Time Asymptotics for the Polyharmonic Wave Equation in Waveguides

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Abstract

Let Ω denote an unbounded domain in $\mathbb{R}^n$ having the form $\Omega = \mathbb{R}^l \times D$ with bounded cross-section $D \subset \mathbb{R}^{n-l}$, and let $m \in \mathbb{N}$ be fixed. This article considers solutions u to the scalar wave equation $\partial_t^2 u(t,x) + (-\Delta)^m u(t,x) = f(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$ satisfying homogeneous Dirichlet boundary condition. The asymptotic behaviour of u as $t \to \infty$ is investigated. Depending on the choice of f, ω and Ω, two cases occur: Either u shows resonance, which means that $|u(t,x)| \to \infty$ as $t \to \infty$ for almost every $x \in \Omega$, or u satisfies the principle of limiting amplitude. Furthermore the spatial operators resolvent and the validity of the principle of limiting absorption is studied.

1 Introduction

Let Ω denote a domain in $\mathbb{R}^n$ $(n \geq 2)$ having the form

$$\Omega = \mathbb{R}^l \times D,\ 1 \leq l \leq n-1,\ D \subset \mathbb{R}^{n-l} \text{ bounded},\ \partial D \in C^\infty. \tag{1.1}$$

If $n = 3,\ l = 1$ and D is connected, then Ω is an infinite tube with cross-section D. In the case $l = n-1$, $D = (a,b)$ with $-\infty < a < b < \infty$, Ω is the domain between two parallel hyperplanes.

Consider the Dirichlet initial-boundary-value problem

$$\left.\begin{array}{rcll} \partial_t^2 u(t,x) + (-\Delta)^m u(t,x) & = & f(x)\,\mathrm{e}^{-\mathrm{i}\omega t} & \text{for } t \geq 0,\ x \in \Omega, \\ \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(t,x) & = & 0 & \text{for } t \geq 0,\ x \in \partial\Omega,\ \alpha \in \mathbb{N}_0^n \text{ with} \\ & & & |\alpha| = \alpha_1 + \ldots + \alpha_n \leq m-1, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) & = & u_1(x) & \text{for } x \in \Omega, \end{array}\right\} \tag{1.2}$$

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where $\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$, $m \in \mathbb{N}$, $\omega \geq 0$. This equation is used in different physical problems. In the case $m = 1$, $n = 2$, the solution $u(t, x)$ describes the perpendicular deviation of a membran stimulated by a periodic perpendicular force, where the membran is fixed at the boundary. In the theory of linear acoustic waves, equation (1.2) with $m = 1$, $n = 3$ is used. The case $m = 2$, $n = 2$ appears in the study of the perpendicular displacement of a thin plate which is clamped at the boundary.

We are interested in the asymptotic behaviour of the solution $u(t, x)$ as $t \to \infty$. It turns out, that the occurence of resonances depends on the number l of unbounded spatial directions and not on the spatial operator's order $2m$: In the cases $l = 1$ or $l = 2$, resonances occur at countable many frequencies of incitation. The amplitude of the resonant solution grows like respectively $\sqrt{t}$ (if $l = 1$) or $\ln t$ (if $l = 2$). These resonances are not connected with eigenvalues of the spatial operator in (1.2). In order to be more precise, suppose that $f, u_0, u_1 \in C_0^\infty(\Omega) := \{\varphi \in C^\infty(\Omega) : \operatorname{supp}\varphi \text{ bounded}, \overline{\operatorname{supp}\varphi} \subset \Omega\}$. Then (1.2) has a solution $u \in C^\infty\big([0,\infty) \times \overline{\Omega}\big)$, which is uniquely determined by an integrability condition (see (5.1) and the remarks there). Set $x = (x', x'')$ with $x' := (x_1, \ldots, x_l) \in \mathbb{R}^l$, $x'' := (x_{l+1}, \ldots, x_n) \in D$ and let $0 < \mu_1 < \mu_2 < \ldots$ denote the eigenvalues of the boundary value problem in cross-direction

$$\left.\begin{aligned} \left(-\partial_{x_{l+1}}^2 - \ldots - \partial_{x_n}^2\right)^m v_{jk}(x'') &= \mu_j\, v_{jk}(x'') && \text{for } x'' \in D, \\ \partial_{x_{l+1}}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_{n-l}} v_{jk}(x'') &= 0 && \text{for } x'' \in \partial D,\ \alpha \in \mathbb{N}_0^{n-l} \\ & && \text{with } |\alpha| \leq m - 1, \end{aligned}\right\} \tag{1.3}$$

where $\{v_{jk}\}$ ($j \in \mathbb{N}$, $1 \leq k \leq K(j) :=$multiplicity of μ_j) is supposed to be a suitable chosen orthonormal system of eigenfunctions being complete in $L_2(D)$. We define the resonance frequencies by $\omega_j := \sqrt{\mu_j}$ ($j \in \mathbb{N}$).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be given by (1.1) and let $f, u_0, u_1 \in C_0^\infty(\Omega)$. Denote by u the unique solution u of (1.2) and (5.1).*

1) *Suppose that $l \geq 3$ or that $\omega \in [0, \infty) \setminus \{\omega_1, \omega_2, \ldots\}$. Then u satisfies the* principle of limiting amplitude*:*

$$u(t, x) = \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega(x) + o_M(1) \qquad \text{as } t \to \infty, \tag{1.4}$$

where the notation $o_M(1)$ means that the remainder turns to 0 uniformly with respect to $x \in M$, where M is an arbitrary bounded subset of $\overline{\Omega}$. Furthermore u_ω is solution of

$$\left.\begin{aligned} ((-\Delta)^m - \omega^2)\, u_\omega &= f && \text{in } \Omega, \\ \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u_\omega &= 0 && \text{on } \partial\Omega \text{ for } |\alpha| \leq m - 1. \end{aligned}\right\} \tag{1.5}$$

2) *If $l = 1$ and $\omega = \omega_j$ ($j \in \mathbb{N}$ fixed), then u shows* resonance *of order $t^{1/2}$:*

$$\begin{aligned} u(t, x', x'') = {} & \frac{1 + \mathrm{i}}{2\sqrt{\pi\omega m}} \sqrt{t}\, \mathrm{e}^{-\mathrm{i}\omega t} \sum_{k=1}^{K(j)} \frac{v_{jk}(x'')}{\|\nabla''^{m-1} v_{jk}\|_D} \int_\Omega f(y', y'') v_{jk}(y'')\, d(y', y'') \\ & + \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega(x', x'') + o_M(1) \qquad \text{as } t \to \infty, \end{aligned} \tag{1.6}$$

u_ω being a solution of (1.5); here the eigenfunctions $\{v_{jk}\}$ of (1.3) have to be chosen according to Definition 3.3 and $\nabla''^k := \Delta''^{k/2}$ for even $k \in \mathbb{N}$ and $\nabla''^k := \nabla''\Delta''^{(k-1)/2}$ for odd $k \in \mathbb{N}$. Note that the unbounded term depends only on $x'' \in D$ and is constant with respect to $x' \in \mathbb{R}^l$.

3) *If $l = 2$ and $\omega = \omega_j$ ($j \in \mathbb{N}$ fixed), then* resonance *of order* $\ln t$ *occurs:*

$$\begin{aligned} u(t,x',x'') &= \frac{\ln t}{4\pi m}\,\mathrm{e}^{-\mathrm{i}\omega t}\sum_{k=1}^{K(j)}\frac{v_{jk}(x'')}{\|\nabla''^{m-1}v_{jk}\|_D^2}\int_\Omega f(y',y'')v_{jk}(y'')\,d(y',y'') \\ &\quad + \mathrm{e}^{-\mathrm{i}\omega t}\,u_\omega(x',x'') + o_M(1) \qquad \text{as } t\to\infty, \end{aligned} \tag{1.7}$$

where u_ω solves (1.5).

Remarks: **1)** The principle of limiting amplitude holds even in the case $l \le 2$ for $\omega = \omega_j$, if f satisfies the orthogonality conditions

$$\int_\Omega f(y',y'')v_{jk}(y'')\,d(y',y'') = 0 \qquad \text{for } 1 \le k \le K(j). \tag{1.8}$$

2) Denote by R_z ($z \in \mathbb{C}\setminus\mathbb{R}$) the resolvent of the spatial operator. The *principle of limiting absorption* holds at $\rho \in \mathbb{R}$, if $v_\rho := \lim_{\tau\downarrow 0} R_{\rho+\mathrm{i}\tau} f$ exists. It turns out, that the principle of limiting absorption holds at $\rho \in \mathbb{R}$ if and only if the solution of (1.2), (5.1) with $\omega := \sqrt{\rho}$ satisfies the principle of limiting amplitude.

