Long-Time Asymptotics for the Wave Equation of Linear Elasticity in Cylindrical Waveguides

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Abstract

Let $\Omega = \mathbb{R} \times D$, $D \subset \mathbb{R}^2$ bounded. Suppose that Ω is filled with a homogeneous and isotropic elastic medium, which is fixed at the boundary, and that a time-harmonic force $f(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$ acts in Ω, where f has bounded support. The resulting elastic motion and its behaviour as $t \to \infty$ is studied. Depending on the choice of ω and f, two different types of time-asymptotic occur: Either the motion is unbounded as $t \to \infty$ at almost every $x \in \Omega$ (resonance case), or the principle of limiting amplitude holds. The resonance is shown to occur for countable many frequencies of incitation. Even two-dimensional elastic motion in a domain $\Omega = \mathbb{R} \times (0,1)$ is considered; here the same phenomena happen. These results are proved using an explicit representation of the motion, which is obtained combinig spectral- and Fourier-transform. The method is presented in a general setting, so that it is applicable also to other translation-invariant wave equations.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be filled with a homogeneous and isotropic elastic medium, which is fixed at the boundary $\partial\Omega$. Suppose that a time-harmonic force $F(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$ (with $F : \Omega \to \mathbb{R}^3$) acts in Ω. Denote by $s(t,x) \in \mathbb{R}^3$ the resulting motion of a point $x \in \Omega$ at time $t \geq 0$. Then s is solution of (see e.g. [1] or [13])

$$\left.\begin{aligned}
&\mu\,\Delta s(t,x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,s(t,x) \\
&\qquad\qquad + \sigma F(t,x)\,\mathrm{e}^{-\mathrm{i}\omega t} = \sigma\,\partial_t^2 s(t,x) \quad \text{for } t \geq 0,\ x \in \Omega, \\
&\qquad\qquad\qquad\qquad s(t,x) = 0 \qquad\qquad \text{for } t \geq 0,\ x \in \partial\Omega, \\
&s(0,x) = s_0(x), \quad \partial_t s(0,x) = s_1(x) \qquad \text{for } x \in \Omega,
\end{aligned}\right\} \tag{1.1}$$

where $\sigma > 0$ (density of mass) and $\mu > 0$, $3\lambda + 2\mu > 0$ (Lamé-constants). More general, we study in a domain $\Omega \subset \mathbb{R}^n$ the solution $u : [0,\infty) \times \Omega \to \mathbb{R}^n$ of the initial-boundary-value problem

$$\left.\begin{aligned}
\partial_t^2 u(t,x) - (\Delta + c_0\,\mathrm{grad}\,\mathrm{div})\,u(t,x) &= f(x)\,\mathrm{e}^{-\mathrm{i}\omega t} && \text{for } t \geq 0,\ x \in \Omega, \\
u(t,x) &= 0 && \text{for } t \geq 0,\ x \in \partial\Omega, \\
u(0,x) = u_0(x), \quad \partial_t u(0,x) &= u_1(x) && \text{for } x \in \Omega,
\end{aligned}\right\} \tag{1.2}$$

where $c_0 > -1$ is supposed. The last condition $c_0 > -1$ is needed to guarantee the coerciveness of the spatial operator. It is satisfied in all physically relevant situations, since $c_0 = 1 + \frac{\lambda}{\mu}$. Let Ω be given by

$$\Omega = \mathbb{R} \times D, \quad D \subset \mathbb{R}^{n-1} \text{ bounded.} \tag{1.3}$$

In the case $n = 3$, if D is connected, then Ω is an infinite tube with cross-section D. If $n = 2$ and $D = (a,b)$ with $-\infty < a < b < \infty$, then Ω is a layer.

We are interested in the asymptotic behaviour as $t \to \infty$ of u solving (1.2). In order to give a brief description of the results, we suppose that $\partial D \in C^\infty$. It will be proved that there exists a set $\omega_{\mathrm{res}} \subset \mathbb{R}$ of resonance frequencies, which is countable and has no finite accumulation point. If the frequency ω of incitation coincides with a resonance frequency, then resonance occurs: There exist $N \in \mathbb{N}$,

$\alpha_1, \dots, \alpha_{N-1} \in (0,1)$ and $f_1, \dots, f_N, u_\omega \in C(\overline{\Omega})$, such that

$$u(t,x) \;=\; \sum_{j=1}^{N-1} \mathrm{e}^{-\mathrm{i}\omega t}\, t^{\alpha_j}\, f_j(x) + \mathrm{e}^{-\mathrm{i}\omega t}\, \ln t \cdot f_N(x) + \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega(x) + o(1) \tag{1.4}$$

as $t \to \infty$, where the precise definition of the meaning of $o(1)$ is given in Theorem 5.2. The functions $f_1, \dots, f_N$ depend on f and can be computed together with the values of $\alpha_1, \dots, \alpha_{N-1} \in (0,1)$ solving a parameter-dependent eigenvalue problem in D. This will be done for a special example of D in a subsequent paper. The present article shows, that at least one of $f_1, \dots, f_N$ does not vanish, if f is chosen suitable, and that the corresponding exponent α_j is element of $[\frac{1}{2}, 1)$.

If $\omega \in [0,\infty) \setminus \omega_{\text{res}}$, then u satisfies the principle of limiting amplitude

$$u(t,x) \;=\; \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega(x) + o(1) \qquad \text{as } t \to \infty, \tag{1.5}$$

where $u_\omega \in C^2(\overline{\Omega})$ solves

$$\left.\begin{aligned} (-\Delta - c_0 \operatorname{grad}\operatorname{div} - \omega^2) u_\omega(x) \;&=\; f(x) \quad \text{in } \Omega, \\ u(x) \;&=\; 0 \qquad\;\; \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.6}$$

These results are obtained using a method, which was developed in [12] and used in [8]. It is based on an explicit representation of the spectral family of the spatial operator in (1.2), which is computed using Fourier-transform with repect to the unbounded variable. This article presents the method in a more general setting, so that it can be applied to other wave equations being translation invariant.

In order to be more precise, let $\mathcal{H}$ be a Hilbert-space given by

$$\mathcal{H} \;:=\; L_2(\mathbb{R}, H), \qquad H = \text{separable Hilbert-space} \tag{1.7}$$

(e.g. $H = L_2(D)^n$, $\mathcal{H} = L_2(\Omega)^n$). The Fourier-transform with respect to the first variable is defined by

$$\mathcal{F} := F \otimes \mathrm{Id} : \mathcal{H} = L_2(\mathbb{R}) \otimes H \to \mathcal{H} \qquad (\otimes = \text{tensor product}), \tag{1.8}$$

with $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ and $\mathrm{Id} : H \to H$ denoting respectively usual Fourier-transform and identity. Let $\mathcal{A}$ be an operator in $\mathcal{H}$, given by

$$\mathcal{A} \;=\; \mathcal{F}^{-1} \circ \int_{\mathbb{R}}^{\oplus} A(\xi)\, d\xi \circ \mathcal{F}, \tag{1.9}$$

where $\{A(\xi)\}_{\xi \in \mathbb{R}}$ denotes a family of self-adjoint operators in H (for the notation see section XIII.16 of [14]). Roughly spoken, (1.9) means that $A(\xi)$ defines the Fourier-transform of $\mathcal{A}$ by

$$\big(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} f\big)(\xi) \;=\; A(\xi)\, f(\xi) \qquad \text{for almost every } \xi \in \mathbb{R},$$

if $f \in D(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1}) \subset L_2(\mathbb{R}, H)$. The operator family is supposed to satisfy the following assumptions:

(A1) For every fixed $\xi \in \mathbb{R}$, $A(\xi)$ is self-adjoint and positive in H.

(A2) For every fixed $\xi \in \mathbb{R}$, $A(\xi)$ has an orthonormal system of eigenfunctions $\{v_j(\xi)\}_{j \in \mathbb{N}}$ being complete in H. The corresponding eigenvalues are denoted by $\lambda_j(\xi)$, where every eigenvalue eventually has to be counted multiple times according to his multiplicity, which is supposed to be finite.

(A3) For every fixed $j \in \mathbb{N}$, $\lambda_j(\xi)$ and $v_j(\xi)$ depend analytically on ξ.

(A4) For every $\lambda \in \mathbb{R}$, the set $\{(j, \xi) \in \mathbb{N} \times \mathbb{R} : \lambda_j(\xi) = \lambda\}$ is empty or finite, and for every fixed $j \in \mathbb{N}$, $\lambda_j(\xi) \to +\infty$ as $\xi \to \pm\infty$.

Then $\mathcal{A}$ is self-adjoint in $\mathcal{H}$ (see [14], sec. XIII.16).

Section 2 studies the spectral family of $\mathcal{A}$. The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists of one infinite interval and is purely absolutely continuous. We say that $\lambda \in \sigma(\mathcal{A})$ is a resonance point, if and only if

$$\exists (j, \xi) \in \mathbb{N} \times \mathbb{R} : \lambda = \lambda_j(\xi) \;\wedge\; \frac{d\lambda_j}{d\xi}(\xi) = 0. \tag{1.10}$$

The set of all resonance points is countable and has no finite accumulation point. The spectral family $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ of $\mathcal{A}$ will be shown to have a Hölder-continuous derivative with respect to λ (in a certain sense) at every $\lambda \in \mathbb{R}$ being no resonance point. If $\lambda_0 \in \mathbb{R}$ is a resonance point, then the derivative with respect to λ of the spectral family has a singularity of type $|\lambda - \lambda_0|^{-\alpha}$ with $\alpha \in (0,1)$. For a more precise statement see (2.15).

