= \frac{\partial \Delta u}{\partial \mathbf{n}}(t,x) &= 0 \quad &\text{on } [0,\infty)\times\partial\Omega; &(1.8) \end{aligned}$$

here $\mathbf{n}$ denotes the unit normal vector on $\partial\Omega$ pointing in the exterior of Ω.

In the case of Dirichlet boundary condition (1.5) it is well known that $u \in C^\infty([0,\infty)\times\overline{\Omega})$. Furthermore, it follows from results obtained by D.M. Eidus in [2] and [3] for a more general class of equations, that the principle of limiting amplitude holds for every odd $n \geq 3$.

The boundary conditions (1.6) and (1.7) were studied by H. Mack and P. Werner in [7] and [8] under the assumption that $\mathbb{R}^n \setminus \overline{\Omega}$ is simple connected. They proved for $n = 3$ that (1.1), (1.6) has a classical solution $u \in C^\infty([0,\infty)\times\overline{\Omega})$ and that

$$u(t,x) = \frac{t^{1/2}}{(2\pi)^{3/2}} \int_\Omega f(y) w_1(y)\, dy\, w_1(x) + u_0(x) + o_M(1) \qquad \text{as } t\to\infty. \tag{1.9}$$

Here $o_M(1)$ means that the remaining terms are of order $o(1)$ uniformly in every bounded subset M of $\overline{\Omega}$. The zero-resonance in (1.9) is closely related to the "admissible standing wave" w_1, which is the unique solution $w_1 \in C^\infty(\overline{\Omega})$ of

$$\left.\begin{aligned} \Delta^2 w_1 &= 0 && \text{in } \Omega,\\ w_1 = \Delta w_1 &= 0 && \text{on } \partial\Omega,\\ D^\alpha\left(w_1(x)-1\right) &= O\Big(|x|^{-|\alpha|-1}\Big) && \text{as } |x|\to\infty,\ 0\leq|\alpha|\leq 3 \end{aligned}\right\} \tag{1.10}$$

($D^\alpha := \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ for $\alpha \in \mathbb{N}_0^n$). Note that $w_1 \notin L_2(\Omega)$. In the case of the boundary condition (1.7), estimate (1.9) remains valid with $w_1 \equiv 1$. For $n \geq 5$ the solutions of (1.1), (1.6) or (1.1), (1.7), respectively, satisfy the principle of limiting amplitude (1.3).

The study of the solution of (1.1), (1.8) is the aim of this paper. In this case the null space N of the related spatial operator

$$\left.\begin{aligned} D(A) &:= \Big\{u \in C^4(\overline{\Omega}) : \Delta u = \tfrac{\partial \Delta u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega,\ D^\alpha u(x) = O\left(|x|^{-(n+1)/2}\right) \\ &\qquad\qquad\qquad\qquad \text{as } |x| \to \infty,\ 0 \le |\alpha| \le 4\Big\}, \\ Au &:= \Delta^2 u \qquad \text{for } u \in D(A) \end{aligned}\right\} \tag{1.11}$$

has infinite dimension, since it contains every potential function with sufficient decay as $|x| \to \infty$. Let $\mathcal{A}$ denote the self-adjoint extension of A introduced in Section 2 and let $P^{(N)} : L_2(\Omega) \to N$ be the orthogonal projection onto N. As a main result of this paper we shall prove that (1.1), (1.8) has a classical solution $u \in C^\infty([0,\infty) \times \overline{\Omega})$ and that

$$u(t,x) = \frac{t^2}{2}\left(P^{(N)} f\right)(x) + \sum_{j=1}^{3} t^{2-j/2}\left(P^{(j)} f\right)(x) + u_0(x) + o_M(1) \qquad \text{as } t \to \infty \tag{1.12}$$

if $n = 3$ or $n = 5$, that

$$u(t,x) = \frac{t^2}{2}\left(P^{(N)} f\right)(x) + t^{1/2}\left(P^{(1)} f\right)(x) + u_0(x) + o_M(1) \qquad \text{as } t \to \infty \tag{1.13}$$

if $n = 7$ and that

$$u(t,x) = \frac{t^2}{2}\left(P^{(N)} f\right)(x) + u_0(x) + o_M(1) \qquad \text{as } t \to \infty \tag{1.14}$$

for $n = 9, 11, \ldots$. The linear operators $P^{(j)}$ $(j = 1,2,3)$ are closely connected with "admissible standing waves" satisfying

$$\Delta w = 0 \qquad \text{in } \Omega.$$

In particular, we shall prove in the case $n = 3$ that

$$P^{(1)} f = \frac{1}{9(2\pi)^{3/2}} \left(2 \sum_{j=1}^{3} \int_\Omega f(y) w_{x_j}(y)\, dy\, w_{x_j} + 3 \int_\Omega f(y) w_{|x|}(y)\, dy\, w_{|x|} \right), \quad (1.15)$$

where w_p denotes the "admissible standing wave" related to the function p (for the definition compare (9.7) and (8.3)). We note that $w_{x_j} \in L_2(\Omega)$ $(j = 1, 2, 3)$ and $w_{|x|} \notin L_2(\Omega)$.

The method used in the proof of (1.12) – (1.14) leads to some new observations concerning the asymptotic behaviour of the solution u in the Dirichlet case (1.1), (1.5). As mentioned above, u satisfies the principle of limiting amplitude (1.3) if the right-hand side f in (1.1) has bounded support. But resonances may occur if the support of f is not bounded. In the case $n = 3$ we prove that for every $s \in \left(0, \frac{3}{4}\right)$ there exists a right-hand side $f \in L_2(\Omega)$, such that the solution of (1.1), (1.5) satisfies

$$u(t, x) = t^s \left(\tilde{P}^{(s)} f \right)(x) + o_M(t^s) \qquad \text{as } t \to \infty \quad (1.16)$$

(compare Theorem 10.1).

In order to prove these results, we represent the solution by a spectral integral. This method is also used in the above mentioned papers [2], [7] and [8]. The time asymptotics of the solution are determined essentially by the behaviour of the spectral family $\{P_\lambda\}$ of the corresponding spatial operator as $\lambda \downarrow 0$, which can be obtained by estimating the resolvent R_z near $z = 0$. The resolvent and its behaviour as $z \to 0$ is computed in [2] and [3] by an indirect method, while in [7] and [8] boundary integral equations are used. In this paper we reduce the resolvent of the operator $\mathcal{A}$ to the resolvent of the corresponding biharmonic Dirichlet operator, which has been studied by Eidus in [2] and [3].

The plan of this paper is the following. In Section 2 the solution is given by a spectral integral. The resolvent $R_z = (\mathcal{A} - zI)^{-1}$ of the related self-adjoint operator $\mathcal{A}$ is studied in Section 3. The computation of R_z can be reduced to the resolvent $R_z^{\mathcal{B}}$ of the self-adjoint Dirichlet operator

$$\left.\begin{aligned} D(\mathcal{B}) &:= \mathring{H}^2(\Omega) \cap H^4(\Omega), \\ \mathcal{B}u &:= \Delta^2 u \qquad \text{for } u \in D(\mathcal{B}), \end{aligned}\right\} \tag{1.17}$$

where $\mathring{H}^2(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the second Sobolev space $H^2(\Omega)$. In particular, we prove

$$R_z f = \frac{1}{z}\left(\Delta R_z^{\mathcal{B}}(\Delta f) - f\right) \qquad \text{for } f \in H^2(\Omega). \tag{1.18}$$

In Section 4 we study the spectral family $\{P_\lambda\}$ of $\mathcal{A}$. It follows from (1.18) that

$$\int_{\mu_1}^{\mu_2} \varphi(\lambda)\, d(P_\lambda f) = \Delta \int_{\mu_1}^{\mu_2} \frac{\varphi(\lambda)}{\lambda}\, d\left(P_\lambda^{\mathcal{B}}(\Delta f)\right) \tag{1.19}$$

for $f \in H^2(\Omega)$ and $\varphi \in C([\mu_1, \mu_2])$, where $P_\lambda^{\mathcal{B}}$ denotes the spectral family of $\mathcal{B}$. This formula allows us to use well-known properties of the operator $\mathcal{B}$ to prove the regularity of our solution in Section 5 and the time asymptotics in Section 6. In Sections 7 – 9 we compute explicit representations of the coefficients in the expansion of $R_z^{\mathcal{B}}$ around $z = 0$. This gives us representations of the operators $P^{(j)}$ used in (1.12). Section 10 is devoted to the study of the Dirichlet case (1.1), (1.5).

2 Spectral representation of the solution

In the following we assume:

Assumption 2.1 *Ω is a connected open subset of $\mathbb{R}^n$, $n = 3, 5, \ldots$, and $\partial\Omega$ consists of a finite number of disjoint $(n-1)$–dimensional surfaces $\Gamma_1, \ldots, \Gamma_m \in C^\infty$.*

The operator A defined by (1.11) is symmetric, positive ($\langle Au, u\rangle \geq 0$ for $u \in D(A)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2(\Omega)$), and $D(A)$ is dense in $L_2(\Omega)$. Hence A has a self-adjoint positive extension $\mathcal{A}$ in $L_2(\Omega)$ according to the theorem of Friedrichs. We note that $\mathcal{A}$ is not coercive. This means that there exist no constants $c_1, c_2 > 0$ such that

$$\langle \mathcal{A}\varphi, \varphi\rangle \geq c_1 \|\varphi\|_2^2 - c_2\|\varphi\|^2 \qquad \text{for every } \varphi \in D(\mathcal{A}),$$

where $\|.\|$ and $\|.\|_2$ denote the norms in $L_2(\Omega)$ and $H^2(\Omega)$, respectively. This can be seen by inserting

$$\varphi(x) := \frac{\partial}{\partial x_1} \frac{1}{|x - (x_0 + \delta\mathbf{n})|^{n-2}} \qquad \text{for } x \in \overline{\Omega}$$

($x_0 \in \partial\Omega$ fixed, $\mathbf{n}$: normal unit vector on $\partial\Omega$ pointing in the exterior of Ω) in this inequality and letting $\delta \downarrow 0$.

We denote the left-continuous spectral family of $\mathcal{A}$ by $\{P_\lambda\}$. Note that $P_\lambda = 0$ for $\lambda \leq 0$, since $\mathcal{A}$ is positive. It is well known (compare [4]) that

$$u(t) := \int_0^\infty \frac{1}{\lambda}\left(1 - \cos\sqrt{\lambda}t\right) d(P_\lambda f) \tag{2.1}$$

for $f \in L_2(\Omega)$ is the unique solution of

$$\left.\begin{aligned} u''(t) + \mathcal{A}u(t) &= f \qquad \text{for } t \geq 0, \\ u(0) = u'(0) &= 0 \end{aligned}\right\} \tag{2.2}$$

with the property

$$u \in \bigcap_{j=0}^{2} C^j\left([0,\infty), D(\mathcal{A}^{1-j/2})\right). \tag{2.3}$$

It remains to show that u is a classical solution of (1.1), (1.8) under sufficient conditions on f. Here the difficulty occurs that the usual elliptic regularity theory does not apply to $\mathcal{A}$, since $\mathcal{A}$ is not coercive. However, we shall prove in section 5 by regularity estimates for the Dirichlet operator $\mathcal{B}$ that u is a classical solution of (1.1), (1.8).