3) The resonance effects are connected to the so-called "admissible standing waves" (compare [9]) given by

$$S_{jk}(x',x'') := \frac{1}{\|\nabla''^{m-1}v_{jk}\|_D^{l/2}}\,v_{jk}(x') \qquad \text{for } (x',x'') \in \Omega.$$

They are solution of the boundary value problem

$$\begin{aligned} ((-\Delta)^m - \mu_j)\,S_{jk} &= 0 \quad \text{in } \Omega, \\ \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} u_\omega &= 0 \quad \text{on } \partial\Omega \text{ for } |\alpha| \le m-1 \end{aligned}$$

Note that $S_{jk} \notin L_2(\Omega)$. The resonance term in (1.6) is like a projection of f on the linear space spanned by $\{S_{j1},\ldots,S_{jK(j)}\}$.

4) The assumptions $\partial D \in C^\infty$ and $f \in C_0^\infty(\Omega)$ can be weakend (cp. Satz 1.22 in [8]), but we skip these details for sake of simplicity.

In the case $m = 1$ these results are already known. They were proved by P. Werner in [12] (for $n = 3$, $l = 2$), in [13] ($n \ge 2$, $l = 1$) and, together with A. G. Ramm, in [10] ($n \ge 2$, $l = n-1$). Furthermore the case $m = 1$ with $n \ge 2$ and arbitrary l is contained in the article [3] of B. A. Iskenderov, Eh. Kh. Ehjvazov and A. N. Ehfendieva (with partly wrong results) and in [7] of the author. The present paper generalizes the results to the case of arbitrary $m \in \mathbb{N}$.

The proofs in these papers are based on the fact, that the spatial operator $-\Delta$ can be decomposed in the sum of a longitudinal part $-\Delta' := -\partial_1^2 - \ldots - \partial_l^2$ and a transversal part $-\Delta'' := -\partial_{l+1}^2 - \ldots - \partial_n^2$. The resolvent of each part can be studied separately to obtain resolvent and spectral family of the full operator. Such a decomposition in longitudinal- and cross-derivatives is not possible for the operator $(-\Delta)^m$ in the case $m \geq 2$. The present article uses another method, which is based on Fourier-transform with respect to the longitudinal variable $(x_1, \ldots, x_l)$. The same method was used by M. D. Groves and the author in [2]. Application to the equations of linear elasticity in an infinite tube will be given in a further paper (see [6]).

This method can roughly be described as follows: Let $\mathcal{A}$ be a self-adjoint extension of $(-\Delta)^m$ in $L_2(\Omega)$ corresponding to Dirichlet boundary condition. Fourier-transform of $\mathcal{A}$ with respect to $x' = (x_1, \ldots, x_l)$ leads to the operator family

$$B_{|\xi|} \;:=\; \left(|\xi|^2 - \Delta''\right)^m, \quad D(B_{|\xi|}) \;:=\; \overset{\circ}{H}{}^m(D) \cap H^{2m}(D)$$

depending on $|\xi| = \sqrt{\xi_1^2 + \ldots + \xi_l^2} =: r \geq 0$ and being self-adjoint in $L_2(D)$. Note that D is bounded. In a first step, the (suitable enumerated) eigenvalues $\mu_1^{(r)}, \mu_2^{(r)}, \ldots$ of B_r are studied. For each fixed $j \in \mathbb{N}$ it is proved that $\mu_j^{(.)}$ depends analytically on r and

$$\frac{d\mu_j^{(r)}}{dr} > 0 \text{ for } r > 0, \qquad \frac{d\mu_j^{(r)}}{dr} = 0 \text{ for } r = 0, \qquad \mu_j^{(r)} \to \infty \text{ as } r \to \infty.$$

Hence the mapping $r \mapsto \mu_j^{(r)}$ has an inverse $r_j : [\mu_j^{(0)}, \infty) \to [0, \infty)$ being differentiable on $(\mu_j^{(0)}, \infty)$. The derivative r_j' has a singularity at $\mu_j^{(0)}$.

The second step contains the heart of the method. Denote by $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ the spectral family of $\mathcal{A}$. The local behaviour of $\frac{dP_\lambda}{d\lambda}$ is studied, which proves to be closely connected to the behaviour of those of the derivatives r_j', which are defined at λ. The derivative of the spectral family is shown to be Hölder-continuous (in a certain sense) on $\mathbb{R} \setminus \{\mu_1^{(0)}, \mu_2^{(0)}, \ldots\}$ and to have singularities at $\lambda = \mu_1^{(0)}, \mu_2^{(0)}, \ldots$.

In the last step, the resolvent of $\mathcal{A}$ and the solution u of (1.2) are given by spectral integrals, which can be estimated by the properties of $\frac{dP_\lambda}{d\lambda}$ given in the second step.

2 The operator family B_r

Set

$$\left.\begin{aligned} D(B_r) \;&:=\; \left\{ v \in \overset{\circ}{H}{}^m(D) : \left(r^2 - \Delta''\right)^m v \in L_2(D) \right\}, \\ B_r v \;&:=\; \left(r^2 - \Delta''\right)^m v \;=\; \sum_{k=0}^{m} \binom{m}{k} r^{2k} \left(-\Delta''\right)^{m-k} v \end{aligned}\right\} \tag{2.1}$$

($\overset{\circ}{H}{}^m(D) :=$ closure of $C_0^\infty(D)$ in $H^m(D)$). For every fixed $r \in \mathbb{R}$, B_r is coercitive, which means that

$$\langle B_r v, v \rangle_D \;\geq\; c_1^{(r)} \, \|v\|_{H^m(D)}^2 - c_2^{(r)} \, \|v\|_D^2 \qquad \text{for } v \in D(B_r) \tag{2.2}$$

with suitable coefficients $c_1^{(r)}, c_2^{(r)} > 0$ depending on r but not on v; here $\langle\,.\,,.\rangle_D$ and $\|.\|_D$ denote the inner product and associated norm in $L_2(D)$, respectively. Standard theory shows that B_r is self-adjoint in $L_2(D)$ and positive (see e.g. [5]).

The aim of this section is to prove the following theorem.

Theorem 2.1 *Let $D \subset \mathbb{R}^{n-l}$ be bounded. Suppose that $m \leq 2$ and D has the segment property or that $m \geq 3$ and $\partial D \in C^{2m}$. Then:*

1) *For every fixed $r \in \mathbb{R}$, B_r has an orthonormal system of eigenfunctions being complete in $L_2(D)$. The eigenfunctions $v_1^{(r)}, v_2^{(r)}, \ldots$ and associated eigenvalues $\mu_1^{(r)}, \mu_2^{(r)}, \ldots$ can be chosen in a way, such that the mappings $r \mapsto v_j^{(r)}$ and $r \mapsto \mu_1^{(r)}$ are analytic and*

$$0 \;<\; \mu_1^{(0)} \;\leq\; \mu_2^{(0)} \;\leq\; \ldots, \qquad \mu_j^{(0)} \to \infty \text{ as } j \to \infty. \tag{2.3}$$

2) *For every $j \in \mathbb{N}$,*

$$\frac{d\mu_j^{(r)}}{dr}(0) \;=\; 0 \qquad \text{and} \qquad \frac{d\mu_j^{(r)}}{dr}(r) \;\geq\; 2m\, r^{2m-1} \text{ if } r \geq 0. \tag{2.4}$$

3) *For every $j \in \mathbb{N}$, the mapping $\mu_j^{(\cdot)} : [0,\infty) \to [\mu_j^{(0)}, \infty) : r \mapsto \mu_j^{(r)}$ has an invers $r_j \in C([\mu_j^{(0)}, \infty)) \cap C^\infty((\mu_j^{(0)}, \infty))$, and*

$$\left.\begin{aligned} r_j(\mu) \;&=\; \frac{1}{\sqrt{m}\left\|\nabla''^{\,m-1} v_j^{(0)}\right\|_D}\sqrt{\mu - \mu_j^{(0)}} + O\Big(\left|\mu - \mu_j^{(0)}\right|^{3/2}\Big) \\ \frac{d}{d\mu}\, r_j(\mu) \;&=\; \frac{1}{2\sqrt{m}\left\|\nabla''^{\,m-1} v_j^{(0)}\right\|_D \sqrt{\mu - \mu_j^{(0)}}} + O\Big(\left|\mu - \mu_j^{(0)}\right|^{1/2}\Big) \end{aligned}\right\} \tag{2.5}$$

as $\mu \downarrow \mu_j^{(0)}$, where $\nabla''^{k} := \Delta''^{k/2}$ for even $k \in \mathbb{N}$ and $\nabla''^{k} := \nabla''\Delta''^{(k-1)/2}$ for odd $k \in \mathbb{N}$.