Section 3 studies the solution of the initial value problem

$$\left.\begin{array}{rcll} u \in C^2([0,\infty),\mathcal{H}), \quad u(t) & \in & D(\mathcal{A}) & \text{for } t \geq 0. \\ u''(t) + \mathcal{A}\,u(t) & = & f\,\mathrm{e}^{-\mathrm{i}\omega t} & \text{for } t \geq 0, \\ u(0) = u_0, \quad u'(0) & = & u_1, & \end{array}\right\} \tag{1.11}$$

where u' denotes the derivative of u. The solution u is given by spectral integrals. Theorem 3.2 shows the connection between the long-time behaviour of u and the properties of the spectral family $\{P_\lambda f\}$ at $\lambda = \omega^2$ and at $\lambda = 0$. If ω^2 is a resonance point and if f is chosen suitable, then u shows resonance. If ω^2 is no resonance point and if the behaviour of the spectral family near $\lambda = 0$ is not too bad, then the principle of limiting amplitude holds.

Section 4 studies spectral properties of the elastic spatial operator. It is shown that the theory developed in Sections 2 and 3 can be applied to problem (1.2). In section 5, the long-time behaviour of the solution u of (1.2) is computed. Application of the general theory yields estimates with respect to some weighted norms. Using elliptic regularity theory, pointwise estimates of $u(t,x)$ as $t \to \infty$ are proved.

Equation (1.1) with other (non homogeneous) boundary conditions in a domain $\Omega = \mathbb{R}^2 \times (0,h)$ is considered in the modelling of seismic waves. In this context, L. Brevdo obtained similar resonance effects in [3], [4] and [5]. He studies the solution using spatial Fourier-transform and Laplace-transform with respect to time. This method was developed by R. Briggs in [6]. The theory in this book leads to the same definition of resonance points as (1.10). The considerations there are not restricted to self-adjoint spatial operators. But one has to pay for this generality: It has to be distinguished between resonance points leading to convective instabilities and those leading to absolute instabilities. The result of the present paper shows in the case of a self-adjoint spatial operator, that at every resonance point an absolute instability occurs.

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2 Spectral properties of $\mathcal{A}$

We study the spectral family $\{P_\lambda\}$ of the self-adjoint operator $\mathcal{A}$ given by (1.9). The main result in this section consists of an explicit representation of the spectral family (see Theorem 2.3).

Suppose that the operator family $\{A(\xi)\}_{\xi\in\mathbb{R}}$ satisfies Assumptions (A1) – (A4) given on page 4. Set

$$\lambda_{\min} \;:=\; \min\left\{\lambda_j(\xi) : j \in \mathbb{N},\ \xi \in \mathbb{R}\right\}. \tag{2.1}$$

From (A3) and (A4), one obtains that the given set has a minimum. Furthermore, (A1) implies that $\lambda_{\min} \geq 0$. Define the set of resonance points by

$$\sigma_{\text{res}}(\mathcal{A}) \;:=\; \left\{\lambda_j(\xi) : j \in \mathbb{N} \;\wedge\; \xi \in \mathbb{R} \;\wedge\; \frac{d\lambda_j}{d\xi}(\xi) = 0\right\}. \tag{2.2}$$

By (A3) and (A4), $\sigma_{\text{res}}(\mathcal{A})$ is countable infinite and has no finite accumulation point. Obviously $\lambda_{\min} \in \sigma_{\text{res}}(\mathcal{A})$. Write $\sigma_{\text{res}}(\mathcal{A})$ as

$$\sigma_{\text{res}}(\mathcal{A}) \;=\; \{\sigma_1, \sigma_2, \dots\} \quad \text{with} \quad \lambda_{\min} = \sigma_1 < \sigma_2 < \dots,\ \sigma_j \to \infty \text{ as } j \to \infty. \tag{2.3}$$

In order to define local inverses of the mapping $\xi \mapsto \lambda_j(\xi)$, let $j \in \mathbb{N}$ be fixed. By the analycity of λ_j, the set

$$S_j \;:=\; \left\{\xi \in \mathbb{R} : \frac{d\lambda_j}{d\xi}(\xi) = 0\right\}$$

is finite or countable with no finite accumulation point. Hence one can write

$$S_j \;=\; \{\rho_{jk} : k \in \mathbb{I}_j\} \quad \text{with} \quad \rho_{jk} < \rho_{j(k+1)} \text{ if } k, k+1 \in \mathbb{I}_j,$$

where $\mathbb{I}_j = \{1, \ldots, N\}$ or $\mathbb{I}_j = \mathbb{N}$ or $\mathbb{I}_j = \mathbb{Z}$. Define intervals I_{jk} by

$$I_{jk} := \begin{cases} (-\infty, \rho_{j(k+1)}] & \text{if } k \notin \mathbb{I}_j,\ k+1 \in \mathbb{I}_j, \\ [\rho_{jk}, \rho_{j(k+1)}] & \text{if } k, k+1 \in \mathbb{I}_j, \\ [\rho_{jk}, \infty) & \text{if } k \in \mathbb{I}_j,\ k+1 \notin \mathbb{I}_j \end{cases} \tag{2.4}$$

for $k \in \tilde{\mathbb{I}}_j$, where $\tilde{\mathbb{I}}_j := \mathbb{I}_j \cup \{\min\{k \in \mathbb{I}_j\} - 1\}$ if $\mathbb{I}_j$ is bounded from below, and $\tilde{\mathbb{I}}_j := \mathbb{I}_j$ otherwise. Then $\{I_{jk}\}_{k \in \tilde{\mathbb{I}}_j}$ defines a decomposition of $\mathbb{R}$ having the property, that the restriction $\lambda_j : I_{jk} \to \lambda_j(I_{jk})$ is one-to-one for every $k \in \tilde{\mathbb{I}}_j$. Let $r_{jk} : \lambda_j(I_{jk}) \to I_{jk}$ denote the inverse mapping. Note that the domain of definition of r_{jk} consists of a closed interval. By the analycity of λ_j, r_{jk} is C^∞ in the interior of $\lambda_j(I_{jk})$. Define $K(\lambda)$ by

$$K(\lambda) := \{(j,k) \in \mathbb{N} \times \mathbb{N} : \lambda \in \lambda_j(I_{jk})\} \qquad \text{for } \lambda \in \mathbb{R}. \tag{2.5}$$

This set is empty, if $\lambda < \lambda_{\min}$. Otherwise, $K(\lambda)$ contains all pairs $(j,k) \in \mathbb{N} \times \mathbb{N}$ having the property, that λ is in the domain of definition of r_{jk}. According to (A4), $K(\lambda)$ is finite for every $\lambda \geq \lambda_{\min}$. Remember that $\sigma_{\text{res}}(\mathcal{A}) = \{\sigma_1, \sigma_2, \ldots\}$ and note that $K(\lambda)$ is constant on (σ_p, σ_{p+1}) for every $p \in \mathbb{N}$. For every $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$ set

$$\left.\begin{aligned} K^+(\sigma_p) &:= \left\{(j,k) \in K(\lambda) \text{ with } \lambda \in (\sigma_p, \sigma_{p+1}) : \frac{d\lambda_j}{d\xi}(r_{jk}(\sigma_p)) = 0\right\}, \\ K^-(\sigma_p) &:= \left\{(j,k) \in K(\lambda) \text{ with } \lambda \in (\sigma_{p-1}, \sigma_p) : \frac{d\lambda_j}{d\xi}(r_{jk}(\sigma_p)) = 0\right\} \ (p \neq 1), \\ K^-(\sigma_1) &:= \emptyset, \end{aligned}\right\} \tag{2.6}$$

$$N(j,k,p) := \min\left\{l \geq 2 : \frac{d^l\lambda_j}{d\xi^l}(r_{jk}(\sigma_p)) \neq 0\right\} \qquad \text{for } (j,k) \in K^+(\sigma_p) \cup K^-(\sigma_p). \tag{2.7}$$

Lemma 2.1 **1)** *If $\sigma_p, \sigma_{p+1} \in \sigma_{\text{res}}(\mathcal{A})$ and $(j,k) \in K(\lambda)$ for $\lambda \in (\sigma_p, \sigma_{p+1})$, then $r_{jk} \in C^\infty(\sigma_p, \sigma_{p+1})$.*

2) *If $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$, $p \geq 2$, and if $(j,k) \in K(\sigma_p) \setminus (K^+(\sigma_p) \cup K^-(\sigma_p))$, then $r_{jk} \in C^\infty(\sigma_{p-1}, \sigma_{p+1})$. Furthermore $K(\sigma_1) = K^+(\sigma_1)$.*

3) *If $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$ and $(j,k) \in K^+(\sigma_p)$, then there exist constants $c_{jkpq} \in \mathbb{R}$, $c_{jkp1} \neq 0$ and $\delta > 0$ such that*

$$\frac{dr_{jk}}{d\lambda}(\lambda) \;=\; \sum_{q=1}^{\infty} \frac{c_{jkpq}}{|\lambda - \sigma_p|^{1-q/N(j,k,p)}} \tag{2.8}$$

for $\lambda \in (\sigma_p, \sigma_p + \delta)$. If $(j,k) \in K^-(\sigma_p)$, then (2.8) holds for $\lambda \in (\sigma_p - \delta, \sigma_p)$ with suitable chosen constants $c_{jkpq} \in \mathbb{R}$, $c_{jkp1} \neq 0$ and $\delta > 0$.