3 The resolvent

We start with some heuristic considerations. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $f \in C_0^\infty(\Omega)$. Then $R_z f = (\mathcal{A} - z\,I)^{-1} f$ satisfies

$$\left.\begin{array}{rcll} (\Delta^2 - z)R_z f & = & f & \text{in } \Omega, \\ \Delta R_z f \;=\; \dfrac{\partial \Delta R_z f}{\partial \mathbf{n}} & = & 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{3.1}$$

Formal differentiation yields

$$\left.\begin{array}{rcll} (\Delta^2 - z)\Delta R_z f & = & \Delta f & \text{in } \Omega, \\ \Delta R_z f \;=\; \dfrac{\partial \Delta R_z f}{\partial \mathbf{n}} & = & 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{3.2}$$

Hence we obtain $\Delta R_z f = R_z^{\mathcal{B}}(\Delta f)$, where $R_z^{\mathcal{B}}$ denotes the resolvent of the Dirichlet operator $\mathcal{B}$ defined by (1.17). By (3.1), we have

$$R_z f = \frac{1}{z}(\Delta^2 R_z f - f) = \frac{1}{z}\left(\Delta(R_z^{\mathcal{B}} \Delta f) - f\right).$$

Now we show that this relation between the resolvents $R_z = (\mathcal{A} - z\,I)^{-1}$ and $R_z^{\mathcal{B}} = (\mathcal{B} - z\,I)^{-1}$ holds for all $f \in H^2(\Omega)$.

Lemma 3.1 *If $z \in \mathbb{C} \setminus \mathbb{R}$ and $f \in H^2(\Omega)$, then*

$$R_z f = \frac{1}{z}\left(\Delta(R_z^{\mathcal{B}} \Delta f) - f\right). \tag{3.3}$$

Proof: Let $f \in C^\infty(\overline{\Omega})$ with bounded support and set

$$u_z = u_z[f] := \frac{1}{z}\left(\Delta(R_z^{\mathcal{B}}\Delta f) - f\right). \tag{3.4}$$

From the elliptic theory for the operator $\mathcal{B}$ (compare [1]) and from Sobolev's lemma we obtain $R_z^{\mathcal{B}}(\Delta f) \in C^\infty(\overline{\Omega})$, and hence $u_z \in C^\infty(\overline{\Omega})$. Since $(\Delta^2 - z)R_z^{\mathcal{B}}\Delta f = \Delta f$, we have

$$\Delta u_z = \frac{1}{z}\left(\Delta^2(R_z^{\mathcal{B}}\Delta f) - \Delta f\right) = R_z^{\mathcal{B}}\Delta f.$$

This implies by (3.4)

$$\Delta^2 u_z = \Delta(R_z^{\mathcal{B}}\Delta f) = z u_z + f. \tag{3.5}$$

Furthermore, since $R_z^{\mathcal{B}}\Delta f \in D(\mathcal{B})$, we obtain

$$\Delta u_z = R_z^{\mathcal{B}}\Delta f = 0, \qquad \frac{\partial \Delta u_z}{\partial \mathbf{n}} = \frac{\partial R_z^{\mathcal{B}}\Delta f}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega. \tag{3.6}$$

Thus u_z is a solution of (3.1). It remains to show that $u_z \in D(A) \subset D(\mathcal{A})$.

Let G_z denote the fundamental solution of $\Delta^2 - z$ in $\mathbb{R}^n$ (compare e.g (2.5) in [5] for a representation). We have for every $z \in \mathbb{C} \setminus \mathbb{R}$ and every multi-index $\alpha \in \mathbb{N}_0^n$

$$D_x^\alpha G_z(|x-y|) = O\left(\mathrm{e}^{-c(z)|x-y|}\right) \qquad \text{as } |x-y| \to \infty$$

with suitable $c(z) > 0$. Furthermore, Greens formula implies

$$\begin{aligned}(R_z^{\mathcal{B}} f)(x) \;&=\; \int_\Omega f(y)\, G_z(|x-y|)\, dy \\ &\quad - \int_{\partial\Omega} \left(\frac{\partial \Delta(R_z^{\mathcal{B}} f)}{\partial \mathbf{n}}(y)\, G_z(|x-y|) - \Delta(R_z^{\mathcal{B}} f)(y)\, \frac{\partial G_z(|x-y|)}{\partial \mathbf{n}_y} \right) ds_y.\end{aligned} \tag{3.7}$$

Since f has bounded support, we obtain

$$D^\alpha(R_z^{\mathcal{B}}\Delta f)(x) = O\left(\mathrm{e}^{-c(z)|x|}\right) \qquad \text{as } |x| \to \infty$$

for every $\alpha \in \mathbb{N}_0^n$. This, together with (3.6) and $u_z \in C^\infty(\overline{\Omega})$ shows that $u_z \in D(A)$. Hence we have $u_z = R_z f$.

Now suppose that $f \in H^2(\Omega)$ and let $\{f_k\}$ be a sequence in $C^\infty(\overline{\Omega})$ such that every f_k has bounded support and $\|f - f_k\|_2 \to 0$ as $k \to \infty$. Since

$$(\mathcal{B} - z)\left(R_z^{\mathcal{B}} \Delta f - R_z^{\mathcal{B}} \Delta f_k\right) = \Delta f - \Delta f_k$$

we obtain by standard estimates that $\|R_z^{\mathcal{B}} \Delta f - R_z^{\mathcal{B}} \Delta f_k\|_2 \to 0$ and hence by (3.4) $\|u_z[f] - u_z[f_k]\| \to 0$ as $k \to \infty$. This yields

$$\mathcal{A} u_z[f_k] = f_k + z u_z[f_k] \to f + z u_z[f] \qquad \text{as } k \to \infty$$

in $L_2(\Omega)$. Since self-adjoint operators are closed, we obtain $u_z[f] \in D(\mathcal{A})$ and $\mathcal{A} u_z[f] = f + z u_z[f]$. This verifies $u_z = R_z f$ for $f \in H^2(\Omega)$. ■

In order study $R_z^{\mathcal{B}}$ more closely, we introduce the weighted Sobolev spaces $H_s^k(\Omega)$ for $k \in \mathbb{N}_0$, $s \in \mathbb{R}$ by

$$\|u\|_{k,s} := \left(\int_\Omega (1 + |x|)^s \sum_{|\alpha| \le k} |D^\alpha u(x)|^2 \, dx \right)^{1/2}, \tag{3.8}$$

$$H_s^k(\Omega) := \{u \in H_{\text{loc}}^k(\Omega) : \|u\|_{k,s} < \infty\}. \tag{3.9}$$

We set $L_{2,s}(\Omega) := H_s^0(\Omega)$. Furthermore, we denote the Banach space of bounded linear operators from $L_{2,s_1}(\Omega)$ into $H_{-s_2}^4(\Omega)$ by $B^{(s_1,s_2)}$. $\mathcal{B}$ satisfies all assumptions of Theorem 1.1 in [3] with $\sigma = \infty$. Hence for every $l \in \mathbb{N}$ there exists a constant $\varepsilon_l > 0$ such that for $|\kappa| < \varepsilon_l$, $0 < \arg \kappa < \frac{\pi}{2}$

$$R_{\kappa^4}^{\mathcal{B}} = \sum_{j=0}^{l-1} \kappa^j F_j + \kappa^l \tilde{F}_l(\kappa) \tag{3.10}$$

whith $F_j \in B^{(s_1(j), s_2(j))}$ and

$$\left\| \tilde{F}_l(\kappa) \right\|_{B^{(s_1(l-1), s_2(l-1))}} \le c \tag{3.11}$$

for

$$s_1(j) > \max\{n, 10 - n + 2j\}, \qquad s_2(j) > 10 - n + 2j,\ s_2(j) \geq 0. \tag{3.12}$$

Furthermore, we have $F_j = 0$ for $1 \leq j \leq n-5$, $\frac{j}{4} \notin \mathbb{N}$ according to (4.10) in [3].

Lemma 3.2 *For sufficiently small $\lambda > 0$, the limits*

$$R^{\mathcal{B}}_{\lambda \pm \mathrm{i}0} := \lim_{\tau \downarrow 0} R^{\mathcal{B}}_{\lambda \pm \mathrm{i}\tau} \tag{3.13}$$

exist with respect to the norm in $B^{(s_1(0), s_2(0))}$. Furthermore,

$$R^{\mathcal{B}}_{\lambda \pm \mathrm{i}0} = \sum_{j=0}^{l} (\kappa_\pm)^j F_j + (\kappa_\pm)^l F_\pm(\lambda), \tag{3.14}$$

where $\kappa_+ = \lambda^{1/4}$, $\kappa_- = \mathrm{i}\lambda^{1/4}$, F_j are the operators appearing in (3.10) and

$$\|F_\pm(\lambda)\|_{B^{(s_1(l), s_2(l))}} \leq c, \tag{3.15}$$

where $c > 0$ does not depend on λ.

Proof: It follows from the argument preceding (3.15) in [3] that $d^l R_z / d\kappa^l$ is uniformly continuous on $\{z \in \mathbb{C} \setminus [0, \infty) : |z| < \varepsilon_l\}$ with respect to the norm in $B^{(s_1(l), s_2(l))}$ for every $l \in \mathbb{N}$. This proves Lemma 3.2. ∎

The low frequency asymptotics of the resolvent R_z of $\mathcal{A}$ are given by the following lemma, which follows immediately from Lemma 3.1 and Lemma 3.2:

Lemma 3.3 *Suppose that $f \in H^2(\Omega)$ and that $\Delta f \in L_{2, s_1(l)}(\Omega)$, where $s_1(l)$ satisfies (3.12) for some $l \in \mathbb{N}$. Then*

$$R_{\kappa^4} f = \frac{1}{\kappa^4} (\Delta F_0 \Delta f - f) + \sum_{j=1}^{l-1} \kappa^{j-4} \Delta F_j \Delta f + \Delta \tilde{F}_l(\kappa) \Delta f \tag{3.16}$$

for $0 < \arg\kappa < \frac{\pi}{2}$, $|\kappa|$ *sufficiently small. Furthermore, the limits*

$$R_{\lambda\pm\mathrm{i}0}f := \lim_{\tau\downarrow 0} R_{\lambda\pm\mathrm{i}\tau}f$$

exist with respect to the norm in $H^2_{-s_2(1)}(\Omega)$ *for sufficiently small* $\lambda > 0$, *and (3.16) holds also in the limit case* $\kappa^4 = \lambda \pm \mathrm{i}0$.

4 The spectral family

We study the spectral family of $\mathcal{A}$ using Stone's formula

$$\frac{1}{2}\langle (P_\lambda + P_{\lambda+0})f - (P_\mu + P_{\mu+0})f, g\rangle = \lim_{\tau\downarrow 0}\frac{1}{2\pi\mathrm{i}}\int_\mu^\lambda \langle (R_{\rho+\mathrm{i}\tau} - R_{\rho-\mathrm{i}\tau})f, g\rangle \, d\rho \quad (4.1)$$

for $\mu < \lambda$ and $f, g \in L_2(\Omega)$. Note that $P_\lambda = 0$ for $\lambda \le 0$ by the positivity of $\mathcal{A}$.

Lemma 4.1 *If* $f \in H^2(\Omega)$ *and* $\varphi \in C([\mu_1, \mu_2])$ *with* $0 < \mu_1 < \mu_2 < \infty$, *then*

$$\int_{\mu_1}^{\mu_2} \varphi(\lambda)\, d(P_\lambda f) = \Delta \int_{\mu_1}^{\mu_2} \frac{\varphi(\lambda)}{\lambda}\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big). \quad (4.2)$$

Remarks: **1)** The integral on the right-hand side of (4.2) belongs to $D(\mathcal{B}^k)$ for every $k \in \mathbb{N}$ if $0 < \mu_1 < \mu_2 < \infty$. By regularity theory for $\mathcal{B}$, this implies that the right-hand side belongs to $H^{4k-2}(\Omega)$ for every $k \in \mathbb{N}$ (compare Theorem 9.8 in [1] and the remark following this theorem; the case of bounded $\partial\Omega$ can be proved by a slight modification of the proof of Theorem 9.8).