Proof: First let $r \in \mathbb{R}$ be fixed. From (2.2) and Rellich's selection theorem one obtains that $\left(B_r + c_2^{(r)} I\right)^{-1} : L_2(D) \to L_2(D)$ is a compact operator. Furthermore it is symmetric and postive. Every eigenfunction of it is an eigenfunction of B_r and vice versa. Application of the theory for compact symmetric operators shows: B_r has a countable set of eigenvalues $\{\mu_1^{(r)}, \mu_2^{(r)}, \ldots\}$, where $\mu_j^{(r)} \to \infty$ as $j \to \infty$, and the eigenfunctions can be chosen such that they form a complete orthonormal basis of $L_2(D)$. Choose the enumeration of eigenvalues, such that that every eigenvalue appears in the enumeration so many times, as his finite multiplicity counts and that $0 < \mu_1^{(r)} \leq \mu_2^{(r)} \leq \ldots$.

Now let r vary. If $m = 1$ or $m = 2$, then

$$D(B_r) \;=\; D(B) \;:=\; \Big\{u \in \overset{\circ}{H}{}^m(D) : \Delta''^{m} u \in L_2(D)\Big\} \quad \text{for } r \in \mathbb{R} \tag{2.6}$$

by definition of $D(B_r)$. This equation holds also in the cases $m \geq 3$, if $\partial D \in C^{2m}$ is supposed, since elliptic regularity theory then shows that

$$D(B_r) \;=\; \overset{\circ}{H}{}^m(D) \cap H^{2m}(D) \;=\; D(B) \qquad \text{for } r \in \mathbb{R}$$

(see e.g. [1]). In the notation of [4], B_r is a self-adjoint holomorphic family of type (A) with compact resolvent, defined for $r \in \mathbb{C}$. According to Theorem 3.9 of Chapter VII in [4], eigenvalues and eigenfunctions can be chosen in a way, such that, for every $j \in \mathbb{N}$, the eigenvalue $\mu_j^{(r)}$ and associated eigenfunction $v_j^{(r)}$ depend analytically on $r \in \mathbb{R}$. Since different graphs $G_j := \left\{(r, \mu_j^{(r)}) : r \in \mathbb{R}\right\} \subset \mathbb{R}^2$ $(j \in \mathbb{N})$ may intersect, the succesion of enumeration has to be changed eventually for $r > 0$. Note that $\mu_1^{(0)} > 0$, since B_0 is positive and $B_0 u = 0$ would imply $u = 0$. This proves Part 1 of Theorem 2.1.

For every fixed $j \in \mathbb{N}$, we have

$$\begin{aligned} \frac{d}{dr}\,\mu_j^{(r)} &= \frac{d}{dr}\left\langle B_r v_j^{(r)}, v_j^{(r)}\right\rangle_D \\ &= \lim_{h\to 0}\frac{1}{h}\left(\left\langle (B_{r+h} - B_r)\, v_j^{(r+h)}, v_j^{(r+h)}\right\rangle_D + \left\langle B_r\left(v_j^{(r+h)} - v_j^{(r)}\right), v_j^{(r+h)}\right\rangle_D \right. \\ &\quad \left. + \left\langle B_r v_j^{(r)}, v_j^{(r+h)} - v_j^{(r)}\right\rangle_D\right). \end{aligned}$$

According to (2.6), $D(B_r)$ is independent of r. Using the symmetry of B_r, we obtain

$$\begin{aligned} \lim_{h\to 0}\frac{1}{h}\left(\left\langle B_r\left(v_j^{(r+h)} - v_j^{(r)}\right), v_j^{(r+h)}\right\rangle_D + \left\langle B_r v_j^{(r)}, v_j^{(r+h)} - v_j^{(r)}\right\rangle_D\right) &= \\ &= 2\,\mathrm{Re}\left\langle \frac{d}{dr}v_j^{(r)}, B_r v_j^{(r)}\right\rangle_D \\ &= \mu_j^{(r)}\,\frac{d}{dr}\left\|v_j^{(r)}\right\|_D^2 \\ &= 0 \end{aligned}$$

(since $\|v_j^{(r)}\|_D = 1$), and hence

$$\begin{aligned} \frac{d}{dr}\,\mu_j^{(r)} &= \lim_{h\to 0}\frac{1}{h}\left\langle (B_{r+h} - B_r)\, v_j^{(r+h)}, v_j^{(r+h)}\right\rangle_D \\ &= \sum_{k=1}^{m}\binom{m}{k} 2k\, r^{2k-1}\left\langle (-\Delta'')^{m-k} v_j^{(r)}, v_j^{(r)}\right\rangle_D \\ &= \sum_{k=1}^{m}\binom{m}{k} 2k\, r^{2k-1}\left\|\nabla''^{\,m-k} v_j^{(r)}\right\|_D^2, \end{aligned} \tag{2.7}$$

This implies that (2.4) holds.

Assertion 1 of Theorem 2.1 and (2.4) imply that the inverse function r_j exists and that $r_j \in C([\mu_j^{(0)}, \infty)) \cap C^\infty((\mu_j^{(0)}, \infty))$. It remains to prove (2.5).

According to (2.1), B_r and $\mu_j^{(r)}$ depend analytically on r^2, since the perturbation argument used in the first step of this proof also holds for the mapping $r \mapsto B_{\sqrt{r}}$. Hence

$$\mu_j^{(r)} = \sum_{j=0}^{\infty} a_j\, r^{2j} \qquad \text{for } r \in \mathbb{R}, \tag{2.8}$$

with suitable constants $a_j \in \mathbb{R}$. Obviously $a_0 = \mu_j^{(0)}$, and $a_1 = m\|\nabla''^{\,m-1} v_j^{(0)}\|_D^2$ by (2.7). Note that $a_1 > 0$, since otherwise $\nabla''^{\,m-1} v_j^{(0)} = 0$ and $v_j^{(0)} \in D(B) \subset \mathring{H}^m(D)$

would imply $v_j^{(0)} = 0$. By the theory of analytic functions,

$$r_j(\mu) \;=\; \sum_{k=0}^{\infty} b_k(\mu-\mu_j^{(0)})^{k/2} \qquad \text{for } \mu_j^{(0)} \le \mu \le \mu_j^{(0)}+\delta$$

with suitable $\delta > 0$ and $b_k \in \mathbb{R}$. Inserting this in the expansion for $\mu_j^{(r)}$, we obtain that $b_0 = 0$ and $b_1 = \frac{1}{\sqrt{m}\left\|\nabla''^{\,m-1}v_j^{(0)}\right\|_D}$. This proves (2.5). $\square$

3 Spectral properties of $(-\Delta)^m$

Consider the operator A given by

$$\left.\begin{array}{rcl} D(\mathcal{A}) & := & \left\{u \in \overset{\circ}{H}{}^m(\Omega) : (-\Delta)^m \in L_2(\Omega)\right\}, \\ \mathcal{A}u & := & (-\Delta)^m\, u. \end{array}\right\} \tag{3.1}$$

Standard theory shows coercivity (cp. (2.2)) and self-adjointness of $\mathcal{A}$ in $L_2(\Omega)$. This section contains representations of the operators resolvent R_z (see (3.5)) and spectral family $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ (see (3.8)), and gives certain properties of $\{P_\lambda\}$ (see Theorem 3.4) needed later.

Define the space S of functions having rapid decrease by

$$S \;:=\; \left\{f \in C^\infty(\overline{\Omega}) \;\middle|\; \forall \alpha,\beta \in \mathbb{N}_0^n : \sup_{x\in\overline{\Omega}} \left|x^\alpha\, \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}\, f(x)\right| < \infty\right\} \tag{3.2}$$

and the partial Fourier-transform $\mathcal{F} : S \to S$ by

$$(\mathcal{F}\varphi)(\xi,x'') \;:=\; \frac{1}{(2\pi)^{l/2}}\int_{\mathbb{R}^l} \mathrm{e}^{-\mathrm{i}\xi\cdot x'}\varphi(x',x'')\,dx' \qquad (\xi\in\mathbb{R}^l,\ x''\in D). \tag{3.3}$$

Theorem 3.1 *Suppose that $f \in C_0^\infty(\Omega)$, $z \in \mathbb{C}\setminus\mathbb{R}$ and set*

$$u_z(\xi,.) \;:=\; \left(B_{|\xi|} - z\,\mathrm{Id}\right)^{-1}(\mathcal{F}f)(\xi,.) \qquad \textit{for } \xi\in\mathbb{R}^l. \tag{3.4}$$

Then $u_z \in S$ and the resolvent $R_z := (\mathcal{A} - z\,\mathrm{Id})^{-1}$ can be represented by

$$(R_zf)(x',x'') \;=\; (\mathcal{F}^{-1}u_z)(x',x'') \;=\; \frac{1}{(2\pi)^{l/2}}\int_{\mathbb{R}^l}\mathrm{e}^{\mathrm{i}\,\xi\cdot x'}\,u_z(\xi,x'')\,d\xi \tag{3.5}$$

for $(x',x'') \in \overline{\Omega}$.