4) *If $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$ and $(j,k) \in K^+(\sigma_p) \cup K^-(\sigma_p)$, then either $(j,k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)$ or $(j,k-1) \in K^+(\sigma_p) \cup K^-(\sigma_p)$. If $(j,k), (j,k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)$, then the coefficients in (2.8) satisfy*

$$c_{jkp1} \;=\; (-1)^{N(j,k,p)+1}\, c_{j(k+1)p1}. \tag{2.9}$$

Proof: Note that $\sigma_{\text{res}}(\mathcal{A}) = \bigcup_{j=1}^{\infty} \lambda_j(S_j)$. If $(j,k) \in K(\lambda)$ for $\lambda \in (\sigma_p, \sigma_{p+1})$, then (σ_p, σ_{p+1}) is subset of the interior of $\lambda_j(I_{jk})$. This implies $r_{jk} \in C^\infty(\sigma_p, \sigma_{p+1})$ by the considerations made before the lemma.

If $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$, $p \geq 2$, and if $(j,k) \in K(\sigma_p) \setminus (K^+(\sigma_p) \cup K^-(\sigma_p))$, then $(\sigma_{p-1}, \sigma_{p+1})$ is subset of the interior of $\lambda_j(I_{jk})$ and $r_{jk} \in C^\infty(\sigma_{p-1}, \sigma_{p+1})$ follows as before. Assume that $(j,k) \in K(\sigma_1)$. Since $\lambda_j(r_{jk}(\sigma_1)) = \sigma_1$ and $\lambda_j(\xi) \geq \sigma_1 = \lambda_{\min}$ for $\xi \in \mathbb{R}$, $\frac{d\lambda_j}{d\xi}(r_{jk}(\sigma_1))$ has to vanish. Hence $(j,k) \in K^+(\sigma_1)$. On the other hand, $K^+(\sigma_1) \subset K(\sigma_1)$ holds by definition.

If $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$ and $(j,k) \in K^+(\sigma_p) \cup K^-(\sigma_p)$, then $r_{jk}(\sigma_p)$ is a boundary point of I_{jk}. This implies that $\sigma_p = \lambda_j(r_{jk}(\sigma_p))$ is a boundary point of $\lambda_j(I_{jk})$, the domain of definition of r_{jk}. According to [2], §21, r_{jk} can be represented by a puiseux-series

$$r_{jk}(\lambda) \;=\; r_{jk}(\sigma_p) + \sum_{q=1}^{\infty} \tilde{c}_{jkpq}\, |\lambda - \sigma_p|^{q/N(j,k,p)}$$

for $\lambda \in \lambda_j(I_{jk}) \cap (\sigma_p - \delta, \sigma_p + \delta)$ with suitable chosen $\tilde{c}_{jkpq} \in \mathbb{R}$ and $\delta > 0$. Hence (2.8) holds. Since $\sigma_p = \lambda_j(r_{jk}(\sigma_p))$ is a boundary point of $\lambda_j(I_{jk})$, σ_p is either a boundary point of $\lambda_j(I_{j(k-1)})$ or of $\lambda_j(I_{j(k+1)})$. This implies that either $(j,k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)$ or $(j,k-1) \in K^+(\sigma_p) \cup K^-(\sigma_p)$. Suppose that $(j,k), (j,k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)$. Then $r_{jk}(\sigma_p) = r_{j(k+1)}(\sigma_p)$ and

$N(j,k,p) = N(j,k+1,p)$. Relation (2.9) follows inserting the puiseux-series representation of r_{jk} and $r_{j(k+1)}$ into the Taylor expansion of λ_j at $\xi = r_{jk}(\sigma_p)$ and using that $\lambda_j(r_{jk}(\lambda)) = \lambda$ for $\lambda \in \lambda_j(I_{jk}) \cup \lambda_j(I_{j(k+1)})$. $\square$

For $s \in \mathbb{R}$ define the Banach space $\mathcal{H}_s$ by

$$\left.\begin{aligned} \|\varphi\|_s &:= \left(\int_{\mathbb{R}} \left(1+|x|^2\right)^s \|\varphi(x)\|_H^2 \, dx\right)^{1/2}, \\ \mathcal{H}_s &:= L_{2,s}(\mathbb{R},H) := \{\varphi \in L_{2,\mathrm{loc}}(\mathbb{R},H) : \|\varphi\|_s < \infty\} \end{aligned}\right\} \tag{2.10}$$

Lemma 2.2 *Suppose that $s, s' > \frac{1}{2}$ and $f \in \mathcal{H}_s$. Set*

$$\psi_j(\xi)(x) := \mathrm{e}^{\mathrm{i}\xi x} \langle (Ff)(\xi), v_j(\xi)\rangle_H \, v_j(\xi) \qquad \textit{for } \xi, x \in \mathbb{R}, \ j \in \mathbb{N}. \tag{2.11}$$

Then $\psi_j(\xi) \in \mathcal{H}_{-s'}$ for $\xi \in \mathbb{R}$ and $\psi_j \in C^{k,\alpha}(\mathbb{R}, \mathcal{H}_{-s'})$ with $k \in \mathbb{N}_0$, $k < \min\{s,s'\} - \frac{1}{2}$ and $\alpha := \min\left\{s-k-\frac{1}{2}, s'-k-\frac{1}{2}, 1\right\}$.

Proof: If $v \in H$, then

$$\langle (\mathcal{F}f)(\xi), v\rangle_H = F\left(\langle f(.), v\rangle_H\right)(\xi)$$

(see (1.8)). The mapping $\xi \mapsto F(\langle f(.), v\rangle_H)(\xi)$ is element of $H^s(\mathbb{R})$, since $f \in \mathcal{H}_s$. By Sobolev's Lemma, $H^s(\mathbb{R}) \subset C^{k,\alpha}(\mathbb{R})$. Since v_j depends analytically on ξ by Assumption (A3), the mapping $\xi \mapsto \langle (\mathcal{F}f)(\xi), v_j(\xi)\rangle_H \, v_j(\xi)$ is element of $C^{k,\alpha}(\mathbb{R}, H)$.

Note that $\mathcal{H}_{-s'}$ is adjoint to $\mathcal{H}_{s'}$ with respect to the inner product in $\mathcal{H}$. If $g \in \mathcal{H}_{s'}$, then

$$\begin{aligned} \langle \psi_j(\xi), g\rangle_{\mathcal{H}} &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi x} \langle (\mathcal{F}f)(\xi), v_j(\xi)\rangle_H \, \langle v_j(\xi), g(x)\rangle_H \, dx \\ &= \sqrt{2\pi} \, \langle (\mathcal{F}f)(\xi), v_j(\xi)\rangle_H \, \langle v_j(\xi), (\mathcal{F}g)(\xi)\rangle_H \, . \end{aligned}$$

Hence $\psi_j \in C^{k,\alpha}(\mathbb{R}, \mathcal{H}_{-s'})$ follows from the first part of the proof. $\square$

Theorem 2.3 *Let $\mathcal{H}$ be a Hilbert-space having the representation (1.7). Suppose that the self-adjoint operator $\mathcal{A}$ is given by (1.9) and that the associated operator family satisfies Assumptions (A1) – (A4) (see page 4). Denote by $\{P_\lambda\}$ the (left-hand continuous) spectral family of $\mathcal{A}$. Then:*

1) *The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is purely absolutely continuous and $\sigma(\mathcal{A}) = [\lambda_{\min}, \infty)$, where $\lambda_{\min}$ is defined by (2.1). In particular, $\mathcal{A}$ has no eigenvalues and $P_\lambda = 0$ for $\lambda \le \lambda_{\min}$.*

2) *Let $s, s' > \frac{1}{2}$ and $f \in \mathcal{H}_s$ be fixed (for the definition of $\mathcal{H}_s$ see (2.10)). Then*

$$P_\lambda f \;=\; \frac{1}{\sqrt{2\pi}} \int_{\lambda_{\min}}^{\lambda} \sum_{(j,k)\in K(\mu)} \left| \frac{dr_{jk}}{d\mu}(\mu) \right| \psi_j(r_{jk}(\mu))\, d\mu \tag{2.12}$$

for $\lambda > \lambda_{\min}$. (For the definition of ψ_j, $K(\mu)$ and r_{jk} see respectively (2.11), (2.5) and the text above (2.5)). Furthermore:

(a) Consider the mapping $P_{\cdot} f : \mathbb{R} \to \mathcal{H}_{-s'} : \lambda \mapsto P_\lambda f$. For every $\sigma_p, \sigma_{p+1} \in \sigma_{\text{res}}(\mathcal{A})$ (see (2.3)),

$$P_{\cdot} f \;\in\; C^{1,\alpha}((\sigma_p, \sigma_{p+1}), \mathcal{H}_{-s'}) \qquad \text{with } \alpha := \min\{s - \tfrac{1}{2}, s' - \tfrac{1}{2}, 1\}. \tag{2.13}$$