2) If $f \in H^2(\Omega)$ and $\varphi \in C([\mu_1, \infty))$ is bounded, then (4.2) also holds for $\mu_2 = \infty$.

Proof: At first we prove (4.2) for $\varphi = 1$. Consider a fixed $\tau > 0$ and set

$$\varphi_{\alpha\beta} := \Delta \int_\alpha^\beta \frac{1}{\lambda - \mathrm{i}\tau}\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \qquad \text{for } 0 < \alpha < \beta < \infty. \quad (4.3)$$

Since

$$R_z^{\mathcal{B}}(\Delta\varphi_{\alpha\beta}) = R_z^{\mathcal{B}}\left(\mathcal{B}\int_\alpha^\beta \frac{1}{\lambda - \mathrm{i}\tau}\, d\big(P_\lambda^{\mathcal{B}}\Delta f\big)\right) = \int_\alpha^\beta \frac{\lambda}{(\lambda - \mathrm{i}\tau)(\lambda - z)}\, d\big(P_\lambda^{\mathcal{B}}\Delta f\big)\,,$$

we obtain by (3.3)

$$R_z\,\varphi_{\alpha\beta} = \Delta\int_\alpha^\beta \frac{1}{(\lambda - \mathrm{i}\tau)(\lambda - z)}\, d\big(P_\lambda^{\mathcal{B}}\Delta f\big)\,.$$

This implies that $R_z\,\varphi_{\alpha\beta}$ is continuous in z for $\operatorname{Re} z < \alpha$ and $\operatorname{Re} z > \beta$. By (4.1), we conclude that $P_\lambda\,\varphi_{\alpha\beta}$ is constant for $\lambda < \alpha$ and $\lambda > \beta$. Hence $P_\lambda\,\varphi_{\alpha\beta} = 0$ for $\lambda \le \alpha$ and $P_\lambda\,\varphi_{\alpha\beta} = \varphi_{\alpha\beta}$ for $\lambda > \beta$. $P_\lambda^{\mathcal{B}}\Delta f$ is continuous in λ, since $\mathcal{B}$ has no eigenvalues. Therefore $\varphi_{\alpha\beta}$ depends continuously on α and β for $0 < \alpha < \beta$ with respect to the norm in $L_2(\Omega)$. Hence we have

$$P_\beta\,\varphi_{\alpha\beta} = \lim_{\gamma\uparrow\beta} P_\beta\, g_{\alpha\gamma} = \lim_{\gamma\uparrow\beta} g_{\alpha\gamma} = \varphi_{\alpha\beta}.$$

Now we obtain for $0 < \alpha < \mu_1 < \mu_2 < \beta$

$$(P_{\mu_2} - P_{\mu_1})\,\varphi_{\alpha\beta} = (P_{\mu_2} - P_{\mu_1})\,(g_{\alpha\mu_1} + g_{\mu_1\mu_2} + g_{\mu_2\beta}) = g_{\mu_1\mu_2}.$$

Since $\varphi_{\alpha\beta} \to \Delta(R_{\mathrm{i}\tau}^{\mathcal{B}}\Delta f)$ in $L_2(\Omega)$ as $\alpha \downarrow 0$ and $\beta \to \infty$, we conclude that

$$(P_{\mu_2} - P_{\mu_1})\left(\Delta(R_{\mathrm{i}\tau}^{\mathcal{B}}\Delta f)\right) = \varphi_{\mu_1\mu_2}. \tag{4.4}$$

By (3.3), we obtain

$$\begin{aligned}(P_{\mu_2} - P_{\mu_1})\left(\Delta(R_{\mathrm{i}\tau}^{\mathcal{B}}\Delta f)\right) &= (P_{\mu_2} - P_{\mu_1})\,(\mathrm{i}\tau R_{\mathrm{i}\tau} f + f) = \\ &= \int_{\mu_1}^{\mu_2} \frac{\lambda}{\lambda - \mathrm{i}\tau}\, d(P_\lambda f) \;\to\; \int_{\mu_1}^{\mu_2} d(P_\lambda f) \qquad \text{as } \tau \downarrow 0\end{aligned} \tag{4.5}$$

in $L_2(\Omega)$. Regularity theory for the operator $\mathcal{B}$ yields

$$g_{\mu_1\mu_2} \to \Delta\int_{\mu_1}^{\mu_2} \frac{1}{\lambda}\, d(P_\lambda f) \qquad \text{as } \tau \downarrow 0 \tag{4.6}$$

in $L_2(\Omega)$. By (4.4) – (4.6), assertion (4.2) with $\varphi = 1$ follows. Approximating an arbitrary $\varphi \in C([\mu_1, \mu_2])$ by piecewise constant functions φ_k we conclude that (4.2) holds, since

$$\left\| \int_{\mu_1}^{\mu_2} \frac{\varphi(\lambda) - \varphi_k(\lambda)}{\lambda} \, d(P_\lambda^{\mathcal{B}} \Delta f) \right\|_4^2 \leq \frac{c}{\mu_1^2} \|\Delta f\|^2 \max_{\mu_1 \leq \lambda \leq \mu_2} |\varphi(\lambda) - \varphi_k(\lambda)|^2$$

by regularity theory for $\mathcal{B}$. ■

Lemma 4.2 *If $f \in H^2(\Omega)$ with $\Delta f \in L_{2,s_1(0)}(\Omega)$, then the projection onto the nullspace of $\mathcal{A}$ is given by*

$$P^{(N)} f = P_{0+} f = f - \Delta(F_0 \Delta f); \tag{4.7}$$

here $F_0 : L_{2,s_1(0)}(\Omega) \to H^4_{-s_2(0)}(\Omega)$ denotes the first summand in (3.10).

Remark: (4.7) implies $\Delta(F_0 \Delta f) \in L_2(\Omega)$.

Proof: From (3.3) and (3.10) we conclude that

$$\begin{aligned} \langle R_{\rho \pm \mathrm{i}\tau} f, g \rangle &= \frac{1}{\rho \pm \mathrm{i}\tau} \left\langle (\Delta R^{\mathcal{B}}_{\rho \pm \mathrm{i}\tau} \Delta f) - f, g \right\rangle \\ &= \frac{1}{\rho \pm \mathrm{i}\tau} \langle (\Delta F_0 \Delta f) - f, g \rangle + r(\rho \pm \mathrm{i}\tau), \end{aligned}$$

where, by (3.11),

$$|r(\rho \pm \mathrm{i}\tau)| = \left| \frac{1}{\rho \pm \mathrm{i}\tau} \left\langle \kappa \Delta \tilde{F}_1(\kappa)(\Delta f), g \right\rangle \right| \leq \frac{c}{|\rho \pm \mathrm{i}\tau|^{3/4}} \|\Delta f\|_{0,s_1(0)} \|g\|_{0,s_2(0)}$$

for every $g \in C_0^\infty(\Omega)$, $|\rho \pm \mathrm{i}\tau|$ sufficiently small, $\kappa^2 = \rho \pm \mathrm{i}\tau$, $0 < \arg \kappa < \pi/2$. Inserting this in (4.1), we obtain

$$\begin{aligned} &\frac{1}{2} \langle (P_\lambda + P_{\lambda+0}) f - (P_0 + P_{0+}) f, g \rangle = \\ &= \lim_{\tau \downarrow 0} \frac{1}{2\pi \mathrm{i}} \int_0^\lambda \left(\frac{1}{\rho + \mathrm{i}\tau} - \frac{1}{\rho - \mathrm{i}\tau} \right) d\rho \, \langle \Delta(F_0 \Delta f) - f, g \rangle + O\left(\lambda^{1/4}\right) \\ &= -\frac{1}{2} \langle \Delta(F_0 \Delta f) - f, g \rangle + O\left(\lambda^{1/4}\right) \qquad \text{as } \lambda \downarrow 0. \end{aligned}$$

Now (4.7) follows as $\lambda \downarrow 0$, since $P_0 f = 0$. ■

The following properties of the spectral family $\{P_\lambda^{\mathcal{B}}\}$ of $\mathcal{B}$ will be needed for the investigation of the regularity and the large time behaviour of the solution u of (1.1), (1.8). We set $\Omega_R := \{x \in \Omega : |x| < R\}$.

Lemma 4.3 *Let $g \in L_{2,s_1(0)}(\Omega)$ and $\varepsilon > 0$ such that (3.10) holds for $|z| < \varepsilon$. Then $P_\lambda^{\mathcal{B}} g : \mathbb{R} \to H^4_{-s_2(0)}(\Omega)$ is continuous with respect to λ for $0 \le \lambda \le \varepsilon$ and differentiable for $0 < \lambda < \varepsilon$. Moreover, there exists a $c > 0$ such that*

$$\left\| \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda} \right\|_{4,-s_2(0)} \le c\,\lambda^{1/4}\,\|g\|_{0,s_1(0)} \qquad \text{for } 0 < \lambda < \varepsilon,\ g \in L_{2,s_1(0)}(\Omega) \tag{4.8}$$

holds. Furthermore, we have $(d/d\lambda) P_\lambda^{\mathcal{B}} g \in H^k(\Omega_R)$ for every $k \in \mathbb{N}$, $R > 0$, and there exist constants $c_{k,R} > 0$ such that

$$\left\| \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda} \right\|_{H^k(\Omega_R)} \le c_{k,R}\,\lambda^{1/4} \|g\|_{0,s_1(0)} \qquad \text{for } 0 < \lambda < \varepsilon,\ g \in L_{2,s_1(0)}(\Omega). \tag{4.9}$$

If in addition $g \in L_{2,s_1(4)}(\Omega)$, then for $k \in \mathbb{N}$ and $R > 0$

$$\left\| \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda} - \sum_{j=1}^{3} \frac{1 - \mathrm{i}^j}{2\pi \mathrm{i}}\,\lambda^{j/4} F_j g \right\|_{H^k(\Omega_R)} \le c_{k,R}\,\lambda^{5/4} \|g\|_{0,s_1(4)} \qquad \text{for } 0 < \lambda < \varepsilon. \tag{4.10}$$

Proof: According to the arguments preceding (3.15) in [3], the resolvent $R_z^{\mathcal{B}} g : \mathbb{C} \to H^4_{-s_2(0)}(\Omega)$ is uniformly continuous with respect to z for $0 \le |z| \le \varepsilon$, $\operatorname{Im} z \ge 0$ and also for $0 \le |z| \le \varepsilon$, $\operatorname{Im} z \le 0$. Hence (4.1) implies

$$\frac{dP_\lambda^{\mathcal{B}} g}{d\lambda} = \frac{1}{2\pi \mathrm{i}} \left(R_{\lambda + \mathrm{i}0}^{\mathcal{B}}\, g - R_{\lambda - \mathrm{i}0}^{\mathcal{B}}\, g \right). \tag{4.11}$$

This, together with (3.14), implies (4.8). By

$$\Delta^2 \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda} = \frac{1}{2\pi \mathrm{i}}\, \Delta^2 \left(R_{\lambda + \mathrm{i}0}^{\mathcal{B}}\, g - R_{\lambda - \mathrm{i}0}^{\mathcal{B}}\, g \right) = 0 \qquad \text{in } \Omega$$

and regularity theory for $\mathcal{B}$, we obtain

$$\left\| \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda} \right\|_{H^{4k}(\Omega_R)} \leq c_k \left\| \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda} \right\|_{L_2(\Omega_{R+1})} \qquad \text{for } k \in \mathbb{N}$$

with suitable constants $c_k > 0$ not depending on g. This yields (4.9) by (4.8).