Proof: From $\left\langle B_{|\xi|}u_z(\xi,.),u_z(\xi,.)\right\rangle_D \in \mathbb{R}$ and $(B_{|\xi|}-z)u_z = \mathcal{F}f$ one obtains that

$$\begin{array}{rcl} \left|-\mathrm{Im}\,z\,\|u_z(\xi,.)\|_D^2\right| & = & \left|\mathrm{Im}\,\left\langle (B_{|\xi|}-z)u_z(\xi,.),u_z(\xi,.)\right\rangle_D\right| \\ & = & \left|\langle(\mathcal{F}f)(\xi,.),u_z(\xi,.)\rangle_D\right| \\ & \le & \|(\mathcal{F}f)(\xi,.)\|_D\,\|u_z(\xi,.)\|_D \end{array}$$

and hence

$$\|u_z(\xi,.)\|_D \;\le\; \frac{1}{|\mathrm{Im}\,z|}\,\|(\mathcal{F}f)(\xi,.)\|_D. \tag{3.6}$$

Elliptic estimates applied to

$$B_0 u_z(\xi,.) \;=\; (B_0 - B_{|\xi|} + z) u_z(\xi,.) + (\mathcal{F}f)(\xi,.)$$

yield

$$\begin{aligned}\|u_z(\xi,.)\|_{H^{2m+j}(D)} \;&\leq\; c_j \left(\|(B_0 - B_{|\xi|} + z)u_z(\xi,.) + (\mathcal{F}f)(\xi,.)\|_{H^j(D)} + \|u_z(\xi,.)\|_D\right) \\ &\leq\; c_j\,|\xi|^{2m}\left(\|u_z(\xi,.)\|_{H^{2m+j-2}(D)} + (\mathcal{F}f)(\xi,.)\|_{H^j(D)}\right) \\ &\leq\; c_j\,|\xi|^{2m}\left(\varepsilon\|u_z(\xi,.)\|_{H^{2m+j-2}(D)} + \frac{1}{\varepsilon}\,\|u_z(\xi,.)\|_D + (\mathcal{F}f)(\xi,.)\|_{H^j(D)}\right),\end{aligned}$$

$c_j \geq$ denoting various constants depending at most on $j \in \mathbb{N}$; for the last estimate compare Theorem 3.4 in [1], which can be extended to unbounded domains of form (1.1). By $\varepsilon := \frac{1}{2c_j|\xi|^{2m}}$, this together with (3.6) proves

$$\|u_z(\xi,.)\|_{H^{2m+j}(D)} \;\leq\; \frac{c_j\,|\xi|^{4m}}{|\mathrm{Im}\, z|}\,\|(\mathcal{F}f)(\xi,.)\|_{H^j(D)} \qquad \text{for each } \xi \in \mathbb{R}^l,\ j \in \mathbb{N}.$$

Sobolev's Lemma shows that $u(\xi,.) \in C^\infty(D)$ and

$$\left|\partial_{x_{l+1}}^{\alpha_{l+1}} \cdots \partial_{x_n}^{\alpha_n} u_z(\xi,x'')\right| \;\leq\; \frac{c_j\,|\xi|^{4m}}{|\mathrm{Im}\, z|}\,\|(\mathcal{F}f)(\xi,.)\|_{H^j(D)} \qquad \text{for } (\xi,x'') \in \overline{\Omega}$$

if $j \geq \alpha_{l+1} + \ldots + \alpha_n + \frac{n-l+1}{2}$.

Set $\tilde{u}(\xi,.) := \left(B_{|\xi|} - z\,\mathrm{Id}\right)^{-1} \partial_{\xi_1}(\mathcal{F}f)(\xi,.)$. From

$$\begin{aligned}(B_{|\xi|} - z)u\left(\tilde{u}_z(\xi,.) - \frac{u_z(\xi + h\,e_1,.) - u_z(\xi,.)}{h}\right) \;&=\; \\ =\; \partial_{\xi_1}(\mathcal{F}f)(\xi,.) - \frac{(\mathcal{F}f)(\xi + h\,e_1,.) - (\mathcal{F}f)(\xi,.)}{h}&\end{aligned}$$

one can conlude that $\tilde{u}_z = \partial_{\xi_1} u_z(\xi,.)$. The same argument used above proves that

$$\left|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_l}^{\alpha_l}\, \partial_{x_{l+1}}^{\alpha_{l+1}} \cdots \partial_n^{\alpha_n} u_z(\xi,x'')\right| \;\leq\; \frac{c_j\,|\xi|^{4m}}{|\mathrm{Im}\, z|}\,\left\|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_l}^{\alpha_l}(\mathcal{F}f)(\xi,.)\right\|_{H^j(D)}$$

for $(\xi,x'') \in \overline{\Omega}$ if $j \geq \alpha_{l+1} + \ldots + \alpha_n + \frac{n-l+1}{2}$. Since $\mathcal{F}f \in S$, this proves $u_z \in S$ and, interchanging differentiation and integration, that

$$((-\Delta)^m - z)\,\mathcal{F}^{-1}u_z \;=\; f.$$

Furthermore $u_z(\xi,.) \in \overset{\circ}{H}{}^m(D)$ for every $\xi \in \mathbb{R}^n$ implies that $\mathcal{F}^{-1}u_z \in \overset{\circ}{H}{}^m(\Omega)$. Since $\mathcal{F}^{-1}u_z \in S \subset H^{2m}(\Omega)$, (3.5) is proved. □

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^n$ and the operator $\mathcal{A}$ be given by (1.1) and (3.1), respectively. Denote by $\{P_\lambda\}$ the left-hand continuous spectral family of $\mathcal{A}$. Then*

$$P_\lambda \;=\; 0 \qquad \textit{for } \lambda \leq \mu_1^{(0)}, \tag{3.7}$$

$\mu_1^{(0)} > 0$ denoting the smallest eigenvalue of B_0 (see Theorem 2.1). If $f \in C_0^\infty(\Omega)$ and $\lambda > \mu_1^{(0)}$, then

$$(P_\lambda f)(x) \;=\; \frac{1}{(2\pi)^{l/2}} \int_{\mu_1^{(0)}}^{\lambda} \sum_{j=1}^{J(\rho)} \frac{d\, r_j(\rho)}{d\rho}\, r_j(\rho)^{l-1} I_j(r_j(\rho), x)\, d\rho, \tag{3.8}$$

with $r_j(\rho)$ being defined in Theorem 2.1,

$$J(\rho) \;:=\; \min\bigl\{j \in \mathbb{N} : \mu_j^{(0)} \ge \rho\bigr\} - 1 \qquad \textit{for } \rho \in \mathbb{R} \tag{3.9}$$

and

$$I_j(r, x', x'') \;:=\; \int_{|\xi|=r} \mathrm{e}^{\mathrm{i}\xi\cdot x'} \bigl\langle (\mathcal{F}f)(\xi, .), v_j^{(r)} \bigr\rangle_D \, dS_\xi \; v_j^{(r)}(x'') \tag{3.10}$$

for $j \in \mathbb{N},\ r \ge 0,\ (x', x'') \in \overline{\Omega}$; here the integral on the right-hand side has to be read as

$$\int_{|\xi|=r} \varphi(\xi)\, dS_\xi \;:=\; \varphi(r) + \varphi(-r) \qquad \textit{if } l = 1. \tag{3.11}$$

Proof: We use Stone's formula

$$\begin{aligned} \frac{1}{2}\, \langle P_{\beta+0} f + P_\beta f - P_{\alpha+0} f - P_\alpha f, g\rangle \;&= \\ =\; \frac{1}{2\pi\mathrm{i}} \lim_{\tau\downarrow 0} \int_\alpha^\beta \langle R_{\rho+\mathrm{i}\tau} f - R_{\rho-\mathrm{i}\tau} f, g\rangle \; d\rho \qquad &\text{for } f, g \in C_0^\infty(\Omega). \end{aligned} \tag{3.12}$$

Note that the Fourier-transform u_z of R_z (cp. (3.5)) can be represented as Fourier-series according to Theorem 2.1:

$$u_z(\xi, x'') \;=\; \sum_{j=1}^\infty \frac{1}{\mu_j^{(|\xi|)} - z} \bigl\langle (\mathcal{F}f)(\xi, .), v_j^{(|\xi|)} \bigr\rangle_D v_j^{(|\xi|)}(x'').$$