(b) If $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$, set

$$N_p \;:=\; \max\left\{ N(j,k,p) : (j,k) \in K^+(\sigma_p) \cup K^-(\sigma_p) \right\} \tag{2.14}$$

and suppose that $s, s' > N_p - \frac{1}{2}$ and $f \in \mathcal{H}_s$ are fixed. Then there exist bounded operators $Q_{jkpl} : \mathcal{H}_s \to \mathcal{H}_{-s'}$, such that

$$\left\| \frac{dP_\lambda f}{d\lambda}(\lambda) - \sum_{(j,k)\in K^+(\sigma_p)} \sum_{l=1}^{N(j,k,p)} \frac{1}{|\lambda - \sigma_p|^{1 - l/N(j,k,p)}}\, Q_{jkpl}(f) \right\|_{-s'} = O\!\left(|\lambda - \sigma_p|^{\alpha/N} \right) \tag{2.15}$$

as $\lambda \downarrow \sigma_p$ with $\alpha := \min\left\{ s - N_p + \frac{1}{2}, s' - N_p + \frac{1}{2}, 1 \right\}$. The same estimate holds as $\lambda \uparrow \sigma_p$, if $K^+(\sigma_p)$ is replaced by $K^-(\sigma_p)$. Furthermore,

$$Q_{jkp1}(f)(x) = \frac{1}{\sqrt{2\pi}} |c_{jkp1}|\, \mathrm{e}^{\mathrm{i}x\, r_{jk}(\sigma_p)} \left\langle (\mathcal{F}f)(r_{jk}(\sigma_p)), v_j(r_{jk}(\sigma_p)) \right\rangle_H v_j(r_{jk}(\sigma_p)) \tag{2.16}$$

for $x \in \mathbb{R}$, where c_{jkp1} denotes the constant of the first summand in (2.8).

Proof: Set

$$\widehat{\mathcal{A}} := \int_{\mathbb{R}}^{\oplus} A(\xi)\, d\xi.$$

According to Theorem XIII.85 in [14], $\widehat{\mathcal{A}}$ is self-adjoint in $\mathcal{H}$. Denote the (left-hand continuous) spectral family in of $\widehat{\mathcal{A}}$ by $\{\widehat{P}_\lambda\}_{\lambda\in\mathbb{R}}$. Consider a fixed $\lambda \in \mathbb{R}$ and set $G(t) := 1$ if $t \le \lambda$, $G(t) := 0$ if $t > \lambda$. Theorem XIII.85 in [14] shows that

$$\widehat{P}_\lambda \;=\; G(\widehat{\mathcal{A}}) \;=\; \int_{\mathbb{R}}^{\oplus} G(A(\xi))\, d\xi \;=\; \int_{\mathbb{R}}^{\oplus} P_\lambda^{(\xi)}\, d\xi$$

with $\{P_\lambda^{(\xi)}\}$ denoting the spectral family of $A(\xi)$ for $\xi \in \mathbb{R}$. If $g \in \mathcal{H}$, then

$$\left(\widehat{P}_\lambda\, g\right)(\xi) \;=\; P_\lambda^{(\xi)} g(\xi) \;=\; \sum_{j\in\{j\in\mathbb{N}:\lambda_j(\xi)<\lambda\}} \langle g(\xi), v_j(\xi)\rangle_H\, v_j(\xi)$$

for every $\xi \in \mathbb{R}$ according to Assumption (A2). Note that $\mathcal{A} = \mathcal{F}^{-1}\widehat{\mathcal{A}}\mathcal{F}$ implies that $P_\lambda = \mathcal{F}^{-1}\widehat{P}_\lambda\mathcal{F}$ for $\lambda \in \mathbb{R}$. Fix $f \in \mathcal{H}_s$, where $s > \frac{1}{2}$. Then

$$(\mathcal{F}P_\lambda f)(\xi) \;=\; \left(\widehat{P}_\lambda\, \mathcal{F}f\right)(\xi) \;=\; \sum_{j\in\{j\in\mathbb{N}:\lambda_j(\xi)<\lambda\}} \langle(\mathcal{F}f)(\xi), v_j(\xi)\rangle_H\, v_j(\xi)\,.$$

The right-hand side consists of a finite sum, has bounded support with respect to ξ (see (A4)) and is a continuous mapping from $\mathbb{R}$ into H (see the proof of Lemma 2.2). Set $L(\lambda) := \{j \in \mathbb{N} : \lambda_j(\xi) < \lambda$ for at least one $\xi \in \mathbb{R}\}$. By (A4), $L(\lambda)$ is empty or finite for every $\lambda \in \mathbb{R}$. Application of $\mathcal{F}^{-1}$ onto the last equation yields that

$$P_\lambda f \;=\; \frac{1}{\sqrt{2\pi}} \sum_{j\in L(\lambda)} \int_{\{\xi\in\mathbb{R}:\lambda_j(\xi)\le\lambda\}} \psi_j(\xi)\, d\xi,$$

where ψ_j depends continuously on ξ (see Lemma 2.2). Obviously $P_\lambda f = 0$ if $\lambda \le \lambda_{\min}$. If $\lambda > \lambda_{\min}$ and $j \in L(\lambda)$,

$$\{\xi \in \mathbb{R} : \lambda_j(\xi) \le \lambda\} \;=\; \bigcup_{k\in\tilde{\mathbb{I}}_j} \left(I_{jk} \cap \{\xi \in \mathbb{R} : \lambda_j(\xi) \le \lambda\}\right)$$

with I_{jk} being defined by (2.4). Note that the sets on the right-hand side have disjoint interiors. Furthermore, $I_{jk} \cap \{\xi \in \mathbb{R} : \lambda_j(\xi) \le \lambda\} \neq \emptyset$ only for finitely many values of k. Substitution $\xi = r_{jk}(\mu)$ yields that

$$\int_{I_{jk}\cap\{\xi\in\mathbb{R}:\lambda_j(\xi)\le\lambda\}} \psi_j(\xi)\,d\xi \;=\; \begin{cases} \displaystyle\int_{\lambda_j(I_{jk})\cap[\lambda_{\min},\lambda]} \psi_j(r_{jk}(\mu)) \left|\frac{dr_{jk}}{d\mu}(\mu)\right| d\mu \\ \qquad\qquad\qquad \text{if } (j,k) \in \tilde{K}(\lambda) := \bigcup_{\mu\le\lambda} K(\mu), \\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

since the integrand on the right-hand side is continuous in the interior of $\lambda_j(I_{jk})$ and has integrable singularities at the boundary points of $\lambda_j(I_{jk})$ according to (2.8). This implies that

$$\begin{aligned} P_\lambda f &= \frac{1}{\sqrt{2\pi}} \sum_{j\in L(\lambda)} \sum_{k\in\tilde{\mathbb{I}}_j} \int_{I_{jk}\cap\{\xi\in\mathbb{R}:\lambda_j(\xi)\le\lambda\}} \psi_j(\xi)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}} \sum_{(j,k)\in\tilde{K}(\lambda)} \int_{\lambda_j(I_{jk})\cap[\lambda_{\min},\lambda]} \psi_j(r_{jk}(\mu)) \left|\frac{dr_{jk}}{d\mu}(\mu)\right| d\mu \end{aligned}$$

Now (2.12) follows. From (2.12) together with Lemmata 2.1 and 2.2 one obtains that (2.13) holds.

Representation (2.12) shows that $\sigma(\mathcal{A})$ is purely absolutely continuous. For every $\lambda \in [\lambda_{\min}, \infty) \setminus \sigma_{\text{res}}(\mathcal{A})$, there exists $f \in \mathcal{H}_s$ such that $\frac{dP_\lambda f}{d\lambda}(\lambda) \ne 0$. This together with $P_\lambda = 0$ for $\lambda \le \lambda_{\min}$ proves that $\sigma(\mathcal{A}) = [\lambda_{\min}, \infty)$.

If the assumptions of case 2b) are satisfied, Lemma 2.2 implies that

$$\left\| \psi_j(\xi) - \sum_{l=0}^{N_p-1} \frac{1}{l!} \frac{d^l\psi_j}{d\xi^l}(r_{jk}(\sigma_p))\,(\xi - r_{jk}(\sigma_p))^l \right\|_{-s'} \;=\; O\Big(|\xi - r_{jk}(\sigma_p)|^{N_p-1+\alpha}\Big)$$

as $\xi \to r_{jk}(\sigma_p)$, where

$$f \;\mapsto\; \frac{d^l\psi_j}{d\xi^l}(r_{jk}(\sigma_p)) :\; \mathcal{H}_s \to \mathcal{H}_{-s'}$$

are bounded linear operators according to Lemma 2.2. Inserting (2.8) and the corresponding puiseux-series representation of r_{jk} at σ_p in the preceding estimate yields (2.15) and (2.16). □

3 Initial-value problems

We study the solution u of (1.11), where $\mathcal{A}$ denotes a positive self-adjoint operator in a Hilbert-space $\mathcal{H}$. The long time behaviour of u is given by Theorem 3.2 and Corollary 3.3.