It follows from

$$\left(\Delta^2 - z\right) R_z^{\mathcal{B}} g = g \text{ in } \Omega, \qquad R_z^{\mathcal{B}} g = \frac{\partial R_z^{\mathcal{B}} g}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega$$

and (3.10) that

$$\Delta^2 F_j g = 0 \text{ in } \Omega, \qquad F_j g = \frac{\partial F_j g}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \qquad \text{for } j = 1, 2, 3. \tag{4.12}$$

Hence (4.10) follows as above from (4.11) and (3.14). ■

In the same way we obtain from [2] (Theorems 3.2, 3.4 and 3.5):

Lemma 4.4 *Let $g \in L_2(\Omega)$ such that $g(x) = 0$ outside some bounded set. Then (4.11) holds for $\lambda > 0$. Furthermore,*

$$\left\| \frac{d}{d\lambda} P_\lambda^{\mathcal{B}} g \Big|_{\lambda=\lambda_1} - \frac{d}{d\lambda} P_\lambda^{\mathcal{B}} g \Big|_{\lambda=\lambda_2} \right\|_{H^k(\Omega(R))} \leq c\, |\lambda_1 - \lambda_2| \tag{4.13}$$

for $R > 0$, $k \in \mathbb{N}$ and $0 < \alpha \leq \lambda_1 < \lambda_2 \leq \beta < \infty$, where $c > 0$ may depend on R, k, α, β and g.

5 Regularity of the solution

By (2.1) and (4.2), we have for every $t \geq 0$

$$u(t) = \frac{t^2}{2} P_{0+} f + \lim_{a \downarrow 0} \Delta \int_a^\infty \frac{1}{\lambda^2} (1 - \cos\sqrt{\lambda} t)\, d\left(P_\lambda^{\mathcal{B}} \Delta f\right). \tag{5.1}$$

In the following we suppose that

$$f \in H^{4k+2}(\Omega), \quad \Delta f \in L_{2,s_1(0)}(\Omega), \quad \Delta^{2j+1} f \in \overset{\circ}{H}{}^2(\Omega) \text{ for } j = 0, 1, \ldots, k-1 \quad (5.2)$$

with $k \in \mathbb{N}$. Note that the third condition in (5.2) is equivalent to $\Delta^{2j+1} f = (\partial/\partial \mathbf{n})\Delta^{2j+1} f = 0$ on $\partial\Omega$ for $j = 0, 1, \ldots, k-1$ (compare Theorem 11.5 in [6]). We need the following elliptic estimate, which is given without proof in [2].

Lemma 5.1 *Let $s \geq 0$ and $k \in \mathbb{N}$ be given. If $\varphi \in H^2_{-s}(\Omega)$, $\varphi = (\partial/\partial \mathbf{n})\varphi = 0$ on $\partial\Omega$ and $\Delta^2 \varphi \in H^k_{-s}(\Omega)$, then $\varphi \in H^{k+4}_{-s}(\Omega)$ and there exists a constant $c > 0$ not depending on φ such that*

$$\|\varphi\|_{k+4,-s} \leq c \left(\|\Delta^2 \varphi\|_{k,-s} + \|\varphi\|_{0,-s} \right). \quad (5.3)$$

Proof: Let $K(R, R') := \{x \in \Omega : R < |x| < R'\}$. A modification of the proof of Theorem 9.8 in [1] yields

$$\|\varphi\|_{H^{k+4}(K(R-1,R+1))} \leq c \left(\|\Delta^2 \varphi\|_{H^k(K(R-2,R+2))} + \|\varphi\|_{L_2(K(R-2,R+2))} \right),$$

where $c > 0$ does not depend on φ and R. Multiplying this estimate with $(R-1)^{-s}$ and taking the sum of the inequalities resulting for $R = 2, 3, \ldots$, we obtain (5.3).

■

Consider the first term on the right-hand side of (5.1). In view of (4.7) we study the regularity of $F_0 \Delta f$. Since $F_0 \Delta f \in H^4_{-s_2(0)}(\Omega)$ and $\Delta^2(F_0 \Delta f) = \Delta f$, $F_0 \Delta f = (\partial/\partial \mathbf{n})(F_0 \Delta f) = 0$ on $\partial\Omega$ (compare (4.12)), we obtain by Lemma 5.1 that $F_0 \Delta f \in H^{4k+4}_{-s_2(0)}(\Omega)$. This implies

$$P_{0+} f = f - \Delta(F_0 \Delta f) \in H^{4k+2}_{-s_2(0)}(\Omega). \quad (5.4)$$

We turn to the second term on the right-hand side of (5.1). In view of (5.2) we have $\Delta f \in D(\mathcal{B}^k)$. By the functional calculus and (5.3) with $s = 0$ we obtain, by regularity theory for $\mathcal{B}$, for every fixed $b > 0$

$$\int_b^\infty \frac{1}{\lambda^2}(1 - \cos\sqrt{\lambda}t)\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \in D(\mathcal{B}^{k+2}) \subset H^{4k+8}(\Omega) \subset H^{4k+8}_{-s_2(0)}(\Omega). \tag{5.5}$$

For $0 < a < b < \varepsilon$ we conclude from Lemma 4.3 that

$$\begin{aligned} &\left\| \mathcal{B}^j \int_a^b \frac{1}{\lambda^2}(1 - \cos\sqrt{\lambda}t)\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \right\|_{-s_2(0)} = \\ &= \left\| \int_a^b \frac{1 - \cos\sqrt{\lambda}t}{\lambda}\, \lambda^{j-1} \frac{dP_\lambda^{\mathcal{B}} \Delta f}{d\lambda}\, d\lambda \right\|_{-s_2(0)} \le c\, t^2\, \|\Delta f\|_{s_1(0)} \left(b^{j+1/4} - a^{j+1/4}\right) \end{aligned}$$

for $j = 0, 1, \ldots$, where $c > 0$ may depend on j. With (5.3), we obtain

$$\begin{aligned} &\left\| \int_a^b \frac{1 - \cos\sqrt{\lambda}t}{\lambda^2}\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \right\|_{4j,-s_2(0)} \le \\ &\le c_j \sum_{k=0}^{j} \left\| \mathcal{B}^k \int_a^b \frac{1 - \cos\sqrt{\lambda}t}{\lambda^2}\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \right\|_{0,-s_2(0)} \le \tilde{c}_j \left(b^{1/4} - a^{1/4}\right) \end{aligned}$$

for $j \in \mathbb{N}$ with suitable constants $c_j, \tilde{c}_j > 0$. This, together with (5.5) proves

$$\lim_{a \downarrow 0} \Delta \int_a^\infty \frac{1}{\lambda^2}(1 - \cos\sqrt{\lambda}t)\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \in H^{4k+6}_{-s_2(0)}(\Omega) \qquad \text{for every } t \ge 0. \tag{5.6}$$

Now (5.4) and (5.6) imply $u(t) \in H^{4k+2}_{-s_2(0)}(\Omega)$ for $t \ge 0$. In the same way we obtain $u'(t) \in H^{4k+2}_{-s_2(0)}(\Omega)$ and

$$u^{(j)}(t) \in H^{4k+6-2j}_{-s_2(0)}(\Omega) \qquad \text{for } j = 2, 3, \ldots, 2k+3 \text{ and } t \ge 0 \tag{5.7}$$

by using $D(\mathcal{B}^{1/2}) \subset H^2(\Omega)$. With similar estimates for $\|u^{(j)}(t_1) - u^{(j)}(t_2)\|_{4k+6-2j, s_2(0)}$ we conclude that

$$u \in \bigcap_{j=2}^{2k+3} C^j\left([0,\infty), H^{4k+6-2j}_{-s_2(0)}(\Omega)\right). \tag{5.8}$$

By the Lemma of Sobolev, this implies $u \in C^4([0,\infty) \times \overline{\Omega})$ if $k \in \mathbb{N}$, $k > \frac{n+4}{8}$. Hence u is a classical solution of (1.1). From

$$\Delta P_{0+} f = \Delta f - \Delta^2 F_0 \Delta f = 0,$$
$$\Delta \lim_{a \downarrow 0} \Delta \int_a^\infty \frac{1}{\lambda^2}(1 - \cos\sqrt{\lambda} t)\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) = \int_0^\infty \frac{1}{\lambda}(1 - \cos\sqrt{\lambda} t)\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \in D(\mathcal{B})$$

and (5.1) we obtain that u satisfies (1.8). Thus we have proved:

Theorem 5.2 *Let $k \in \mathbb{N}$, $k > \frac{n+4}{8}$ and suppose that f satisfies (5.2). Then (1.1), (1.8) has a classical solution $u \in C^4([0,\infty) \times \overline{\Omega})$, which is given by (5.1). Furthermore, (5.8) holds.*

Remark: This solution is the only solution of (1.1), (1.8) satisfying (2.3).

6 Time asymptotics

We rewrite (5.1) as

$$u(t) = \frac{t^2}{2} P^{(N)} f + I_1(t) + I_2 + I_3(t), \tag{6.1}$$

where

$$I_1(t) := \lim_{a \downarrow 0} \Delta \int_a^\delta \frac{1}{\lambda^2}(1 - \cos\sqrt{\lambda} t)\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big), \tag{6.2}$$
$$I_2 := \Delta \int_\delta^\infty \frac{1}{\lambda^2}\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big), \tag{6.3}$$
$$I_3(t) := -\Delta \int_\delta^\infty \frac{1}{\lambda^2} \cos\sqrt{\lambda} t\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big) \tag{6.4}$$

and $\delta > 0$ is chosen so that (4.10) holds for $0 < \lambda \leq \delta$.

Lemma 6.1 *Assume that* $f \in H^2(\Omega)$, $\Delta f \in L_{2,s_1(4)}(\Omega)$ *and that* $\delta > 0$ *is sufficiently small. Then*

$$I_1(t,x) = \sum_{j=1}^{3} \left(E_j\, t^{2-j/2} - \frac{1-\mathrm{i}^j}{2\pi \mathrm{i}} \frac{1}{\left(1-\frac{j}{4}\right)\delta^{1-j/4}} \right) \Delta(F_j \Delta f)(x) + \varphi_1(t,x) + O_M\left(\frac{1}{t}\right) \qquad \text{as } t \to \infty, \tag{6.5}$$

where

$$E_j = \frac{-\mathrm{i}(1-\mathrm{i}^j)}{(4-j)\,\Gamma\left(\frac{4-j}{2}\right)\sin\left(\frac{4-j}{4}\,\pi\right)} \qquad (j = 1,2,3) \tag{6.6}$$

and

$$|\varphi_1(t,x)| \leq c_M\, \delta^{1/4} \|\Delta f\|_{0,s_1(4)} \qquad \text{for } t \geq 0,\ x \in M, \tag{6.7}$$

if M *is an arbitrary bounded subset of* $\overline{\Omega}$.

Remark: Recall that $O_M\left(\frac{1}{t}\right)$ in (6.5) means that the remaining term is of order $O\left(\frac{1}{t}\right)$ uniformly in every bounded subset M of $\overline{\Omega}$.