Using elliptic estimates, pointwise convergence for $\xi \in \mathbb{R}^l,\ x'' \in \overline{D}$ can be shown. This implies that

$$\begin{aligned} &(R_{\rho+\mathrm{i}\tau} f)(x', x'') - (R_{\rho-\mathrm{i}\tau} f)(x', x'') \;= \\ &=\; \frac{1}{(2\pi)^{l/2}} \int_{\mathbb{R}^l} \mathrm{e}^{\mathrm{i}\xi\cdot x'} \left(u_{\rho+\mathrm{i}\tau}(\xi, x'') - u_{\rho-\mathrm{i}\tau}(\xi, x'')\right) d\xi \\ &=\; \frac{1}{(2\pi)^{l/2}} \int_{\mathbb{R}^l} \mathrm{e}^{\mathrm{i}\xi\cdot x'} \left(\sum_{j=1}^\infty \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \bigl\langle (\mathcal{F}f)(\xi, .), v_j^{(|\xi|)} \bigr\rangle_D v_j^{(|\xi|)}(x'') \right) d\xi \end{aligned} \tag{3.13}$$

for $(\xi, x'') \in \overline{\Omega}$, $z \in \mathbb{C} \setminus \mathbb{R}$. Set $L(\beta) := \min\bigl\{j \in \mathbb{N} : \mu_j^{(0)} > \beta\bigr\}$ and

$$\psi_\rho(\xi, x'') \;:=\; \sum_{j=L(\beta)}^\infty \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \bigl\langle (\mathcal{F}f)(\xi, .), v_j^{(|\xi|)} \bigr\rangle_D v_j^{(|\xi|)}(x'')$$

for $(\xi, x'') \in \Omega$ and $\rho \leq \beta$. With $\mu_j^{(|\xi|)} \geq \mu_j^{(0)} \geq \mu_{L(\beta)}^{(0)}$ for $j \geq L(\beta)$ one obtains

$$\begin{aligned}
\|\psi_\rho(\xi,.)\|_D^2 &= \sum_{j=L(\beta)}^{\infty} \left(\frac{2\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \right)^2 \left| \left\langle (\mathcal{F}f)(\xi,.), v_j^{(|\xi|)} \right\rangle_D \right|^2 \\
&\leq \frac{4\tau^2}{\left(\left(\mu_k^{(0)} - \rho\right)^2 + \tau^2 \right)^2} \sum_{j=L(\beta)}^{\infty} \left| \left\langle (\mathcal{F}f)(\xi,.), v_j^{(|\xi|)} \right\rangle_D \right|^2 \\
&\leq \frac{4\tau^2}{\left(\left(\mu_k^{(0)} - \beta\right)^2 + \tau^2 \right)^2} \|(\mathcal{F}f)(\xi,.)\|_D^2
\end{aligned}$$

and

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\psi_\rho\right\|_\Omega^2 &= \|\psi_\rho\|_\Omega^2 = \int_{\mathbb{R}^l} \|\psi_\rho(\xi,.)\|_D^2 \, d\xi \\
&\leq \frac{4\tau^2}{\left(\left(\mu_k^{(0)} - \beta\right)^2 + \tau^2 \right)^2} \int_{\mathbb{R}^l} \|(\mathcal{F}f)(\xi,.)\|_D^2 \, d\xi \\
&= \frac{4\tau^2}{\left(\left(\mu_k^{(0)} - \beta\right)^2 + \tau^2 \right)^2} \|f\|_\Omega^2
\end{aligned}$$

for every $\xi \in \mathbb{R}^l$. This implies that

$$\lim_{\tau\downarrow 0} \int_{\mathbb{R}^l} \mathrm{e}^{\mathrm{i}\xi\cdot x'} \sum_{j=L(\beta)}^{\infty} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \left\langle (\mathcal{F}f)(\xi,.), v_j^{(|\xi|)} \right\rangle_D v_j^{(|\xi|)}(x'') \, d\xi = 0 \qquad (3.14)$$

in sense of $L_2(\Omega)$-norm uniformly with respect to $\rho \leq \beta$. In case of $\beta < \mu_1^{(0)}$, this proves that the right-hand side of (3.12) vanishes and hence that $P_\lambda f = 0$ for $\lambda < \mu_1^{(0)}$, $f \in C_0^\infty(\Omega)$. By the left-hand continuity of P_λ, (3.7) follows.

In case of $\beta \geq \mu_1^{(0)}$, (3.12), (3.13) and (3.14) imply that

$$\begin{aligned}
&\frac{1}{2} \left\langle P_{\beta+0}f + P_\beta f - P_{\alpha+0}f - P_\alpha f, g \right\rangle = \\
&= \frac{1}{\mathrm{i}\,(2\pi)^{1+l/2}} \lim_{\tau\downarrow 0} \int_\alpha^\beta \left(\int_{\mathbb{R}^l} \left(\sum_{j=1}^{L(\beta)-1} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \varphi_j(\xi) \right) d\xi \right) d\rho \qquad (3.15)
\end{aligned}$$

with

$$\varphi_j(\xi) := \int_\Omega \mathrm{e}^{\mathrm{i}\xi\cdot x''} \left\langle (\mathcal{F}f)(\xi,.), v_j^{(|\xi|)} \right\rangle_D v_j^{(|\xi|)}(x'') \, \overline{g(x',x'')} \, d(x',x'') \qquad (3.16)$$

for $(x', x'') \in \overline{\Omega}$. Let $j \in \mathbb{N}$ be fixed and suppose that $\rho > \mu_j^{(0)}$. We prove that

$$\int_{\mathbb{R}^l} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \varphi_j(\xi) \, d\xi = 2\pi\mathrm{i} \frac{dr_j(\rho)}{d\rho} \int_{|\xi|=r_j(\rho)} \varphi_j(\xi) \, dS_\xi + o(1) \qquad (3.17)$$

as $\tau \downarrow 0$ uniformly with respect to ρ in every compact subset of $(\mu_j^{(0)}, \infty)$. Note that r_j is a strictly increasing function. Fix $\delta \in (0, r_j(\rho))$ and set

$$\Phi_j(\xi, x'') := \begin{cases} 0 & \text{if } |\xi - r_j(\rho)| < \delta, \\ \dfrac{2\mathrm{i}\tau}{\left(\lambda_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \left\langle (\mathcal{F}f)(\xi, .), v_j^{(|\xi|)} \right\rangle_D v_j^{(|\xi|)}(x'') & \text{otherwise.} \end{cases}$$

Since $\left|\lambda_j^{(|\xi|)} - \rho\right| \geq c > 0$ for $|\xi - r_j(\rho)| \geq \delta$, one obtains

$$\|\Phi_j(\xi, .)\|_D \;\leq\; \frac{2\tau}{c^2} \left| \left\langle (\mathcal{F}f)(\xi, .), v_j^{(|\xi|)} \right\rangle_D \right| \;\leq\; \frac{2\tau}{c^2} \, \|(\mathcal{F}f)(\xi, .)\|_D$$

for $\xi \in \mathbb{R}^l$ and

$$\left\|\mathcal{F}^{-1}\Phi_j\right\|_\Omega^2 \;=\; \|\Phi_j\|_\Omega^2 \;\leq\; \frac{4\,\tau^2}{c^4} \, \|\mathcal{F}f\|_\Omega^2 \;=\; \frac{4\,\tau^2}{c^4} \, \|f\|_\Omega^2 \qquad (1 \leq j \leq L(\rho)).$$

This shows that

$$\int_{|\xi - r_j(\rho)| \geq \delta} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \, \varphi_j(\xi) \, d\xi \;=\; O(\tau)$$

as $\tau \downarrow 0$. The substitution $\mu = \mu_j^{(r)}$ ($\Leftrightarrow r = r_j(\mu)$) and $\delta_1 := \rho - \mu_j^{(r_j(\rho) - \delta_0)} > 0$, $\delta_2 := \mu_j^{(r_j(\rho) + \delta_0)} - \rho > 0$ yields

$$\begin{aligned} &\int_{|\xi - r_j(\rho)| \leq \delta} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \, \varphi_j(\xi) \, d\xi \;= \\ &= \int_{r_j(\rho) - \delta_0}^{r_j(\rho) + \delta_0} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(r)} - \rho\right)^2 + \tau^2} \left(\int_{|\xi| = r} \varphi(\xi) \, dS_\xi \right) dr \\ &= \int_{\rho - \delta_1}^{\rho + \delta_2} \frac{2\mathrm{i}\tau}{(\mu - \rho)^2 + \tau^2} \, \frac{dr_j(\mu)}{d\mu} \left(\int_{|\xi| = r_j(\mu)} \varphi(\xi) \, dS_\xi \right) d\mu \\ &= 2\pi\mathrm{i} \, \frac{dr_j(\rho)}{d\rho} \int_{|\xi| = r_j(\rho)} \varphi_j(\xi) \, dS_\xi + o(1) \qquad \text{as } \tau \downarrow 0. \end{aligned}$$