Denote by $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ the (left-hand continuous) spectral family of $\mathcal{A}$. Then $P_\lambda = 0$ if $\lambda \le 0$. For $r > 0$, set

$$\left.\begin{aligned} D(\mathcal{A}^r) &:= \{\varphi \in \mathcal{H} : \int_0^\infty \lambda^{2r}\, d\big(\|P_\lambda\varphi\|^2\big) < \infty\}, \\ \mathcal{A}^r\,\varphi &:= \int_0^\infty \lambda^r\, d(P_\lambda\varphi) \qquad \text{if } \varphi \in D(\mathcal{A}^r). \end{aligned}\right\} \tag{3.1}$$

Theorem 3.1 *Let $\mathcal{A}$ be a self-adjoint positive operator in the Hilbert-space $\mathcal{H}$. Then:*

1) *There is at most one solution u of the initial-value problem (1.11).*

2) *If $\omega \ge 0$ and $u_0 \in D(\mathcal{A}^{k/2})$, $u_1 \in D(\mathcal{A}^{(k-1)/2})$, $f \in D(\mathcal{A}^{(k-2)/2})$ for some $k \in \mathbb{N}$, $k \ge 2$, then (1.11) has a solution u. Furthermore,*

$$\left.\begin{aligned} \mathcal{A}^{(k-j)/2}\,u &\in C^j([0,\infty),\mathcal{H}) \quad (j \le k), \\ \frac{d^j(\mathcal{A}^{i/2}u)}{dt^j} &= \mathcal{A}^{i/2}\left(\frac{d^j u}{dt^j}\right) \quad (j+i \le k). \end{aligned}\right\} \tag{3.2}$$

Proof: If u solves (1.11) with vanishing data, then $\frac{d}{dt}\left(\|u'(t)\|^2 + \langle \mathcal{A}u(t), u(t)\rangle\right) = 0$ for $t > 0$. This implies that $u = 0$.

By functional calculus, the solution of (1.11) is given by

$$\begin{aligned} u(t) &= \int_0^\infty \psi_\omega(t,\lambda)\, d(P_\lambda f) \\ &\quad + \int_0^\infty \cos\sqrt{\lambda}t\, d(P_\lambda u_0) + \int_0^\infty \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d(P_\lambda u_1), \end{aligned} \tag{3.3}$$

where in the case $\omega > 0$

$$\psi_\omega(t,\lambda) \;=\; \begin{cases} \dfrac{1}{\lambda-\omega^2}\left(\mathrm{e}^{-\mathrm{i}\omega t} - \cos\sqrt{\lambda}t + \dfrac{\mathrm{i}\omega}{\sqrt{\lambda}}\sin\sqrt{\lambda}t\right) & \text{if } \lambda \in \mathbb{R}\setminus\{\omega^2, 0\}, \\ \dfrac{\mathrm{i}}{2\omega}\left(t\,\mathrm{e}^{-\mathrm{i}\omega t} - \dfrac{1}{\omega}\sin\omega t\right) & \text{if } \lambda = \omega^2, \\ \dfrac{1}{\omega^2}\left(1 - \mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{i}\omega t\right) & \text{if } \lambda = 0, \end{cases} \tag{3.4}$$

and

$$\psi_0(t,\lambda) \;=\; \begin{cases} \dfrac{1}{\lambda}\left(1 - \cos\sqrt{\lambda}t\right) & \text{if } \lambda \in \mathbb{R}\setminus\{0\}, \\ \dfrac{t^2}{2} & \text{if } \lambda = 0. \end{cases} \tag{3.5}$$

From this (3.2) follows by standard calculations. □

Note that $\psi_\omega(t,\omega^2)$ and $\psi_\omega(t,0)$ are unbounded as $t \to \infty$. The behaviour of u as $t \to \infty$ depends crucially on the behaviour of P_λ as $\lambda \to \omega^2$ and $\lambda \downarrow 0$. We require, that the operator $\mathcal{A}$ satisfies the following assumption:

(P1) $\mathcal{H}$ is continuously imbedded in some Banach-space $\mathcal{B}$ and there is a subset $\tilde{\mathcal{B}} \subset \mathcal{H}$, such that for every fixed $g \in \tilde{\mathcal{B}}$ the mapping $P.g : \mathbb{R} \mapsto \mathcal{B} : \lambda \mapsto P_\lambda g$ is differentiable almost everywhere with derivative $\frac{dP_\lambda g}{d\lambda} \in L_1((0,M),\mathcal{B})$ for every $M > 0$.

If $f \in \tilde{\mathcal{B}}$ and $\varphi \in C([0,M])$, then

$$\int_0^M \varphi(\lambda)\, d(P_\lambda f) \;=\; \int_0^M \varphi(\lambda)\, \frac{dP_\lambda f}{d\lambda}(\lambda)\, d\lambda \tag{3.6}$$

$(M > 0)$, where the right-hand side has to be read as a Bochner-integral, see eg. [16]. We will make use of the following theorem:

Theorem 3.2 *Suppose that $\mathcal{A}$ is a positive and self-adjoint operator in the Hilbert space $\mathcal{H}$ and that $\mathcal{A}$ obeyes condition (P1) given above. Let $\omega \geq 0, f \in \tilde{\mathcal{B}}, u_1 \in D(\mathcal{A}^{1/2}) \cap \tilde{\mathcal{B}}, u_2 \in D(\mathcal{A}) \cap \tilde{\mathcal{B}}$ be given. Furthermore suppose that the following two assumptions are satisfied:*

(P2) *There exist* $N_f, N_{u_1} \in \mathbb{N}_0$ *and* $f_1^{(0)}, \ldots, f_{[N_f/2]}^{(0)}, (u_1)_1^{(0)}, \ldots, (u_1)_{[N_{u_1}/2]}^{(0)} \in \mathcal{B}$ *(with* $\left[\frac{N}{2}\right] := \max\left\{j \in \mathbb{N}_0 : j \le \frac{N}{2}\right\}$*), such that for respectively* $g := f$ *and* $g := u_1$

$$\int_0^1 \frac{1}{\sqrt{\lambda}} \left\| \frac{dP_\lambda g}{d\lambda}(\lambda) - \sum_{j=1}^{[N_g/2]} \frac{g_j^{(0)}}{\lambda^{1-j/N_g}} \right\|_{\mathcal{B}} d\lambda \;<\; \infty.$$

(P3) *There exist* $N \in \mathbb{N}_0$ *and* $f_1^{\pm}, \ldots, f_N^{\pm} \in \mathcal{B}$*, such that*

$$\int_0^{\omega^2} \frac{1}{|\lambda - \omega^2|} \left\| \frac{dP_\lambda f}{d\lambda}(\lambda) - \sum_{j=1}^{N} \frac{f_j^-}{|\lambda - \omega^2|^{1-j/N}} \right\|_{\mathcal{B}} d\lambda \;<\; \infty \qquad (\text{if } \omega^2 > 0),$$

$$\int_{\omega^2}^{\omega^2+1} \frac{1}{\lambda - \omega^2} \left\| \frac{dP_\lambda f}{d\lambda}(\lambda) - \sum_{j=1}^{N} \frac{f_j^+}{(\lambda - \omega^2)^{1-j/N}} \right\|_{\mathcal{B}} d\lambda \;<\; \infty.$$

Then the solution u *of (1.11) given by (3.3) has the following asymptotic behaviour:*

1) *If* $\omega \neq 0$*, then*

$$\lim_{t\to\infty} \left\| u(t) - \mathrm{e}^{-\mathrm{i}\omega t} I_1(t) - I_2(t) - \mathrm{e}^{-\mathrm{i}\omega t} u_\omega \right\|_{\mathcal{B}} \;=\; 0, \tag{3.7}$$

where

$$\begin{aligned} I_1(t) \;&:=\; \sum_{j=1}^{N-1} t^{1-j/N} \, \frac{D_1(1-\frac{j}{N})(f_j^+ - f_j^-) + \mathrm{i}\, D_2(1-\frac{j}{N})(f_j^+ + f_j^-)}{(2\omega)^{1-j/N}} \\ &\quad + \ln t \cdot (f_N^+ - f_N^-), \end{aligned} \tag{3.8}$$

$$D_1(\beta) \;:=\; \frac{\pi}{2\Gamma(\beta+1)\, \sin\frac{\beta\pi}{2}} \qquad \text{for} \quad 0 < \beta < 2, \tag{3.9}$$

$$D_2(\beta) \;:=\; \frac{\pi}{2\Gamma(\beta+1)\, \cos\frac{\beta\pi}{2}} \qquad \text{for} \quad -1 < \beta < 1, \tag{3.10}$$

$$\begin{aligned} I_2(t) \;&:=\; -\sum_{j=1}^{[N_f/2]} t^{1-2j/N_f} \, \frac{2\mathrm{i}\, D_2(1-\frac{2j}{N_f})}{\omega} \, f_j^{(0)} \\ &\quad + \sum_{j=1}^{[N_{u_1}/2]} t^{1-2j/N_{u_1}} \, 2\, D_2(1-\tfrac{2j}{N_{u_1}})\, (u_1)_j^{(0)}, \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
u_\omega \;:=\; & \lim_{\varepsilon\downarrow 0}\Bigg(\int_{|\lambda-\omega^2|\geq\varepsilon}\frac{1}{\lambda-\omega^2}\,d(P_\lambda f) \\
& -\sum_{j=1}^{N-1}\frac{1}{(1-\frac{j}{N})\,\varepsilon^{1-j/N}}(f_j^+-f_j^-)+\ln\varepsilon\cdot(f_N^+-f_N^-)\Bigg) \\
& +\frac{\mathrm{i}\pi}{2}(f_N^++f_N^-)+(C_e-\ln(2\omega))(f_N^+-f_N^-) \qquad (3.12)
\end{aligned}
$$

(C_e :=Euler-Mascheroni´s constant, the limit has to be taken in $\mathcal{B}$).