Proof: With (4.9) and the Lemma of Sobolev we conclude that

$$I_1(t,x) = \Delta \int_0^\delta \frac{1}{\lambda^2}(1-\cos\sqrt{\lambda}t)\, \frac{dP_\lambda^{\mathcal{B}} f}{d\lambda}(x)\, d\lambda.$$

Note that by (2.16) and (2.19) in [5]

$$\begin{aligned} I_4(t,x) &:= \Delta \int_0^\delta \frac{1}{\lambda^2}(1-\cos\sqrt{\lambda}t) \left(\sum_{j=1}^{3} \frac{1-\mathrm{i}^j}{2\pi \mathrm{i}}\, (F_j\Delta f)(x)\, \lambda^{j/4} \right) d\lambda \\ &= \sum_{j=0}^{3} \left(E_j\, t^{2-j/2} - \frac{1-\mathrm{i}^j}{2\pi \mathrm{i}} \frac{1}{\left(1-\frac{j}{4}\right)\delta^{1-j/4}} \right) \Delta(F_j\Delta f)(x) + O_M\left(\frac{1}{t}\right) \end{aligned}$$

as $t \to \infty$. Hence we obtain by (4.10) and Sobolev's Lemma that (6.7) holds with $\varphi_1(t,x) = I_4(t,x) - I_1(t,x)$. ■

Lemma 6.2 *Suppose that* $\Delta f \in L_{2,s_1(4)}(\Omega)$ *and that* $\delta > 0$ *is sufficiently small. Then*

$$I_2 = \sum_{j=1}^{3} \frac{1-\mathrm{i}^j}{2\pi \mathrm{i}} \frac{1}{\left(1-\frac{j}{4}\right)\delta^{1-j/4}} \Delta(F_j \Delta f) + \Delta(F_4 \Delta f) + \varphi_2 \tag{6.8}$$

in every bounded subset M *of* $\overline{\Omega}$ *with*

$$|\varphi_2(x)| \le c_M\, \delta^{1/4}\, \|\Delta f\|_{s_1(4)} \qquad \text{for } x \in M. \tag{6.9}$$

Proof: By (6.3) and (4.2), we have

$$I_2 = \int_\delta^\infty \frac{1}{\lambda}\, d(P_\lambda f) = \lim_{\tau \downarrow 0} \int_\delta^\infty \frac{1}{\lambda - \mathrm{i}\tau}\, d(P_\lambda f) = \lim_{\tau\downarrow 0} \left(R_{\mathrm{i}\tau} f - \int_0^\delta \frac{1}{\lambda - \mathrm{i}\tau}\, d(P_\lambda f) \right), \tag{6.10}$$

where the limit has to be understood in the sense of $L_2(\Omega)$. In particular, this limit holds in $L_{2,-s}(\Omega)$ for every $s \ge 0$. By (3.3) and (3.10), we have

$$R_{\mathrm{i}\tau} f = \frac{1}{\mathrm{i}\tau}\left(\Delta(F_0 \Delta f) - f\right) + \sum_{j=1}^{4} \frac{-\mathrm{i}\, \mathrm{e}^{\mathrm{i}j\pi/8}}{\tau^{1-j/4}} \Delta(F_j \Delta f) - \tau^{1/4} \mathrm{i}\, \mathrm{e}^{\mathrm{i}\pi/8} \Delta\Big(\tilde{F}_5(\tau^{1/4} \mathrm{e}^{\mathrm{i}\pi/8}) \Delta f\Big), \tag{6.11}$$

where

$$\Big\| \Delta\Big(\tilde{F}_5(\tau^{1/4}\mathrm{e}^{\mathrm{i}\pi/8})\Delta f\Big) \Big\|_{2,-s_2(4)} \le c\, \|\Delta f\|_{0,s_1(4)}. \tag{6.12}$$

Consider the second term on the right-hand side of (6.10). For $j = 1, 2, 3$ we have

$$\begin{aligned} \int_0^\delta \frac{1}{(\lambda - \mathrm{i}\tau)\lambda^{1-j/4}}\, d\lambda &= \frac{1}{\tau^{1-j/4}} \left(\int_0^\infty \frac{d\xi}{(\xi - \mathrm{i})\xi^{1-j/4}} - \int_{\delta/\tau}^\infty \frac{d\xi}{\xi^{2-j/4}} + O\left(\tau^{2-j/4}\right) \right) \\ &= \frac{1}{\tau^{1-j/4}} \left(\frac{\pi}{2\cos\left(\frac{j\pi}{8}\right)} + \frac{\mathrm{i}\pi}{2\sin\left(\frac{j\pi}{8}\right)} \right) - \frac{1}{\left(1-\frac{j}{4}\right)\delta^{1-j/4}} + O(\tau) \end{aligned}$$

as $\tau \downarrow 0$. Note that $\frac{\pi}{2\cos(j\pi/8)} + \frac{\mathrm{i}\pi}{2\sin(j\pi/8)} = \frac{2\pi}{1-\mathrm{i}^j}\, \mathrm{e}^{\mathrm{i}j\pi/8}$ for $j = 1, 2, 3$. From this and (4.2) we obtain

$$\int_0^\delta \frac{1}{\lambda - \mathrm{i}\tau}\, d(P_\lambda f) = \frac{1}{-\mathrm{i}\tau} P^{(N)} f + \lim_{a \downarrow 0} \Delta \int_a^\delta \frac{1}{\lambda(\lambda - \mathrm{i}\tau)}\, d\Big(P_\lambda^{\mathcal{B}} \Delta f\Big)$$

$$= \quad -\frac{1}{\mathrm{i}\tau} P^{(N)} f + \sum_{j=1}^{3} \left(-\frac{\mathrm{i} e^{\mathrm{i}j\pi/8}}{\tau^{1-j/4}} - \frac{1-\mathrm{i}^j}{2\pi \mathrm{i} \left(1-\frac{j}{4}\right) \delta^{1-j/4}} \right) \Delta(F_j \Delta f)$$
$$+ \varphi_2 + O(\tau) \qquad \text{as } \tau \downarrow 0, \tag{6.13}$$

where

$$\varphi_2 = \Delta \int_0^\delta \frac{1}{\lambda} \left(\frac{dP_\lambda^{\mathcal{B}} \Delta f}{d\lambda} - \sum_{j=1}^{3} \frac{1-\mathrm{i}^j}{2\pi \mathrm{i}} \lambda^{j/4} F_j \Delta f \right) d\lambda. \tag{6.14}$$

From (6.10) – (6.14) we obtain by (4.10) that (6.8) and (6.9) hold. ■

Lemma 6.3 *Let $\Delta f \in D(\mathcal{B}^k)$ for some $k > \frac{n}{8}$. Then we have for every fixed $\delta > 0$*

$$I_3(t,x) = o_M(1) \qquad \textit{as } t \to \infty. \tag{6.15}$$

Proof: By

$$\|\varphi\|_{4k} \le c_k \sum_{j=0}^{k} \|B^j \varphi\| \qquad \text{for } \varphi \in D(B^k), \tag{6.16}$$

we conclude that there exists an $A > 0$ such that

$$\left\| \int_A^\infty \frac{\cos\sqrt{\lambda}t}{\lambda^2} \, d\left(P_\lambda^{\mathcal{B}} \Delta f\right) \right\|_{4k} \le \varepsilon \tag{6.17}$$

for given $\varepsilon > 0$. Let $g \in C_0^\infty(\Omega)$ such that $\|g - \Delta f\| < \varepsilon$. Then we obtain by (6.16) that

$$\left\| \int_\delta^A \frac{\cos\sqrt{\lambda}t}{\lambda^2} \, d\left(P_\lambda^{\mathcal{B}} g\right) - \int_\delta^A \frac{\cos\sqrt{\lambda}t}{\lambda^2} \, d\left(P_\lambda^{\mathcal{B}} \Delta f\right) \right\|_{4k} \le \frac{c}{\delta^2} \varepsilon. \tag{6.18}$$

It follows from Lemma 4.4 that

$$\Delta \int_\delta^A \frac{\cos\sqrt{\lambda}t}{\lambda^2} \, d\left(P_\lambda^{\mathcal{B}} g\right)(x) = \int_\delta^A \frac{\cos\sqrt{\lambda}t}{\lambda^2} \, \Delta \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda}(x) \, d\lambda =: \int_\delta^A \frac{\cos\sqrt{\lambda}t}{\sqrt{\lambda}} \, \varphi(x,\lambda) \, d\lambda. \tag{6.19}$$

We prove

$$\left| \int_\delta^A \frac{\cos\sqrt{\lambda}t}{\sqrt{\lambda}} \, \varphi(x,\lambda) \, d\lambda \right| < \varepsilon \qquad \text{for } t > T_\varepsilon, \ x \in M. \tag{6.20}$$

Then (6.15) follows from (6.17) – (6.20) by Sobolev's Lemma.

According to Lemma 4.4 we suppose that

$$|\varphi(x,\lambda_1)-\varphi(x,\lambda_2)|<\tilde{\varepsilon} \qquad \text{for } |\lambda_1-\lambda_2|<\tilde{\delta},\ x\in M.$$

We set $\lambda_n := \left(\frac{\pi}{t}\right)^2\left(n+\frac{1}{2}\right)^2$ and obtain for sufficiently large t and n

$$\int_{\lambda_{2n+1}}^{\lambda_{2n+3}} \frac{\cos\sqrt{\lambda}t}{\sqrt{\lambda}}\,\varphi(x,\lambda)\,d\lambda \;\le$$
$$\le\; \frac{4}{t}\left(\max_{[\lambda_{2n+1},\lambda_{2n+2}]\times M}\{\varphi(x,\lambda)\} - \min_{[\lambda_{2n+2},\lambda_{2n+3}]\times M}\{\varphi(x,\lambda)\}\right) \;\le\; \frac{4\tilde{\varepsilon}}{t}.$$

Taking the sum of these estimates for $\frac{t\sqrt{\delta}}{2\pi}-\frac{3}{4}\le n\le\frac{t\sqrt{A}}{2\pi}-\frac{7}{4},\ n\in\mathbb{N}$, we obtain

$$\int_{\delta}^{A}\frac{\cos\sqrt{\lambda}t}{\sqrt{\lambda}}\,\varphi(x,\lambda)\,d\lambda \le \frac{\sqrt{A}-\sqrt{\delta}}{\pi}\,4\tilde{\varepsilon}+\frac{16}{t}\left(\max_{[\delta,A]\times M}|\varphi(\lambda,x)|\right).$$

The same argument can be applied to estimate the integral from below. This proves (6.20). ∎

From Lemmata 6.1 – 6.3 and the remark following (3.11) we obtain:

Theorem 6.4 *Assume that* $k\in\mathbb{N}$, $k>\frac{n+4}{8}$ *and that* f *satisfies (5.2) and* $\Delta f\in L_{2,s_1(4)}(\Omega)$. *Then (1.1), (1.8) has a unique solution* u *satisfying (2.3). Furthermore, we have* $u\in C^4([0,\infty)\times\overline{\Omega})$ *and*

$$u(t,x)=\frac{t^2}{2}\left(P^{(N)}f\right)(x)+\sum_{j=1}^{3}t^{2-j/2}E_j\,\Delta(F_j\Delta f)(x)+\Delta(F_4\Delta f)(x)+o_M(1)$$
$$\text{as } t\to\infty. \qquad (6.21)$$

F_j are the operators appearing in the expansion (3.10). Furthermore, $F_1=F_2=0$ *if* $n=7$ *and* $F_1=F_2=F_3=0$ *if* $n\ge 9$.