The same argument shows that

$$\int_{\mathbb{R}^l} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \varphi_j(\xi) \, d\xi \;=\; +o(1) \tag{3.18}$$

as $\tau \downarrow 0$ uniformly with respect to ρ in every compact subset of $(-\infty, \mu_j^{(0)})$ and that

$$\left| \int_{\mathbb{R}^l} \frac{2\mathrm{i}\tau}{\left(\mu_j^{(|\xi|)} - \rho\right)^2 + \tau^2} \varphi_j(\xi) \, d\xi \right| \;\leq\; \frac{c_j}{\rho - \mu_j^{(0)}} \tag{3.19}$$

for $0 \le \rho \le \mu_j^{(1)}$. Insert (3.17), (3.18) and (3.19) to obtain

$$\begin{aligned}
\frac{1}{2}\,\langle P_{\beta+0}f + P_\beta f - P_{\alpha+0}f - P_\alpha f, g\rangle \;&=\\
&= \frac{1}{(2\pi)^{l/2}} \int_\alpha^\beta \left(\sum_{j=1}^{J(\rho)} \frac{dr_j(\rho)}{d\rho} \int_{|\xi|=r_j(\rho)} \varphi_j(\xi)\, dS_\xi \right) d\rho.
\end{aligned}$$

Since the integrand on the right-hand side depends continuously on $\rho \in \mathbb{R} \setminus \{\mu_1^{(0)}, \mu_2^{(0)}, \dots\}$ and is integrabel at $\rho = \mu_1^{(0)}, \mu_2^{(0)}, \dots$, this proves

$$\langle P_\lambda f, g\rangle \;=\; \int_{\mu_1^{(0)}}^\lambda \left(\sum_{j=1}^{J(\rho)} \frac{dr_j(\rho)}{d\rho} \int_{|\xi|=r_j(\rho)} \varphi_j(\xi)\, dS_\xi \right) d\rho$$

for $\lambda > \mu_1^{(0)}$. Note that φ_j is defined by (3.16). Since $g \in C_0^\infty(\Omega)$ is arbitrary, (3.8) follows . □

For the next result we have to define a new enumeration of eigenvalues $\mu_j^{(0)}$ and eigenfunctions $v_j^{(0)}$ of B_0 (the old one was defined in Theorem 2.1).

Definition 3.3 *Let $0 < \mu_1 < \mu_2 < \dots$ denote all eigenvalues of B_0. Here every eigenvalue appears only one time in contrary to (2.3). Each eigenvalue μ_j has finite multiplicity, denoted by $K(j)$. Let $\{v_{j1}, \dots, v_{j\,K(j)}\}$ be an enumeration of the eigenfunctions $\{v_i^{(0)} : \mu_i^{(0)} = \mu_j\}$.*

Theorem 3.4 *If $f \in C_0^\infty(\Omega)$, then, for every $x \in \Omega$, $(P_\lambda f)(x)$ is differentiable with respect to λ for $\lambda \in \mathbb{R} \setminus \{\mu_1, \mu_2, \dots\}$ and*

$$\frac{dP_\lambda f}{d\lambda}(x) - \frac{dP_\mu f}{d\mu}(x) \;=\; O_M(|\lambda - \mu|) \qquad \text{as } \lambda \to \mu \tag{3.20}$$

if $\mu \in \mathbb{R} \setminus \{\mu_1, \mu_2, \dots\}$; here the notation $O_M(|\lambda - \mu|)$ means that the remainder is of order $O(|\lambda - \mu|)$ uniformly with respect to x in every compact subset of $\overline{\Omega}$. Furthermore, for every $j \in \mathbb{N}$ and every $x \in \Omega$, $\lim_{\lambda \uparrow \mu_j} \frac{dP_\lambda f}{d\lambda}(x)$ exists and

$$\begin{aligned}
\frac{dP_\lambda f}{d\lambda}(x) - \lim_{\lambda \uparrow \mu_j} \frac{dP_\lambda f}{d\lambda}(x) \;&=\\
&= \begin{cases}
O_M(|\lambda - \mu_j|) & \text{as } \lambda \uparrow \mu_j \ (l \in \mathbb{N}),\\
\dfrac{1}{\sqrt{\lambda - \mu_j}}\,(P^{(j)}f)(x) + O_M\Big(\sqrt{\lambda - \mu_j}\Big) & \text{as } \lambda \downarrow \mu_j \ (l = 1),\\
(P^{(j)}f)(x) + O_M\big((\lambda - \mu_j)^{3/4}\big) & \text{as } \lambda \downarrow \mu_j \ (l = 2),\\
O_M(\lambda - \mu_j) & \text{as } \lambda \downarrow \mu_j \ (l \ge 3),
\end{cases}
\end{aligned} \tag{3.21}$$

where

$$\big(P^{(j)}f\big)(x', x'') \;:=\; \frac{1}{\pi(4m)^{l/2}} \sum_{k=1}^{K(j)} \frac{1}{\|\nabla''^{m-1} v_{jk}\|_D^l} \int_\Omega f(y', y'') v_{jk}(y'')\, d(y', y'')\, v_{jk}(x'') \tag{3.22}$$

for $(x', x'') \in \overline{\Omega}$.

Proof: According to (3.8),

$$\frac{dP_\lambda f}{d\lambda}(x) \;=\; \frac{1}{(2\pi)^{l/2}} \sum_{\{k\in\mathbb{N}:\mu_k^{(0)}<\lambda\}} \frac{d\,r_k(\lambda)}{d\lambda}\, r_k(\lambda)^{l-1} I_k(r_k(\lambda),x)\,d\lambda \tag{3.23}$$

holds for $\lambda \in [\mu_1,\infty)\setminus\{\mu_1,\mu_2,\dots\}$. From (3.10), $\mathcal{F}f \in S$ and analycity of $r \mapsto v_j^{(r)}$, one obtains that

$$I_j(r,x) - I_j(s,x) \;=\; O_M(|r-s|) \qquad \text{as } r \to s > 0.$$

This together with Theorem 2.1 proves (3.20). Furthermore the same argument shows that

$$\lim_{\lambda\uparrow\mu_j} \frac{dP_\lambda f}{d\lambda}(x) \;=\; \frac{1}{(2\pi)^{l/2}} \sum_{\{k\in\mathbb{N}:\mu_k^{(0)}<\mu_j\}} \frac{d\,r_k(\lambda)}{d\lambda}\, r_k(\lambda)^{l-1} I_k(r_k(\lambda),x)\,d\lambda$$

and

$$\frac{dP_\lambda f}{d\lambda}(x) - \lim_{\lambda\uparrow\mu_j} \frac{dP_\lambda f}{d\lambda}(x) \;=\; O_M(|\lambda-\mu_j|) \qquad \text{as } \lambda \uparrow \mu_j$$

for each $k \in \mathbb{N}$. Hence the first estimate in (3.21) is proved.

Note that $v_j^{(r)} = v_j^{(0)} + O(r^2)$ (cp. (2.8)). If $l = 1$, this implies that

$$\begin{aligned} I_j(r,x) \;&=\; \Big(\mathrm{e}^{\mathrm{i}rx'} \big\langle (\mathcal{F}f)(r,.), v_j^{(r)} \big\rangle_D + \mathrm{e}^{-\mathrm{i}rx'} \big\langle (\mathcal{F}f)(-r,.), v_j^{(-r)} \big\rangle_D \Big)\, v_j^{(r)}(x'') \\ &=\; 2\int_\Omega f(x',x'')\, v_j^{(0)}(x'')\, d(x',x'') + O_M\big(r^2\big) \qquad \text{as } r \downarrow 0. \end{aligned}$$

If $l = 2$, then

$$I_j(r,x) \;=\; 2\pi\sqrt{r} \int_\Omega f(x',x'')\, v_j^{(0)}(x'')\, d(x',x'') + O_M\big(r^{3/2}\big) \qquad \text{as } r \downarrow 0.$$

Finally, if $l \geq 3$, then

$$I_j(r,x) \;=\; O_M\big(r^{l-1}\big) \qquad \text{as } r \downarrow 0.$$

Inserting these estimates and (2.5) in (3.23) proves the remaining three parts of (3.21). $\square$

4 Limiting absorption principle

Theorem 4.1 *Suppose that Ω and $\mathcal{A}$ are given by (1.1) and (3.1), respectively, and that $f \in C_0^\infty(\Omega)$. Let $R_z := (\mathcal{A} - z\,\mathrm{Id})^{-1}$ denote the resolvent of $\mathcal{A}$. Then:*

1) *If $l \leq 2$ and $\rho \in \mathbb{R}\setminus\{\mu_1,\mu_2,\dots\}$, or if $l \geq 3$ and $\rho \in \mathbb{R}$,*

$$v_\rho(x) \;:=\; \lim_{\varepsilon\downarrow 0} \int_{|\lambda-\rho|\geq\varepsilon} \frac{1}{\lambda-\rho} \frac{dP_\lambda f}{d\lambda}(x)\,d\lambda + \mathrm{i}\pi \frac{dP_\rho f}{d\rho}(x) \quad \textit{for } x \in \overline{\Omega} \tag{4.1}$$

exists with $v_\rho \in C^\infty(\overline{\Omega})$, and the principle of limiting absorption

$$(R_{\rho+\mathrm{i}\tau} f)(x) \;=\; v_\rho(x) + O_M\big(\tau^{1/4}\big) \qquad \textit{as } \tau \downarrow 0 \tag{4.2}$$

holds.