2) *If $\omega = 0$, then*

$$
\lim_{t\to\infty}\|u(t)-I_3(t)-u_0\|_{\mathcal{B}} \;=\; 0, \qquad (3.13)
$$

where

$$
\begin{aligned}
I_3(t) \;:=\; & \sum_{j=1}^{N-1} t^{2-2j/N}\,2\,D_1(2-\tfrac{2j}{N})\,f_j^+ + \ln t\cdot 2f_N^+ \\
& + \sum_{j=1}^{[N_{u_1}/2]} t^{1-2j/N_{u_1}}\,2\,D_2(1-\tfrac{j}{N_{u_1}})\,(u_1)_j^{(0)}, \qquad (3.14) \\
u_0 \;:=\; & \lim_{\varepsilon\downarrow 0}\Bigg(\int_\varepsilon^\infty\frac{1}{\lambda}\,d(P_\lambda f)-\sum_{j=1}^{N-1}\frac{f_j^+}{(1-\frac{j}{N})\,\delta^{1-j/N}}+\ln\delta\cdot f_N^+\Bigg) \\
& + 2C_e\,f_N^+ \qquad (3.15)
\end{aligned}
$$

(the limit has to be taken with respect to the norm in $\mathcal{B}$).

Proof: Note that

$$
\left.\begin{aligned}
D_1(\beta) &= \int_0^\infty\frac{1-\cos\mu}{\mu^{1+\beta}}\,d\mu \quad \text{for} \quad 0<\beta<2, \\
D_2(\beta) &= \int_0^\infty\frac{\sin\mu}{\mu^{1+\beta}}\,d\mu \quad \text{for} \quad -1<\beta<1.
\end{aligned}\right\} \qquad (3.16)
$$

Let $\varepsilon > 0$ be given. The proof proceeds in several steps.

Step 1: Since $\mathcal{H}$ is supposed to be continuously imbedded in $\mathcal{B}$, we can choose

$M > \omega^2 + 1$, such that

$$
\begin{aligned}
& \left\| \int_M^\infty \psi_\omega(t,\lambda)\, d(P_\lambda f) + \int_M^\infty \cos\sqrt{\lambda}t\, d(P_\lambda u_0) + \int_M^\infty \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d(P_\lambda u_1) \right\|_{\mathcal{B}}^2 \\
\leq\ & c \left\| \int_M^\infty \psi_\omega(t,\lambda)\, d(P_\lambda f) + \int_M^\infty \cos\sqrt{\lambda}t\, d(P_\lambda u_0) + \int_M^\infty \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d(P_\lambda u_1) \right\|_{\mathcal{H}}^2 \\
\leq\ & 3c \left(\int_M^\infty 9\, d(\|P_\lambda f\|_{\mathcal{H}}^2) + \int_M^\infty 1\, d(\|P_\lambda u_0\|_{\mathcal{H}}^2) + \int_M^\infty 1\, d(\|P_\lambda u_1\|_{\mathcal{H}}^2) \right) \\
<\ & \varepsilon^2
\end{aligned}
$$

and

$$
\left\| \int_M^\infty \frac{1}{\lambda - \omega^2}\, d(P_\lambda f) \right\|_{\mathcal{B}} \ <\ \varepsilon.
$$

Step 2: By the Riemann-Lebesgue-Lemma (see eg. [9]), we have

$$
\left\| \int_0^M \cos\sqrt{\lambda}t\, d(P_\lambda f) \right\|_{\mathcal{B}} \ =\ \left\| \mathrm{Re} \int_0^{\sqrt{M}} \mathrm{e}^{\mathrm{i}\mu t} \frac{dP_\lambda f}{d\lambda}(\mu^2)\, 2\mu\, d\mu \right\|_{\mathcal{B}} \ <\ \varepsilon
$$

for $t > T_1$. Here and in the following, $T_1, T_2, \ldots$ denote suitable chosen positive real numbers.

Step 3: By (P2), we can choose $\delta > 0$, such that

$$
\begin{aligned}
& \left\| \int_0^\delta \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \left(\frac{dP_\lambda u_1}{d\lambda}(\lambda) - \sum_{j=1}^{[N_{u_1}/2]} \frac{(u_1)_j^{(0)}}{\lambda^{1-j/N_{u_1}}} \right) d\lambda \right\|_{\mathcal{B}} \\
\leq\ & \int_0^\delta \frac{1}{\sqrt{\lambda}} \left\| \frac{dP_\lambda u_1}{d\lambda}(\lambda) - \sum_{j=1}^{[N_{u_1}/2]} \frac{(u_1)_j^{(0)}}{\lambda^{1-j/N_{u_1}}} \right\|_{\mathcal{B}} d\lambda \\
<\ & \varepsilon.
\end{aligned}
$$

Riemann-Lebesgue´s Lemma yields as above

$$
\left\| \int_\delta^M \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d(P_\lambda f) \right\|_{\mathcal{B}} \ <\ \varepsilon \qquad \text{for } t > T_2.
$$

With (3.16), we obtain for $j \in \{1, \ldots, \left[\frac{N_{u_1}}{2}\right]\}$

$$
\begin{aligned}
\int_0^\delta \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \frac{1}{\lambda^{1-j/N_{u_1}}}\, d\lambda \ &=\ 2\, t^{1-2j/N_{u_1}} \int_0^\infty \frac{\sin\mu}{\mu^{2-2j/N_{u_1}}}\, d\mu + O\left(\frac{1}{t}\right) \\
&=\ 2\, t^{1-2j/N_{u_1}}\, D_2\left(1 - \frac{2j}{N_{u_1}}\right) + O\left(\frac{1}{t}\right)
\end{aligned}
$$

as $t \to \infty$. Combining all estimates of this step, we conclude that

$$\left\| \int_0^M \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d(P_\lambda u_1) - \sum_{j=1}^{[N_{u_1}/2]} t^{1-2j/N_{u_1}}\, 2D_2(1-\frac{2j}{N_{u_1}})\,(u_1)_j^{(0)} \right\|_{\mathcal{B}} \;<\; 3\varepsilon$$

for $t > T_3$.

Step 4: From here to Step 6, the case $\omega \neq 0$ will be considered. Note that

$$\psi_\omega(t,\lambda) \;=\; \mathrm{e}^{-\mathrm{i}\omega t}\,\frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2} - \frac{\mathrm{i}\,\sin\sqrt{\lambda}t}{\omega\sqrt{\lambda}} + \frac{\mathrm{i}\,\sin\sqrt{\lambda}t}{\omega\left(\sqrt{\lambda}+\omega\right)}$$

The same argument used in Step 3 shows that

$$\Bigg\| \int_0^M \left(-\frac{\mathrm{i}\,\sin\sqrt{\lambda}t}{\omega\sqrt{\lambda}} + \frac{\mathrm{i}\,\sin\sqrt{\lambda}t}{\omega\left(\sqrt{\lambda}+\omega\right)} \right) d(P_\lambda f) + \sum_{j=1}^{[N_f/2]} t^{1-2j/N_f}\,\frac{2\mathrm{i}\,D_2(1-\frac{2j}{N_f})}{\omega}\, f_j^{(0)} \Bigg\|_{\mathcal{B}} \;<\; \varepsilon$$

for $t > T_4$.

Step 5: Assumption (P3) implies, that the limit

$$\mu_\omega^- \;:=\; \lim_{\delta\downarrow 0}\left(\int_0^{\omega^2-\delta} \frac{1}{\lambda-\omega^2}\,\frac{dP_\lambda f}{d\lambda}(\lambda)\,d\lambda + \sum_{j=1}^{N-1} \frac{f_j^-}{(1-\frac{j}{N})\,\delta^{1-j/N}} - \ln\delta\cdot f_N^- \right)$$

exists in $\mathcal{B}$. Choose $\delta > 0$ so small, such that

$$\left\| \int_0^{\omega^2-\delta} \frac{1}{\lambda-\omega^2}\,\frac{dP_\lambda f}{d\lambda}(\lambda)\,d\lambda + \sum_{j=1}^{N-1} \frac{f_j^-}{(1-\frac{j}{N})\,\delta^{1-j/N}} - \ln\delta\cdot f_N^- - \mu_\omega^- \right\|_{\mathcal{B}} \;<\; \varepsilon$$

and

$$\int_{\omega^2-\delta}^{\omega^2} \frac{|1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}|}{|\lambda-\omega^2|}\,\left\| \frac{dP_\lambda f}{d\lambda}(\lambda) - \sum_{j=1}^{N} \frac{f_j^-}{|\lambda-\omega^2|^{1-j/N}} \right\|_{\mathcal{B}} d\lambda \;<\; \varepsilon$$