7 Properties and implicit representations of F_j

Assume that $f \in C^\infty(\overline{\Omega})$ with bounded support. Inserting the expansion (3.10) in $(\mathcal{B} - \kappa^4) R^{\mathcal{B}}_{\kappa^4} f = f$ we obtain

$$\Delta^2 F_j f = \begin{cases} f & \text{in } \Omega \text{ for } j = 0, \\ 0 & \text{in } \Omega \text{ for } j = 1, 2, 3, \end{cases} \tag{7.1}$$

$$F_j f = \frac{\partial F_j f}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega \text{ for } j = 0, 1, 2, 3. \tag{7.2}$$

The fundamental solution G_z of Δ^2 in $\mathbb{R}^n$ has the expansion

$$G_{\kappa^4}(\xi) = \sum_{j=0}^{\infty} C_j \frac{\kappa^{4j}}{\xi^{n-4-4j}} + \kappa^{n-4} \sum_{j=0}^{\infty} D_j \kappa^{2j} \xi^{2j} \qquad \text{for } \xi \in \mathbb{R}^n, \tag{7.3}$$

where

$$C_j = \frac{(-1)^{\sigma+1/2}}{\pi^\sigma \, 2^{4j+4} \, (2j+1)! \, \Gamma(2j+2-\sigma)}, \tag{7.4}$$

$$D_j = \frac{\mathrm{i} \, (-1)^j}{\pi^\sigma \, 2^{2\sigma+2j+3} \, j! \, \Gamma(\sigma+j+1)} \left(1 + (-1)^j \, \mathrm{i}^n\right) \tag{7.5}$$

with $\sigma = \frac{n}{2} - 1$ (compare e.g. (2.5) – (2.9) and (2.28) – (2.29) in [5]). Both series in (7.3) converge uniformly with respect to $\kappa \in M$ for every bounded $M \subset \{\kappa \in \mathbb{C} : 0 < \arg \kappa < \frac{\pi}{2}\}$.

In order to give implicit representations of F_j for $j \geq 0$, we set

$$I(F_j f, \varphi) := \int_{\partial\Omega} \left(\frac{\partial \Delta F_j f}{\partial \mathbf{n}}(y) \, \varphi(y) - (\Delta F_j f)(y) \, \frac{\partial \varphi}{\partial \mathbf{n}}(y) \right) ds_y. \tag{7.6}$$

Consider the case $n = 3$. We insert (3.10) and (7.3) into (3.7) and compare the coefficients of $\kappa^{-1}, \kappa^0, \kappa^1, \ldots$. This yields

$$0 = \int_\Omega f(y) \, dy - \int_{\partial\Omega} \frac{\partial \Delta F_0 f}{\partial \mathbf{n}} \, ds, \tag{7.7}$$

$$F_0f(x) = C_0\int_\Omega f(y)|x-y|\,dy - C_0\,I(F_0f,|x-.|) - D_0\int_{\partial\Omega}\frac{\partial\Delta F_1f}{\partial\mathbf{n}}\,ds, \qquad (7.8)$$

$$F_1f(x) = D_1\int_\Omega f(y)|x-y|^2\,dy - C_0\,I(F_1f,|x-.|) - \sum_{j=0}^{1} D_j\,I\Big(F_{2-2j}f,|x-.|^{2j}\Big), \qquad (7.9)$$

$$F_2f(x) = -\,C_0\,I(F_2f,|x-.|) - \sum_{j=0}^{1} D_j\,I\Big(F_{3-2j}f,|x-.|^{2j}\Big), \qquad (7.10)$$

$$F_3f(x) = D_2\int_\Omega f(y)|x-y|^4\,dy - C_0\,I(F_3f,|x-.|) - \sum_{j=0}^{2} D_j\,I\Big(F_{4-2j}f,|x-.|^{2j}\Big). \qquad (7.11)$$

Note that

$$C_0 = -\frac{1}{8\pi},\quad D_0 = \frac{1+\mathrm{i}}{8\pi},\quad D_1 = \frac{1-\mathrm{i}}{48\pi},\quad D_3 = \frac{1+\mathrm{i}}{960\pi} \qquad \text{(in the case } n=3). \qquad (7.12)$$

We need some special properties of F_0f. Using

$$|x-y| = |x|\left(1+\frac{|y|^2-2xy}{|x|^2}\right)^{1/2} = |x|\left(1-\frac{xy}{|x|^2}\right) + O\left(\frac{1}{|x|}\right) \qquad \text{as } |x|\to\infty, \qquad (7.13)$$

we obtain from (7.7) and (7.8)

$$D^\alpha(F_0f)(x) = D^\alpha\left(c_1 + \sum_{j=1}^{3} c_2^{(j)}\frac{x_j}{|x|}\right) + O\left(\frac{1}{|x|^{1+|\alpha|}}\right) \qquad \text{as } |x|\to\infty,\ |\alpha|\ge 0 \qquad (7.14)$$

with

$$c_1 = -\frac{1+\mathrm{i}}{8\pi}\int_{\partial\Omega}\frac{\partial\Delta F_1f}{\partial\mathbf{n}}\,ds, \qquad (7.15)$$

$$c_2^{(j)} = \frac{1}{8\pi}\left(\int_\Omega f(y)y_j\,dy - \int_{\partial\Omega}\left(\frac{\partial\Delta F_0f}{\partial\mathbf{n}}(y)\,y_j - (\Delta F_0f)(y)\frac{\partial y_j}{\partial\mathbf{n}_y}\right)ds_y\right). \qquad (7.16)$$

In the cases $n=5,7,\ldots$ representations similar to (7.8) – (7.11) hold. We note that, for $f\in C^\infty(\overline{\Omega})$ with bounded support, F_0f belongs to $C^\infty(\overline{\Omega})$ and F_0f is the unique solution of

$$\left.\begin{aligned} \Delta^2 F_0 f &= f && \text{in } \Omega, \\ F_0 f = \frac{\partial F_0 f}{\partial \mathbf{n}} &= 0 && \text{on } \partial\Omega, \\ D^\alpha(F_0 f)(x) &= O\left(\frac{1}{|x|^{L+|\alpha|}}\right) && \text{as } |x| \to \infty,\ 0 \le |\alpha| \le 3 \end{aligned}\right\} \tag{7.17}$$

with

$$L := \begin{cases} 0 & \text{if } n = 3, \\ n-4 & \text{if } n = 5, 7, \ldots. \end{cases} \tag{7.18}$$

(compare Theorem 3.6 in [2]).

8 Special solutions of $\Delta^2 w = 0$

We study solutions of $\Delta^2 w = 0$, which play an essential role by the description of the operators F_j appearing in (3.10). Throughout this section we denote by R_0 a positive real number such that

$$\partial\Omega \ \subset \ \{x \in \mathbb{R}^n : |x| < R_0\}. \tag{8.1}$$

Furthermore, we set

$$\Omega_R \ := \ \{x \in \Omega : |x| < R\}. \tag{8.2}$$

By a (real) polynomial p we denote a function of the form

$$p(x) = \sum_{|\alpha| \le k} c_\alpha x^\alpha$$

with $c_\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ $(x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n})$. We prove:

Lemma 8.1 *Assume that p is a polynomial with $\Delta^2 p = 0$. Furthermore, in the case $n = 3$, $p(x) = |x|$ is admitted. Then there exists a unique solution $W_p \in$*

$C^\infty(\overline{\Omega})$ of

$$\left.\begin{array}{rclll} & & \Delta^2 W_p & = 0 & \text{in } \Omega, \\ W_p & = & \dfrac{\partial W_p}{\partial \mathbf{n}} & = 0 & \text{on } \partial\Omega, \\ D^\alpha(W_p(x) - p(x)) & & & = O\left(\dfrac{1}{|x|^{L+|\alpha|}}\right) & \text{as } |x| \to \infty,\ 0 \le |\alpha| \le 3, \end{array}\right\} \qquad (8.3)$$

where L is defined by (7.18).

Remark: If $n = 3$ and $p(x) = \text{const.}$, then $W_p = 0$. But in all other cases W_p does not vanish.

Proof: Uniqueness is well known (compare Lemmata 3.9 and 3.10 in [2]). In order to prove existence let $\psi \in C^\infty(\overline{\Omega})$ such that $\psi(x) = 1$ for $|x| \le R_0$, $\psi(x) = 0$ for $|x| \ge R_0 + 1$. If p is a polynomial, then the assertion follows from (7.17) by setting

$$W_p := F_0(\Delta^2(\psi p)) + (1 - \psi)p. \qquad (8.4)$$

If $n = 3$ and $p(x) = |x|$, we choose $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and set $\tilde{p}(x) := |x - x_0|$. Then $W_{|x|}$ is given by (8.4) with p being replaced by $\tilde{p}$. ■

The following two lemmata give essential properties of the functions W_p.

Lemma 8.2 *Assume that p is a polynomial with $\Delta^2 p = 0$, and that $p \neq \text{const}$ if $n = 3$. If W_p is the associated solution of (8.3) and $f \in C_0^\infty(\Omega)$, then*

$$\langle f, W_p \rangle = \int_\Omega f\, p\, dx - \int_{\partial\Omega} \left(\frac{\partial \Delta F_0 f}{\partial \mathbf{n}}\, p - (\Delta F_0 f)\, \frac{\partial p}{\partial \mathbf{n}} \right) ds. \qquad (8.5)$$

Lemma 8.3 *Suppose that $n = 3$ and $W_{|x|}$ is the solution of (8.3) with $p(x) = |x|$. Then we have for $f \in C_0^\infty(\Omega)$*

$$\left\langle f, W_{|x|} \right\rangle = -(1 + \mathrm{i}) \int_{\partial\Omega} \frac{\partial \Delta F_1 f}{\partial \mathbf{n}}\, ds. \qquad (8.6)$$

Proof of Lemma 8.2: Let $R > R_0$ such that $f(x) = 0$ for $|x| > R$. From (7.1) and Greens formula we obtain

$$\begin{aligned}\int_{\Omega_R} f(W_p - p)\,dx &= \int_{\Omega_R} (\Delta^2 F_0 f)(W_p - p)\,dx \\ &= \int_{\Omega_R} (F_0 f)\Delta^2 (W_p - p)\,dx - \int_{\partial\Omega}\left(\frac{\partial \Delta F_0 f}{\partial \mathbf{n}}\,p - (\Delta F_0 f)\,\frac{\partial p}{\partial \mathbf{n}}\right) ds + O\Big(\frac{1}{R}\Big)\end{aligned}$$

as $R \to \infty$. This proves (8.5). ■

Proof of Lemma 8.3: By (7.1) and (8.3), we have for sufficiently large $R > R_0$

$$\begin{aligned}\int_{\Omega_R} f\,W_{|x|}\,dx &= \int_{\Omega_R} (\Delta^2 F_0 f)\,W_{|x|}\,dx \\ &= \sum_{j=0}^{1}\int_{|x|=R}\left(\frac{\partial \Delta^{1-j} F_0 f}{\partial \mathbf{n}}\,\Delta^j W_{|x|} - (\Delta^{1-j} F_0 f)\,\frac{\partial \Delta^j W_{|x|}}{\partial \mathbf{n}}\right) ds.\end{aligned}$$

We insert (7.14) with (7.15), $D^\alpha W_{|x|} = D^\alpha |x| + O\big(1/|x|^{|\alpha|}\big)$ as $|x| \to \infty$ and use that

$$\int_{|x|=R}\left(\frac{\partial}{\partial \mathbf{n}}\Delta^j \frac{x_k}{|x|}\right)\Delta^{1-j}|x|\,ds \;=\; \int_{|x|=R}\left(\Delta^j \frac{x_k}{|x|}\right)\frac{\partial \Delta^{1-j}|x|}{\partial \mathbf{n}}\,ds \;=\; 0$$

$(j = 0, 1$ and $k = 1, 2, 3)$ by symmetry. We conclude that

$$\int_{\Omega_R} f\,W_{|x|}\,dx \;=\; \frac{\mathrm{i}+1}{8\pi}\int_{\partial\Omega}\frac{\partial \Delta F_1 f}{\partial \mathbf{n}}\,ds \int_{|x|=R}\frac{\partial \Delta |x|}{\partial \mathbf{n}}\,ds + O\Big(\frac{1}{R}\Big)$$

as $R \to \infty$. This proves (8.6). ■

Remark: We can characterize the coefficients $c_1, c_2^{(j)}$ in the expansion (7.14) of $F_0 f$. In particular, we obtain from (7.15) – (7.16) with (8.6) and (8.5) that

$$c_1 \;=\; \frac{1}{8\pi}\big\langle f, W_{|x|}\big\rangle, \qquad c_2^{(j)} \;=\; \frac{1}{8\pi}\big\langle f, W_{x_j}\big\rangle. \tag{8.7}$$