2) *If $l = 1$ and $j \in \mathbb{N}$, set*

$$v_{\mu_j}(x) \;:=\; \lim_{\varepsilon\downarrow 0}\left(\int_{|\lambda-\mu_j|\ge\varepsilon} \frac{1}{\lambda-\mu_j}\,\frac{dP_\lambda f}{d\lambda}(x)\,d\lambda - \frac{2}{\sqrt{\varepsilon}}\,(P^{(j)}f)(x)\right) + \mathrm{i}\pi \lim_{\lambda\uparrow\mu_j}\frac{dP_\lambda f}{d\lambda}(x) \tag{4.3}$$

for $x \in \overline{\Omega}$ with $P^{(j)}f$ being defined by (3.22). Then $v_{\mu_j} \in C^\infty(\overline{\Omega})$ and

$$(R_{\mu_j+\mathrm{i}\tau}f)(x) \;=\; \frac{1}{\sqrt{\tau}}\,\frac{(1+\mathrm{i})\pi}{\sqrt{2}}\,(P^{(j)}f)(x) + v_{\mu_j}(x) + O_M\big(\sqrt{\tau}\big) \qquad \text{as } \tau\downarrow 0. \tag{4.4}$$

3) *If $l = 2$ und $j \in \mathbb{N}$, set*

$$v_{\mu_j}(x) \;:=\; \lim_{\varepsilon\downarrow 0}\left(\int_{|\lambda-\mu_j|\ge\varepsilon} \frac{1}{\lambda-\mu_j}\,\frac{dP_\lambda f}{d\lambda}(x)\,d\lambda + \ln\varepsilon\cdot(P^{(j)}f)(x)\right) + \mathrm{i}\pi \lim_{\lambda\uparrow\mu_j}\frac{dP_\lambda f}{d\lambda}(x) + \frac{\mathrm{i}\pi}{2}\,(P^{(j)}f)(x) \tag{4.5}$$

for $x \in \overline{\Omega}$. Then $v_{\mu_j} \in C^\infty(\overline{\Omega})$ and

$$(R_{\mu_j+\mathrm{i}\tau}f)(x) \;=\; \ln\!\left(\frac{1}{\tau}\right)(P^{(j)}f)(x) + v_{\mu_j}(x) + O_M\big(\tau^{1/4}\big) \qquad \text{as } \tau\downarrow 0. \tag{4.6}$$

Remarks: **1)** Let $j \in \mathbb{N}$ be fixed. If f satisfies the orthogonality condition $P^{(j)}f = 0$ (cp. (3.22)), then the principle of limiting absorption holds also in the cases $l = 1, 2$ at $\rho = \mu_j$.

2) It is easy to show that v_ρ given by (4.1) is solution of

$$\left.\begin{aligned} ((-\Delta)^m - \rho)\,v_\rho &= f \quad \text{in } \Omega,\\ \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\,v_\rho &= 0 \quad \text{on } \partial\Omega \text{ for } |\alpha| \le m-1. \end{aligned}\right\} \tag{4.7}$$

In the case $m = 1$, if $\rho \in \mathbb{R}\setminus\{\mu_1,\mu_2,\ldots\}$, v_ρ can be uniquely characterized by this equation and some radiation conditions (see e.g. [7]). The radiation conditions show that $u_\rho(x)\mathrm{e}^{-\mathrm{i}\omega t}$ is an outgoing wave. In the case $m \ge 2$, the question is open, if such a characterization is possible.

Proof: Suppose that $\rho \in \mathbb{R}$, $\tau \ne 0$ and $k \in \mathbb{N}$. Then

$$\mathcal{A}^k\,R_{\rho+\mathrm{i}\tau}f \;=\; \int_0^\infty \frac{\lambda^k}{\lambda-\rho-\mathrm{i}\tau}\,d(P_\lambda f).$$

Since $f \in C_0^\infty(\Omega) \subset D(\mathcal{A}^k)$ for every $k \in \mathbb{N}$, elliptic regularity theory shows that $R_{\rho+\mathrm{i}\tau}f \in C^\infty(\overline{\Omega})$ and

$$(R_{\rho+\mathrm{i}\tau}f)(x) \;=\; \int_0^\infty \frac{1}{\lambda-\rho-\mathrm{i}\tau}\,\frac{dP_\lambda f}{d\lambda}(x) \qquad \text{for } x \in \overline{\Omega}.$$

If $\delta > 0$ is fixed, then

$$\left\| \mathcal{A}^k \left(\int_{|\lambda-\rho|\geq\delta} \frac{1}{\lambda - \rho - \mathrm{i}\tau}\, d(P_\lambda f) - \int_{|\lambda-\rho|\geq\delta} \frac{1}{\lambda-\rho}\, d(P_\lambda f) \right) \right\|^2 \;=$$

$$= \int_{|\lambda-\rho|\geq\delta} \frac{\tau^2\,\lambda^{2k}}{|\lambda-\rho-\mathrm{i}\tau|^2\,|\lambda-\rho|^2}\, d\big(\|P_\lambda f\|^2\big)$$

$$\leq \frac{\tau^2}{\delta^4}\,\|\mathcal{A}^k f\|^2.$$

Again using elliptic regularity theory, one obtains that

$$\int_{|\lambda-\rho|\geq\delta} \frac{1}{\lambda-\rho-\mathrm{i}\tau}\,\frac{dP_\lambda f}{d\lambda}(x) \;=\; \int_{|\lambda-\rho|\geq\delta} \frac{1}{\lambda-\rho}\,\frac{dP_\lambda f}{d\lambda}(x) + O(\tau) \qquad \text{as } \tau\downarrow 0 \tag{4.8}$$

uniformly with respect to $x \in \overline{\Omega}$. Hence it is sufficient to study

$$I_\delta(\tau,x) \;:=\; \int_{\rho-\delta}^{\rho+\delta} \frac{1}{\lambda-\rho-\mathrm{i}\tau}\,\frac{dP_\lambda f}{d\lambda}(x) \qquad \text{for } x\in\overline{\Omega} \tag{4.9}$$

with fixed $\delta > 0$. If $\rho < \mu_1$ and $\delta \leq \mu_1 - \lambda$, this integral vanishes, since $P_\lambda = 0$ for $\lambda \leq \mu_1 = \mu_1^{(0)}$.

First suppose that $\rho \in [\mu_1,\infty)\setminus\{\mu_1,\mu_2,\ldots\}$ and choose $\delta > 0$ and $j\in\mathbb{N}$ such that $[\rho-\delta,\rho+\delta]\subset(\mu_j,\mu_{j+1})$. From

$$\int_{\rho-\delta}^{\rho+\delta} \frac{1}{\lambda-\rho-\mathrm{i}\tau}\,d\lambda \;=\; \mathrm{i}\pi + O(\tau) \qquad \text{as } \tau\downarrow 0,$$

$$\int_{\rho-\delta}^{\rho+\delta} |\lambda-\rho| \left| \frac{1}{\lambda-\rho-\mathrm{i}\tau} - \frac{1}{\lambda-\rho} \right| d\lambda \;=\; O\!\left(\tau\ln\frac{1}{\tau}\right) \qquad \text{as } \tau\downarrow 0$$

one obtains by (3.20) that

$$I_\delta(\tau,x) \;=\; \mathrm{i}\pi \left[\frac{dP_\lambda f}{d\lambda}(x)\right]_{\lambda=\rho} + \int_{\rho-\delta}^{\rho+\delta} \frac{1}{\lambda-\rho} \left(\frac{dP_\lambda f}{d\lambda}(x) - \left[\frac{dP_\lambda f}{d\lambda}(x)\right]_{\lambda=\rho} \right) d\lambda$$

$$+\, O_M\!\left(\tau\ln\frac{1}{\tau}\right) \qquad \text{as } \tau\downarrow 0.$$

Using

$$\lim_{\varepsilon\downarrow 0} \left(\int_{\rho-\delta}^{\rho-\varepsilon} \frac{1}{\lambda-\rho}\,d\lambda + \int_{\rho+\varepsilon}^{\rho+\delta} \frac{1}{\lambda-\rho}\,d\lambda \right) \;=\; 0,$$

one can rewrite this as

$$I_\delta(\tau,x) \;=\; \mathrm{i}\pi\frac{dP_\rho f}{d\rho}(x) + \lim_{\varepsilon\downarrow 0} \left(\int_{\rho-\delta}^{\rho-\varepsilon} \frac{1}{\lambda-\rho}\,\frac{dP_\lambda f}{d\lambda}(x) + \int_{\rho+\varepsilon}^{\rho+\delta} \frac{1}{\lambda-\rho}\,\frac{dP_\lambda f}{d\lambda}(x) \right)$$

$$+\, O_M\!\left(\tau\ln\frac{1}{\tau}\right) \qquad \text{as } \tau\downarrow 0. \tag{4.10}$$

This together with (4.8) proves (4.2) in the case of arbitrary $l\in\mathbb{N}$, if $\rho\in\mathbb{R}\setminus\{\mu_1,\mu_2,\ldots\}$.