(see (P3)). Let $\beta \in (0,1)$ be fixed. Standard calculations (see e.g. [11], (6.2)–(6.9) and the equation following (6.29)) show, that for every $\tilde{\varepsilon} > 0$, there exists $\delta(\tilde{\varepsilon}) > 0$, such that for every $\delta \in (0, \delta(\tilde{\varepsilon}))$ there is an $T_5(\tilde{\varepsilon},\delta) > 0$ with

$$\left| \int_{\omega^2-\delta}^{\omega^2} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{(\lambda-\omega^2)|\lambda-\omega^2|^\beta}\,d\lambda - \frac{1}{\beta\,\delta^\beta} + \frac{t^\beta}{(2\omega)^\beta}\int_0^\infty \frac{1-\mathrm{e}^{\mathrm{i}\mu}}{\mu^{(1+\beta)}}\,d\mu \right| \;<\; \tilde{\varepsilon}$$

and

$$\left| \int_{\omega^2-\delta}^{\omega^2} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\, d\lambda + \ln\delta + \ln t + C_e - \ln(2\omega) - \frac{\mathrm{i}\pi}{2} \right| \;<\; \tilde{\varepsilon}$$

for $t > T_5(\tilde{\varepsilon}, \delta)$. By Riemann-Lebesgue,

$$\left\| \int_0^{\omega^2-\delta} \frac{\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\, d(P_\lambda f) \right\|_{\mathcal{B}} \;<\; \varepsilon \qquad \text{for } t > T_6,$$

if $\delta > 0$ is fixed. This, together with (3.16), proves

$$\Bigg\| \int_0^{\omega^2} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\, d(P_\lambda f) - \mu_\omega^- + \left(C_e - \ln(2\omega) - \frac{\mathrm{i}\pi}{2} \right) f_N^-$$
$$+ \sum_{j=1}^{N-1} t^{1-j/N} \frac{D_1(1-\frac{j}{N}) - \mathrm{i}D_2(1-\frac{j}{N})}{(2\omega)^{1-j/N}}\, f_j^- + \ln t \cdot f_N^- \Bigg\|_{\mathcal{B}} \;<\; 5\varepsilon$$

for $t > T_7$.

Step 6: As above,

$$\Bigg\| \int_{\omega^2}^{M} \frac{1-\mathrm{e}^{-\mathrm{i}(\sqrt{\lambda}-\omega)t}}{\lambda-\omega^2}\, d(P_\lambda f) - \mu_\omega^+ - \left(C_e - \ln(2\omega) + \frac{\mathrm{i}\pi}{2} \right) f_N^+$$
$$- \sum_{j=1}^{N-1} t^{1-j/N} \frac{D_1(1-\frac{j}{N}) + \mathrm{i}D_2(1-\frac{j}{N})}{(2\omega)^{1-j/N}} f_j^+ + \ln t \cdot f_N^+ \Bigg\|_{\mathcal{B}} \;<\; \varepsilon$$

for $t > T_8$ with

$$\mu_\omega^+ \;:=\; \lim_{\delta\downarrow 0} \left(\int_{\omega^2+\delta}^{M} \frac{1}{\lambda-\omega^2} \frac{dP_\lambda f}{d\lambda}(\lambda)\, d\lambda - \sum_{j=1}^{N-1} \frac{f_j^+}{(1-\frac{j}{N})\,\delta^{1-j/N}} + \ln\delta \cdot f_N^+ \right)$$

is proved. Combining the results of Step 1 to Step 6, we obtain that (3.7) holds.

The case $\omega = 0$ is proved in the same way by

$$\left| \int_0^{\delta} \frac{1-\cos\sqrt{\lambda}t}{\lambda^{1+\beta}}\, d\lambda - 2\, t^{2\beta} \int_0^{\infty} \frac{1-\cos\mu}{\mu^{1+2\beta}}\, d\mu + \frac{1}{\beta\,\delta^{\beta}} \right| \;<\; \tilde{\varepsilon} \qquad (\beta \in (0,1)),$$

$$\left| \int_0^{\delta} \frac{1-\cos\sqrt{\lambda}t}{\lambda}\, d\lambda - 2\ln t - \ln\delta - 2C_e \right| \;<\; \tilde{\varepsilon}$$

for $t > T_9(\tilde{\varepsilon}, \delta)$ (use (3.67) in [15]). $\square$

Corollary 3.3 *(Regularity estimates) Let all Assumptions of Theorem 3.2 be satisfied. In addition, suppose that* $u_0 \in D(\mathcal{A}^k)$, $u_1 \in D(\mathcal{A}^{k-1/2})$, $f \in D(\mathcal{A}^{k-1})$ *for a fixed* $k \in \mathbb{N}$. *Denote by u the solution of (1.11).*

1) *General case: If* $\omega \neq 0$, *then*

$$\lim_{t\to\infty} \left\| A^j u(t) - \mathrm{e}^{-\mathrm{i}\omega t}\,\omega^{2j}\, I_1(t) - \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega^{(j)} \right\|_{\mathcal{B}} \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, k \tag{3.17}$$

with $I_1(t)$ *being defined in Theorem 3.2 and*

$$\begin{aligned} u_\omega^{(j)} \;:=\; \lim_{\varepsilon\downarrow 0} \Bigg(& \int_{|\lambda-\omega^2|\geq\varepsilon} \frac{\lambda^j}{\lambda-\omega^2}\, d(P_\lambda f) \\ & - \sum_{l=1}^{N-1} \frac{\omega^{2j}}{(1-\frac{l}{N})\,\varepsilon^{1-l/N}}\,(f_l^+ + f_l^-) + \ln\varepsilon\cdot\omega^{2j}(f_N^+ - f_N^-)\Bigg) \\ & + \frac{i\pi\,\omega^{2j}}{2}\,(f_N^+ + f_N^-) + \omega^{2j}\,(C_e - \ln(2\omega))\,(f_N^- - f_N^+) \end{aligned} \tag{3.18}$$

($j \leq k$), *where the limit has to be taken in* $\mathcal{B}$. *If* $\omega = 0$, *then*

$$\lim_{t\to\infty} \left\| A^j u(t) - A^{j-1} f \right\|_{\mathcal{B}} \;=\; 0 \qquad \text{for } j = 1, 2, \ldots, k. \tag{3.19}$$

2) *Principle of limiting Amplitude: Suppose that additionally* $N_f = N_{u_1} = 0$ *in (P2) of Theorem 3.2 and that* $N = 1$, $f_1^+ = f_1^-$ $(= \frac{dP_\lambda}{d\lambda}(\omega^2))$ *in (P3). Then*

$$\lim_{t\to\infty} \left\| \mathcal{A}^j\, u(t) - \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega^{(j)} \right\|_{\mathcal{B}} \;=\; 0 \qquad \text{for } j = 0, 1, \ldots, k \tag{3.20}$$

with

$$u_\omega^{(j)} \;:=\; \lim_{\varepsilon\downarrow 0} \int_{|\lambda-\omega^2|\geq\varepsilon} \frac{\lambda^j}{\lambda-\omega^2}\, d(P_\lambda f) + \mathrm{i}\pi\,\omega^{2j}\, f_1^+ \tag{3.21}$$

($j \leq k$). *The limit in (3.21) has to be taken in* $\mathcal{B}$.

Proof: For the first part, use the arguments of the proof of Theorem 3.2. The additional assumption is needed only in Step 1. From Step 2 on, replace $\frac{dP_\lambda f}{d\lambda}(\lambda)$ by $\lambda^k\, \frac{dP_\lambda f}{d\lambda}(\lambda)$.

The assertion of the second part follows directly from part 1, if $j \geq 1$. The case $j = 0$ is obtained from Theorem 3.2. Note that $f_1^+ = f_1^- = 0$ if $\omega = 0$, since $P_\lambda f = 0$ for $\lambda \leq 0$. □

The following theorem characterizes the limiting amplitude (3.21) by the principle of limiting absorption.

Theorem 3.4 *Suppose that $\mathcal{A}$ is a positive and self-adjoint operator in the Hilbert space $\mathcal{H}$ and that $\mathcal{A}$ obeyes the condition (P1) given before Theorem 3.2. Let $\omega \geq 0$ and $f \in \tilde{B}$ be given and assume that (P3) of Theorem 3.2 is satisfied with $N = 1$ and $f_1^+ = f_1^-$. Then $u_\omega^{(0)}$ defined by (3.21) exists in $\mathcal{B}$ and*

$$\lim_{\tau \downarrow 0} \left\| R_{\omega^2 + \mathrm{i}\tau} f - u_\omega^{(0)} \right\|_{\mathcal{B}} \;=\; 0. \tag{3.22}$$

Proof: Conclude from (P1) and (P3) (with $N = 1$ and $f_1^+ = f_1^-$), that the limit $u_\omega^{(0)}$ exists in $\mathcal{B}$. Recall that

$$R_{\omega^2 + \mathrm{i}\tau} f \;=\; \int_0^\infty \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau}\, d(P_\lambda f).$$