Assume that p, q are polynomials satisfying $\Delta^2 p = \Delta^2 q = 0$ or $p = |.|$ or $q = |.|$. We define the constants $C_{p,q}$ by

$$C_{q,p} \;:=\; -\frac{1}{8\pi}\int_{\partial\Omega}\left(\frac{\partial \Delta W_q}{\partial \mathbf{n}}\,p - \Delta W_q\,\frac{\partial p}{\partial \mathbf{n}}\right) ds \qquad \text{for } p \neq |.|, \tag{8.8}$$

$$C_{|x|,|x|} := C_{|x|,|x-x_0|} \qquad \text{for arbitrary } x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}, \tag{8.9}$$

$$C_{q,|x|} := C_{|x|,q}. \tag{8.10}$$

Lemma 8.4 *Suppose that $n = 3$ and that p is a polynomial satisfying $\Delta^2 p = 0$ or $p = |.|$. Then*

$$W_p(x) = p(x) + C_{|x|,p} + \sum_{j=1}^{3} C_{x_j,p} \frac{x_j}{|x|} + O\left(\frac{1}{|x|}\right) \qquad \text{as } |x| \to \infty. \tag{8.11}$$

Proof: If $p \neq |.|$, we apply (7.14) and (8.7) to (8.4). Greens formula yields (8.11). In the case $p = |.|$ the coefficient $C_{|x|,|x|}$ can be computed analogous. Furthermore, we obtain from (7.14), (8.4), (8.7) and an expansion of W_{x_j} around x_0 for the coefficient of $\frac{x_j}{|x|}$

$$\begin{aligned}
&\frac{1}{8\pi} \int_\Omega \Delta^2(\psi(x)|x-x_0|)\, W_{x_j}\, dx - (x_0)_j \; = \\
&= \frac{1}{8\pi} \sum_{j=0}^{1} \int_{\partial\Omega} \left(\frac{\partial \Delta_x^{1-j}|x-x_0|}{\partial \mathbf{n}_x} \Delta^j W_{x_j} - \Delta_x^{1-j}|x-x_0| \frac{\partial \Delta^j W_{x_j}}{\partial \mathbf{n}} \right) ds_x - (x_0)_j \\
&= -\frac{1}{8\pi} \int_{|x-x_0|=R} \frac{\partial \Delta_x |x-x_0|}{\partial \mathbf{n}_x} \left(C_{|x|,x_j} + (x_0)_j \right) ds_x - (x_0)_j + O\left(\frac{1}{R}\right) \\
&= C_{|x|,x_j} + O\left(\frac{1}{R}\right) \qquad \text{as } R \to \infty.
\end{aligned}$$

■

Remarks: **1)** Note that $C_{x_j,x_j} \neq 0$ $(j = 1,2,3)$, since

$$\begin{aligned}
C_{x_j,x_j} &= -\frac{1}{8\pi} \int_{\partial\Omega} \left(\frac{\partial \Delta(W_{x_j} - x_j)}{\partial \mathbf{n}} (x_j - W_{x_j}) - \Delta(W_{x_j} - x_j) \frac{\partial x_j - W_{x_j}}{\partial \mathbf{n}} \right) ds \\
&= -\frac{1}{8\pi} \int_\Omega |\Delta W_{x_j}|^2 dx.
\end{aligned}$$

In the same way we obtain that $C_{|x|,|x|} \neq 0$.

2) It would be possible to give an analogous expansion of W_p in the cases $n = 5, 7, \ldots$. But this is not necessary for the proof of the representations of

F_1, F_2, F_3. We only note that $C_{1,1} \neq 0$, since

$$C_{1,1} \;=\; C_0 \int_{\partial\Omega} \left(\frac{\partial \Delta W_1}{\partial \mathbf{n}}\, 1 - \Delta W_1 \,\frac{\partial 1}{\partial \mathbf{n}} \right) ds \;=\; C_0 \int_{\Omega} |\Delta W_1|^2 \, dx. \tag{8.12}$$

9 Representation of the operators F_j and $P^{(j)}$

Theorem 9.1 *Let $n = 3$. For $f \in C_0^\infty(\Omega)$, we have*

$$F_1 f \;=\; \frac{\mathrm{i}-1}{48\pi} \left(2 \sum_{j=1}^{3} \left\langle f, W_{x_j} \right\rangle W_{x_j} + 3 \left\langle f, W_{|x|} \right\rangle W_{|x|} \right), \tag{9.1}$$

$$\begin{aligned} F_2 f \;=\; & -\frac{\mathrm{i}}{144\pi} \left(3 \left\langle f, W_{|x|} \right\rangle W_{|x|^2} + 2 \sum_{j=1}^{3} \left\langle f, W_{x_j}^{(1)} \right\rangle W_{x_j} \right. \\ & \left. + 3 \left\langle f, W_{|x|^2} + W_{|x|}^{(1)} \right\rangle W_{|x|} \right), \end{aligned} \tag{9.2}$$

$$\begin{aligned} F_3 f \;=\; & -\frac{1+\mathrm{i}}{960\pi} \left(4 \sum_{j=1}^{3} \left\langle f, W_{x_j} \right\rangle W_{x_j |x|^2} - 4 \sum_{j,k=1}^{3} \left\langle f, W_{x_j x_k} \right\rangle W_{x_j x_k} \right. \\ & - \frac{1}{3} \left\langle f, 16 W_{|x|^2} + 10 W_{|x|}^{(1)} \right\rangle W_{|x|^2} + \frac{1}{9} \sum_{j=1}^{3} \left\langle f, 20 W_{x_j}^{(2)} - 36 W_{x_j |x|^2} \right\rangle W_{x_j} \\ & \left. - \frac{10}{3} \left\langle f, W_{|x|}^{(2)} + W_{|x|^2}^{(1)} \right\rangle W_{|x|} \right), \end{aligned} \tag{9.3}$$

where W_p is given by Lemma 8.1 and

$$W_p^{(1)} \;=\; 2 \sum_{j=1}^{3} C_{x_j,p} W_{x_j} + 3\, C_{|x|,p} W_{|x|} \tag{9.4}$$

$$W_p^{(2)} \;=\; 3 C_{|x|,p} \left(W_{|x|^2} + W_{|x|}^{(1)} \right) + 2 \sum_{j=1}^{3} C_{x_j,p} W_{x_j}^{(1)} + 3 C_{|x|^2,p} W_{|x|} \tag{9.5}$$

Note that the operators F_j are bounded operators from $L_{2,s_1(j)}(\Omega)$ into $H^4_{s_2(j)}(\Omega)$ according to [3], if $s_1(j), s_2(j)$ satisfy (3.12). Hence it follows by approximation that (9.1) – (9.3) hold for $F_j f$ with $f \in L_{2,s_1(j)}$. Moreover, (9.1) – (9.3) imply:

Corollary 9.2 *If $n = 3$, then F_j ($j = 1, 2, 3$) can be extended by continuity to a bounded operator from $L_{2,s_1(j)}(\Omega)$ into $H^4_{-s_2(j)}(\Omega)$ for every*

$$s_1(j), s_2(j) > 3 + 2j; \tag{9.6}$$

$F_j f$ is given for $f \in L_{2,s_1(j)}(\Omega)$ by (9.1) – (9.3).

We apply the representations of F_1, F_2, F_3 to the time asymptotics in Theorem 6.4. We have for $f \in H^2_s(\Omega)$ with $s > 9$

$$\langle \Delta f, W_p \rangle \;=\; \langle f, \Delta W_p \rangle$$

for all functions W_p appearing on the right-hand sides of (9.1) – (9.5). We set

$$w_p \;:=\; \Delta W_p. \tag{9.7}$$

Then obviously $\Delta w_p = 0$ in Ω. We obtain from Theorem 6.4:

Corollary 9.3 *Suppose that $n = 3$, that all assumptions of Theorem 6.4 are satisfied and that in addition $f \in H^2_s(\Omega)$ for some $s > 9$. Then the time asymptotics of the solution u of (1.1), (1.8) are given by (1.12), where*

$$P^{(1)} f \;=\; \frac{\mathrm{i}-1}{48\pi}\, E_1 \left(2 \sum_{j=1}^{3} \left\langle f, w_{x_j} \right\rangle w_{x_j} + 3 \left\langle f, w_{|x|} \right\rangle w_{|x|} \right). \tag{9.8}$$

Analogous representations of $\frac{1}{E_j} P^{(j)} f$ $(j = 2, 3)$ are obtained from the representations (9.2) – (9.5) of $F_j f$ by replacing W_p by w_p. The constants E_j are given by (6.6).

Remarks: **1)** We note that by (8.3) $w_{x_j} \in L_2(\Omega)$ for $j = 1, 2, 3$, $w_{|x|} \notin L_2(\Omega)$.

2) If $n = 5, 7, \ldots$, then representations of $P^{(j)}$ follow in a similar way from Theorem 9.5 below.

In order to prove Theorem 9.1, we use the following lemma, which is an immediate consequence of Lemma 8.1 and of Lemma 3.10 in [2].

Lemma 9.4 *Assume that p_j $(j = 1, \ldots, k)$ are polynomials with $\Delta^2 p_j = 0$, p_j not being constant. If $v \in C^4(\overline{\Omega})$ is a solution of*

$$\begin{aligned} \Delta^2 v &= 0 \quad \text{in } \Omega, \\ v = \frac{\partial v}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with

$$v(x) = \sum_{j=1}^{k} c_j\, p_j(x) + c_{k+1} + \frac{1}{8\pi} I(v, |x - .|)$$

$(c_1, \ldots, c_{k+1} \in \mathbb{C})$, then

$$v = \sum_{j=1}^{k} c_j\, W_{p_j} + \frac{1}{8\pi} \int_{\partial\Omega} \frac{\partial \Delta v}{\partial \mathbf{n}}\, ds\, W_{|x|},$$

where W_{p_j} and $W_{|x|}$ are given by Lemma 8.1.

Proof of Theorem 9.1: Using (7.7) and (8.5), we rewrite (7.9) as

$$\begin{aligned} F_1 f(x) &= \frac{1-\mathrm{i}}{48\pi} \left(\sum_{j=1}^{3} \left\langle f, W_{x_j} \right\rangle (-2x_j) + \left\langle f, W_{|x|^2} \right\rangle \right) - \frac{1+\mathrm{i}}{8\pi} \int_{\partial\Omega} \frac{\partial \Delta F_2 f}{\partial \mathbf{n}}\, ds \\ &\quad + \frac{1}{8\pi} I(F_1 f, |x - .|). \end{aligned} \tag{9.9}$$

By (7.1) – (7.2), Lemma 9.4 and (8.6), we conclude that (9.1) holds.