Second suppose that $\rho = \mu_j$ for a fixed $j \in \mathbb{N}$ and choose $\delta > 0$ such that $[\rho - \delta, \rho + \delta] \subset (\mu_{j-1}, \mu_{j+1})$. Note that

$$\int_{\rho-\delta}^{\rho+\delta} |\lambda - \rho|^{\alpha} \left| \frac{1}{\lambda - \rho - \mathrm{i}\tau} - \frac{1}{\lambda - \rho} \right| d\lambda \;=\; O(\tau^{\alpha}) \qquad \text{as } \tau \downarrow 0 \tag{4.11}$$

$(0 < \alpha < 1)$. If $l \geq 3$, then (4.2) is proved in the same way as above using the last equation with $\alpha = \frac{1}{4}$.

In the case $l = 1$, set

$$\Psi(\lambda, x) \;:=\; \begin{cases} \dfrac{dP_\lambda f}{d\lambda}(x) & \text{for } \lambda \in [\rho - \delta, \rho), \\ \lim\limits_{\lambda \uparrow \rho} \dfrac{dP_\lambda f}{d\lambda}(x) & \text{for } \lambda = \rho, \\ \dfrac{dP_\lambda f}{d\lambda}(x) - \dfrac{1}{\sqrt{\lambda - \rho}}\,(P^{(j)} f)(x) & \text{for } \lambda \in (\rho, \rho + \delta] \end{cases} \tag{4.12}$$

for $x \in \overline{\Omega}$. Using

$$\Psi(\lambda, x) - \Psi(\rho, x) \;=\; O_M\Big(\sqrt{|\lambda - \rho|}\Big) \qquad \text{as } \lambda \to \rho$$

(see (3.21) with $\rho = \mu_j$) and (4.11) with $\alpha = \frac{1}{2}$, one obtains as above that

$$\begin{aligned} &\int_{\rho-\delta}^{\rho+\delta} \frac{1}{\lambda - \rho - \mathrm{i}\tau}\, \Psi(\lambda, x)\, d\lambda \;= \\ &= \; \mathrm{i}\pi \Psi(\rho, x) + \lim_{\varepsilon \downarrow 0} \left(\int_{\rho-\delta}^{\rho-\varepsilon} \frac{1}{\lambda - \rho}\, \Psi(\lambda, x)\, d\lambda + \int_{\rho+\varepsilon}^{\rho+\delta} \frac{1}{\lambda - \rho}\, \Psi(\lambda, x)\, d\lambda \right) \\ &\quad + O_M\big(\sqrt{\tau}\big) \\ &= \; \lim_{\varepsilon \downarrow 0} \left(\int_{\rho-\delta}^{\rho-\varepsilon} \frac{1}{\lambda - \rho}\, \frac{dP_\lambda f}{d\lambda}(x)\, d\lambda + \int_{\rho+\varepsilon}^{\rho+\delta} \frac{1}{\lambda - \rho}\, \frac{dP_\lambda f}{d\lambda}(x)\, d\lambda - \frac{2}{\sqrt{\varepsilon}}\,(P^{(j)} f)(x) \right) \\ &\quad + \mathrm{i}\pi \lim_{\lambda \uparrow \rho} \frac{dP_\lambda f}{d\lambda}(x) + \frac{2}{\sqrt{\delta}}\,(P^{(j)} f)(x) + O_M\big(\sqrt{\tau}\big) \end{aligned}$$

as $\tau \downarrow 0$. By

$$\int_{\rho}^{\rho+\delta} \frac{1}{\lambda - \rho - \mathrm{i}\tau}\, \frac{1}{\sqrt{\lambda - \rho}}\, d\lambda \;=\; \frac{\pi(1 + \mathrm{i})}{\sqrt{2}}\, \frac{1}{\sqrt{\tau}} - \frac{2}{\sqrt{\delta}} + O(\tau) \qquad \text{as } \tau \downarrow 0,$$

(see e.g. (6.16), (6.17) in [7]), this implies that

$$\begin{aligned} I_\delta(\tau, x) \;&=\; \int_{\rho-\delta}^{\rho+\delta} \frac{1}{\lambda - \rho - \mathrm{i}\tau}\, \Psi(\lambda, x)\, d\lambda + \int_{\rho}^{\rho+\delta} \frac{1}{\lambda - \rho - \mathrm{i}\tau}\, \frac{1}{\sqrt{\lambda - \rho}}\, d\lambda\, \big(P^{(j)} f\big)(x) \\ &=\; \frac{\pi(1 + \mathrm{i})}{\sqrt{2}}\, \frac{1}{\sqrt{\tau}}\,(P^{(j)} f)(x) + \mathrm{i}\pi \lim_{\lambda \uparrow \rho} \frac{dP_\lambda f}{d\lambda}(x) \\ &\quad + \lim_{\varepsilon \downarrow 0} \left(\int_{\rho-\delta}^{\rho-\varepsilon} \frac{1}{\lambda - \rho}\, \frac{dP_\lambda f}{d\lambda}(x)\, d\lambda + \int_{\rho+\varepsilon}^{\rho+\delta} \frac{1}{\lambda - \rho}\, \frac{dP_\lambda f}{d\lambda}(x)\, d\lambda - \frac{2}{\sqrt{\varepsilon}}(P^{(j)} f)(x) \right) \\ &\quad + O_M\big(\sqrt{\tau}\big) \qquad \text{as } \tau \downarrow 0, \end{aligned} \tag{4.13}$$

which together with (4.8) proves (4.4).

In the case $l = 2$, similar calculations and

$$\int_{\rho}^{\rho+\delta} \frac{1}{\lambda - \rho - \mathrm{i}\tau}\, d\lambda \;=\; \ln\frac{1}{\tau} + \frac{\mathrm{i}\pi}{2} + \ln\delta + O(\tau) \qquad \text{as } \tau \downarrow 0 \tag{4.14}$$

show that

$$\begin{aligned} I_\delta(\tau, x) \;=\; & \ln\left(\frac{1}{\tau}\right)(P^{(j)}f)(x) + \mathrm{i}\pi \lim_{\lambda\uparrow\rho} \frac{dP_\lambda f}{d\lambda}(x) \\ & + \lim_{\varepsilon\downarrow 0}\left(\int_{\rho-\delta}^{\rho-\varepsilon} \frac{1}{\lambda-\rho}\frac{dP_\lambda f}{d\lambda}(x) \int_{\rho+\varepsilon}^{\rho+\delta} \frac{1}{\lambda-\rho}\frac{dP_\lambda f}{d\lambda}(x)\left(\frac{\mathrm{i}\pi}{2} + \ln\varepsilon\right)(P^{(j)}f)(x)\right) \\ & + O_M\big(\tau^{1/4}\big) \qquad \text{as } \tau \downarrow 0. \end{aligned} \tag{4.15}$$

This proves (4.6). □

5 Long-time asymptotics

Consider the solution u of

$$u \;\in\; \bigcap_{j=0}^{2} C^j\big([0,\infty), H^{2m-jm}(\Omega)\big)\,, \tag{5.1}$$

$$u(t) \;\in\; D(\mathcal{A}) \qquad \text{for } t \geq 0, \tag{5.2}$$

$$u''(t) + \mathcal{A}\,u(t) \;=\; f\,\mathrm{e}^{-\mathrm{i}\omega t} \qquad \text{for } t \geq 0, \tag{5.3}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{5.4}$$

given by

$$\begin{aligned} u(t) \;=\; & \int_0^\infty \frac{1}{\lambda - \omega^2}\left(\mathrm{e}^{-\mathrm{i}\omega t} - \cos\sqrt{\lambda}t + \frac{\mathrm{i}\omega}{\sqrt{\lambda}}\sin\sqrt{\lambda}t\right) d(P_\lambda f) \\ & + \int_0^\infty \cos\sqrt{\lambda}t\, d(P_\lambda u_0) + \int_0^\infty \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d(P_\lambda u_1). \end{aligned} \tag{5.5}$$

Standard theory shows, that this solution is unique. If $u_0, u_1, f \in C_0^\infty(\Omega)$ is supposed, then $u \in C^\infty\big([0,\infty)\times\overline{\Omega}\big)$ by elliptic regularity theory, and u solves (1.2). On the other hand, every solution of (1.2) satisfying (5.1) is solution of (5.2) – (5.4). Hence (1.2) has a unique solution satisfying (5.1).

By Theorems 3.2, 3.4 and 4.1, we can estimate the spectral integrals in (5.5) by the same arguments used in [13] and [7] to prove Theorem 1.1.

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