In the next step we prove that

$$\int_{\partial\Omega} \frac{\partial \Delta F_2 f}{\partial \mathbf{n}}\, ds = -\frac{\mathrm{i}}{6} \left\langle f, W_{|x|^2} + W^{(1)}_{|x|} \right\rangle \tag{9.10}$$

with $W^{(1)}_{|x|}$ being given by (9.4). Using (9.1) and (8.11), we obtain

$$\begin{aligned} F_1 f(x) - \frac{\mathrm{i}-1}{48\pi} \left(2 \sum_{j=1}^{3} \left\langle f, W_{x_j} \right\rangle x_j + 3 \left\langle f, W_{|x|} \right\rangle |x| \right) &= \\ = \frac{\mathrm{i}-1}{48\pi} \Bigg(2 \sum_{j=1}^{3} \left\langle f, W_{x_j} \right\rangle \left(C_{|x|,x_j} + \sum_{k=1}^{3} C_{x_k,x_j} \frac{x_k}{|x|} \right) & \\ + 3 \left\langle f, W_{|x|} \right\rangle \left(C_{|x|,|x|} + \sum_{k=1}^{3} C_{x_k,|x|} \frac{x_k}{|x|} \right) \Bigg) + O\left(\frac{1}{|x|}\right) & \end{aligned}$$

as $|x| \to \infty$. On the other hand we conclude from (9.9) and (7.13), (8.6) that

$$\begin{aligned} F_1 f(x) - \frac{\mathrm{i}-1}{48\pi}\left(2\sum_{j=1}^{3}\left\langle f, W_{x_j}\right\rangle x_j + 3\left\langle f, W_{|x|}\right\rangle |x|\right) &= \\ = \frac{1-\mathrm{i}}{48\pi}\left\langle f, W_{|x|^2}\right\rangle - \frac{1+\mathrm{i}}{8\pi}\int_{\partial\Omega}\frac{\partial \Delta F_2 f}{\partial \mathbf{n}}\, ds & \\ - \frac{1}{8\pi}\sum_{j=1}^{3}\frac{x_j}{|x|}\int_{\partial\Omega}\left(\frac{\partial \Delta F_1 f}{\partial \mathbf{n}}(y)\, y_j - (\Delta F_1 f)(y)\,\frac{\partial y_j}{\partial \mathbf{n}_y}\right) ds_y + O\left(\frac{1}{|x|}\right) & \end{aligned}$$

as $|x| \to \infty$. Comparing the terms being independent of x, we conclude that (9.10) holds.

We conclude from (9.1), (9.4) and (8.8) that for every polynomial p with $\Delta^2 p = 0$

$$\int_{\partial\Omega}\left(\frac{\partial \Delta F_1 f}{\partial \mathbf{n}}\, p - (\Delta F_1 f)\,\frac{\partial p}{\partial \mathbf{n}}\right) ds \;=\; \frac{1-\mathrm{i}}{6}\left\langle f, W_p^{(1)}\right\rangle. \tag{9.11}$$

Now we prove (9.2). Inserting (9.11) and (8.6) in (7.10), we obtain

$$\begin{aligned} F_2 f(x) \;=\; & -\frac{\mathrm{i}}{48\pi}\left\langle f, W_{|x|}\right\rangle |x|^2 - \frac{\mathrm{i}}{72\pi}\sum_{j=1}^{3}\left\langle f, W_{x_j}^{(1)}\right\rangle x_j + \frac{\mathrm{i}}{144\pi}\left\langle f, W_{|x|^2}^{(1)}\right\rangle \\ & - \frac{1+\mathrm{i}}{8\pi}\int_{\partial\Omega}\frac{\partial \Delta F_3 f}{\partial \mathbf{n}}\, ds + \frac{1}{8\pi} I(F_2 f, |x - .|). \end{aligned} \tag{9.12}$$

It follows from Lemma 9.4 and (9.10) that (9.2) holds.

Now we study $F_3 f$. By the argument used in the proof of (9.10) we obtain, by comparing the spatial expansions of (9.2) and (9.12), that

$$\int_{\partial\Omega}\frac{\partial \Delta F_3 f}{\partial \mathbf{n}}\, ds \;=\; \frac{1+\mathrm{i}}{36}\left\langle f, W_{|x|}^{(2)} + W_{|x|^2}^{(1)}\right\rangle. \tag{9.13}$$

It follows from (9.2), (8.8) and (9.5) that

$$\int_{\partial\Omega}\left(\frac{\partial \Delta F_2 f}{\partial \mathbf{n}}\, p - (\Delta F_2 f)\frac{\partial p}{\partial \mathbf{n}}\right) ds \;=\; \frac{\mathrm{i}}{18}\left\langle f, W_p^{(2)}\right\rangle. \tag{9.14}$$

Inserting (7.7), (8.5), (9.10) and (9.14) into (7.11), we obtain

$$F_3 f(x) \;=\; \frac{1+\mathrm{i}}{960\pi}\left(-4\sum_{j=1}^{3}\left\langle f, W_{x_j}\right\rangle x_j |x|^2 + 4\sum_{j,k=1}^{3}\left\langle f, W_{x_j,x_k}\right\rangle x_j x_k\right.$$

$$+2\left\langle f, W_{|x|^2}\right\rangle |x|^2 - 4\sum_{j=1}^{3}\left\langle f, W_{x_j|x|^2}\right\rangle x_j\Bigg)$$

$$+\frac{1-\mathrm{i}}{48\pi}\left(\frac{\mathrm{i}}{6}\left\langle f, W_{|x|^2} + W^{(1)}_{|x|}\right\rangle |x|^2 + \frac{\mathrm{i}}{9}\sum_{j=1}^{3}\left\langle f, W^{(3)}_{x_j}\right\rangle x_j\right)$$

$$+c + I(F_3 f, |x - .|)$$

with suitable $c \in \mathbb{C}$. From this we conclude by Lemma 9.4 and (9.13) that (9.3) holds. ■

If $n = 5, 7, 9, \ldots$, we have $F_j = 0$ for $\frac{j}{4} \notin \mathbb{N}$, $j \le n-5$. The same method as above can be used to compute $F_{n-4}, F_{n-3}, F_{n-2}$ (for $n = 5, 9, 13, \ldots$) or $F_{n-4}, F_{n-2}, F_{n-1}$ (for $n = 7, 11, 15, \ldots$), respectively.

Theorem 9.5 *If $n = 5, 7, 9, \ldots$ and $f \in C_0^\infty(\Omega)$, then*

$$F_{n-4}f = D_0 \langle f, W_1\rangle W_1, \tag{9.15}$$

$$F_{n-2}f = D_1\left(\langle f, W_1\rangle W_{|x|^2} - 2\sum_{j=1}^{n}\left\langle f, W_{x_j}\right\rangle W_{x_j} + \left\langle f, W^{(3)}_{|x|^2}\right\rangle W_1\right) \tag{9.16}$$

with

$$W^{(3)}_{|x|^2} = \begin{cases} W_{|x|^2} + \dfrac{D_0^3\, C_{1,1}^2}{C_0^2} W_1 & \text{if } n = 5, \\ W_{|x|^2} & \text{if } n = 7, 9, \ldots. \end{cases} \tag{9.17}$$

Furthermore,

$$F_{n-3}f = \begin{cases} -\dfrac{D_0^2\, C_{1,1}}{C_0}\langle f, W_1\rangle W_1 & \text{if } n = 5, \\ 0 & \text{if } n = 9, 13, \ldots \end{cases} \tag{9.18}$$

and

$$F_{n-1}f = \begin{cases} -\dfrac{D_0^2\, C_{1,1}}{C_0}\langle f, W_1\rangle W_1 & \text{if } n = 7, \\ 0 & \text{if } n = 11, 15, \ldots. \end{cases} \tag{9.19}$$

Remarks: **1)** If $n = 7, 11, 15, \ldots$, then $\Delta^2 F_{n-3} f = F_{n-7} f \neq 0$ for $f \neq 0$, since $\Delta^{(n-3)/2} F_{n-7} f = f$.

2) Note that $C_{1,1} \neq 0$ by (8.12).

3) We mention without proof, that it is possible to represent F_j with $\frac{j}{4} \notin \mathbb{N}$ for arbitrary j in a similar way by solutions W_p of equations of the form $\Delta^{2k} W_p = 0$.

10 The Dirichlet problem

Consider the mixed problem (1.1), (1.5). The unique solution satisfying

$$u \in \bigcap_{j=0}^{2} C^j \left([0,\infty), D(\mathcal{B}^{1-j/2})\right) \tag{10.1}$$

is given by

$$u(t) = \int_0^\infty \frac{1}{\lambda}(1 - \cos\sqrt{\lambda}t)\, d\left(P_\lambda^{\mathcal{B}} f\right). \tag{10.2}$$

We suppose in the following that $g \in C_0^\infty(\Omega)$ and set

$$f := \int_0^\infty \frac{1}{\lambda^s}\, d\left(P_\lambda^{\mathcal{B}} g\right) \tag{10.3}$$

with $0 \leq s < \frac{5}{8}$. By (4.8), we have for $0 < \delta < \varepsilon$

$$\begin{aligned} \left\| \int_\delta^\varepsilon \frac{1}{\lambda^s}\, d\left(P_\lambda^{\mathcal{B}} g\right) \right\|^2 &= \int_\delta^\varepsilon \frac{1}{\lambda^{2s}}\, d\left(\left\|P_\lambda^{\mathcal{B}} g\right\|^2\right) = \left\langle \int_\delta^\varepsilon \frac{1}{\lambda^{2s}} \frac{dP_\lambda^{\mathcal{B}} g}{d\lambda}\, d\lambda, g \right\rangle \\ &\leq c_g \int_\delta^\varepsilon \frac{1}{\lambda^{2s-1/4}}\, d\lambda\, \|g\|^2, \end{aligned}$$

since g has bounded support. Hence $f \in L_2(\Omega)$. By (10.3), we can rewrite (10.2) as

$$u(t) = \int_0^\infty \frac{1}{\lambda^{1+s}}(1 - \cos\sqrt{\lambda}t)\, d\left(P_\lambda^{\mathcal{B}} g\right). \tag{10.4}$$

It follows analogously to Theorem 6.4 (compare also (2.16) and (2.21) in [5]):

Theorem 10.1 *Suppose that* $g \in C_0^\infty(\Omega)$. *If* $0 \le s < \frac{5}{8}$ *and* $f \in L_2(\Omega)$ *is defined by (10.3), then (1.1), (1.5) has a classical solution* $u \in C^\infty([0,\infty) \times \overline{\Omega})$. *Furthermore, the following estimates hold with suitable non-vanishing constants* $E_1(s), E_2(s) \in \mathbb{C}$:

1) *If* $\frac{1}{2} < s < \frac{5}{8}$, *then*

$$u(t,x) = t^{2s-1/2} E_1(s)\, F_1 g(x) + t^{2s-1} E_2(s)\, F_2 g(x) + u_0^{(s)}(x) + o_M(1) \qquad \text{as } t \to \infty. \qquad (10.5)$$

2) *If* $s = \frac{1}{2}$, *then*

$$u(t,x) = t^{1/2} E_1(\frac{1}{2})\, F_1 g(x) - (\ln t)\, \frac{2\mathrm{i}}{\pi}\, F_2 g(x) + u_0^{(1/2)}(x) + o_M(1) \qquad \text{as } t \to \infty. \qquad (10.6)$$

3) *If* $\frac{1}{4} < s < \frac{1}{2}$, *then*

$$u(t,x) = t^{2s-1/2} E_1(s)\, F_1 g(x) + u_0^{(s)}(x) + o_M(1) \qquad \text{as } t \to \infty. \qquad (10.7)$$

4) *If* $s = \frac{1}{4}$, *then*

$$u(t,x) = -(\ln t)\, \frac{1+\mathrm{i}}{\pi}\, F_1 g(x) + u_0^{(1/4)}(x) + o_M(1) \qquad \text{as } t \to \infty. \qquad (10.8)$$

5) *If* $0 \le s < \frac{1}{4}$, *then* u *satisfies the principle of limiting amplitude (compare (1.3)).*

Remark: $F_1 g$ does not vanish in general if $n = 3$ or $n = 5$. This follows from the representation of F_1 given in (9.1) or (9.15), respectively.

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