Let $\varepsilon > 0$ be given. Choose $\delta > 0$, such that

$$\begin{aligned} &\left\| \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau} \left(\frac{dP_\lambda f}{d\lambda}(\lambda) - f_1^+ \right) d\lambda \right\|_{\mathcal{B}} \\ &\leq \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{|\lambda - \omega^2|} \left\| \frac{dP_\lambda f}{d\lambda}(\lambda) - f_1^+ \right\|_{\mathcal{B}} d\lambda \\ &< \varepsilon \end{aligned}$$

and

$$\left\| \int_{|\lambda-\omega^2| \geq \delta} \frac{1}{\lambda - \omega^2}\, d(P_\lambda f) + \mathrm{i}\pi f_1^+ - u_\omega^{(0)} \right\|_{\mathcal{B}} \;<\; \varepsilon$$

with $u_\omega^{(0)}$ being defined by (3.21). There exists $\tau_0 > 0$, such that for $\tau \in (0, \tau_0)$

$$\begin{aligned} &\left\| \int_{|\lambda-\omega^2| \geq \delta} \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau}\, d(P_\lambda f) - \int_{|\lambda-\omega^2| \geq \delta} \frac{1}{\lambda - \omega^2}\, d(P_\lambda f) \right\|_{\mathcal{B}} \\ &\leq c \left\| \int_{|\lambda-\omega^2| \geq \delta} \frac{\tau}{(\lambda - \omega^2)(\lambda - \omega^2 - \mathrm{i}\tau)}\, d(P_\lambda f) \right\|_{\mathcal{H}} \\ &< \varepsilon \end{aligned}$$

and

$$\left\| \int_{\omega^2-\delta}^{\omega^2+\delta} \frac{1}{\lambda - \omega^2 - \mathrm{i}\tau}\, f_1^+\, d\lambda - \mathrm{i}\pi\, f_1^+ \right\|_{\mathcal{B}} \;<\; \varepsilon.$$

This proves (3.22). □

4 Spectral properties of elastic operator

Let $\Omega \subset \mathbb{R}^n$ be given by (1.3) and set

$$\left.\begin{aligned} D(\mathcal{A}) &:= \{\varphi \in \mathring{H}^1(\Omega)^n : (\Delta + c_0 \operatorname{grad}\operatorname{div})\varphi \in L_2(\Omega)^n\}\,, \\ \mathcal{A}\varphi &:= -(\Delta + c_0 \operatorname{grad}\operatorname{div})\varphi \qquad \text{for } \varphi \in D(\mathcal{A}), \end{aligned}\right\} \tag{4.1}$$

where $c_0 > -1$ is supposed, the derivatives have to be taken in distributional sense and $\mathring{H}^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Standard calculations show that

$$\langle \mathcal{A}\varphi, \varphi\rangle \;\geq\; \min\{1, 1+c_0\}\left(\|\varphi\|_1^2 - \|\varphi\|_0^2\right) \qquad \text{for } \varphi \in D(\mathcal{A}) \tag{4.2}$$

($\|.\|_k :=$ norm in $H^k(\Omega)^n$) and that $\mathcal{A}$ is self-adjoint in $L_2(\Omega)^n$.

Note that $L_2(\Omega)^n = L_2(\mathbb{R}, L_2(D)^n)$, since Ω is given by (1.3). In this section we define an operator family $\{A(\xi)\}_{\xi\in\mathbb{R}}$ in $L_2(D)$, such that $\mathcal{A}$ can be represented by (1.9). We show that the theory developed in section 2 can be applied to obtain a representation of the spectral family of $\mathcal{A}$.

Let the variable in D be denoted by $(x_2, \ldots, x_n)$ and set

$$\operatorname{grad}_D := (\partial_2, \ldots, \partial_n)^T, \quad \operatorname{div}_D := (\partial_2, \ldots, \partial_n), \quad \Delta_D := \partial_2^2 + \ldots + \partial_n^2.$$

Consider the dyadic product $\operatorname{grad}_D \operatorname{div}_D$ defining a $(n-1)\times(n-1)$-matrix. The $n \times n$-matrix given by

$$\begin{pmatrix} -\xi^2 & \mathrm{i}\xi\operatorname{div}_D \\ \mathrm{i}\xi\operatorname{grad}_D & \operatorname{grad}_D \operatorname{div}_D \end{pmatrix}$$

contains $(-\xi^2, \mathrm{i}\xi\partial_2, \mathrm{i}\xi\partial_3, \ldots, \mathrm{i}\xi\partial_n)$ in the first line and $(-\xi^2, \mathrm{i}\xi\partial_2, \mathrm{i}\xi\partial_3, \ldots, \mathrm{i}\xi\partial_n)^T$ in the first column. E.g. in the case $n = 3$,

$$\begin{pmatrix} -\xi^2 & \mathrm{i}\xi\operatorname{div}_D \\ \mathrm{i}\xi\operatorname{grad}_D & \operatorname{grad}_D \operatorname{div}_D \end{pmatrix} = \begin{pmatrix} -\xi^2 & \mathrm{i}\xi\partial_2 & \mathrm{i}\xi\partial_3 \\ \mathrm{i}\xi\partial_2 & \mathrm{i}\xi\partial_2^2 & \mathrm{i}\xi\partial_2\partial_3 \\ \mathrm{i}\xi\partial_3 & \mathrm{i}\xi\partial_3\partial_2 & \mathrm{i}\xi\partial_3^2 \end{pmatrix}.$$

Define the operator $A(\xi)$ for $\xi \in \mathbb{R}$ by

$$\left.\begin{aligned} D(A(\xi)) &:= \left\{\varphi \in \mathring{H}^1(D)^n : \left(\Delta_D + c_0\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{grad}_D\,\mathrm{div}_D \end{pmatrix}\right)\varphi \in L_2(D)^n\right\}, \\ A(\xi)\varphi &:= -\left(\Delta_D - \xi^2 + c_0\begin{pmatrix} -\xi^2 & \mathrm{i}\xi\mathrm{div}_D \\ \mathrm{i}\xi\mathrm{grad}_D & \mathrm{grad}_D\,\mathrm{div}_D \end{pmatrix}\right)\varphi \quad \text{for } \varphi \in D(A(\xi)). \end{aligned}\right\} \tag{4.3}$$

Let $\xi \in \mathbb{R}$ be fixed. As above,

$$\langle A(\xi)\varphi, \varphi\rangle_D \;\geq\; \min\{1, 1 + c_0\}\left(\sum_{j=1}^{n} \|\mathrm{grad}_D\,\varphi_j\|_D^2 + \xi^2\,\|\varphi\|_D^2\right) \;\geq\; 0 \tag{4.4}$$

for $\varphi = (\varphi_1, \ldots, \varphi_n) \in D(A(\xi))$ is proved. Furthermore, $A(\xi)$ is self-adjoint in $L_2(D)^n$.

Lemma 4.1 *Suppose that D has the segment property and that $A(\xi)$ is defined by (4.3). Then:*

1) *For every fixed $\xi \in \mathbb{R}$, $A(\xi)$ has an orthonormal system of eigenfunctions $\{v_j(\xi)\}_{j\in\mathbb{N}}$ being complete in $L_2(D)^n$. Eigenfunctions and associated eigenvalues $\{\lambda_j(\xi)\}_{j\in\mathbb{N}}$ can be chosen in a way, such that for every fixed $j \in \mathbb{N}$ the mappings $\xi \mapsto v_j(\xi)$, $\xi \mapsto \lambda_j(\xi)$ are analytic on $\mathbb{R}$.*

2) *For every $\xi \in \mathbb{R}$, $\lambda_j(\xi) > 0$ for $j \in \mathbb{N}$ and $\lambda_j(\xi) \to \infty$ as $j \to \infty$.*

3) *For every $j \in \mathbb{N}$, there exists $k \in \mathbb{N}$, such that $\lambda_j(\xi) = \lambda_k(-\xi)$ for $\xi \in \mathbb{R}$.*

Proof: Let $\xi \in \mathbb{R}$ be fixed. Using Rellich's selection theorem, one obtains from (4.4) that $(A(\xi) + \mathrm{Id})^{-1} : L_2(D)^n \to L_2(D)^n$ is compact. Furthermore this operator is symmetric and positive. The theory of compact symmetric operators shows that $(A(\xi) + \mathrm{Id})^{-1}$ has an orthonormal system $\{v_1(\xi), v_2(\xi), \ldots\}$ of eigenfunctions being complete in $L_2(D)^n$. Denote by $\mu_j(\xi)$ the eigenvalue of $(A(\xi) + \mathrm{Id})^{-1}$ associated to $v_j(\xi)$ ($j \in \mathbb{N}$). Then $\mu_j(\xi) \downarrow 0$ as $j \to \infty$. The operator $A(\xi)$ has the same eigenfunctions with associated eigenvalues $\lambda_j(\xi) = \frac{1-\mu_j(\xi)}{\mu_j(\xi)} \to \infty$ as $j \to \infty$.

Now let $\xi \in \mathbb{R}$ vary. In the notation of [10], $\{A(\xi)\}$ is a self-adjoint holomorphic operator family of type (A) with compact resolvent. According to Theorem 3.9 of Chapter VII there, eigenvalues and eigenfunctions can be chosen in a way, such that they depend analytically on ξ. Since the graphs of $\lambda = \lambda_j(\xi)$ may intersect, enumeration has to be changed eventually. Note that this change doesn't touch the property $\lambda_j(\xi) \to \infty$ as $j \to \infty$ for fixed $\xi \in \mathbb{R}$. Finally, $\lambda_j(\xi) > 0$ is obtained from (4.4) and from $D(A(\xi)) \subset \mathring{H}^1(D)^n$.

Fix $j_0 \in \mathbb{N}$ and consider $v_{j_0}(\xi) = \binom{u_1(\xi)}{u_D(\xi)}$, where $u_D := (v_{j_0 2}, \dots, v_{j_0 n})^T$. From $A(\xi)\binom{u_1(\xi)}{u_D(\xi)} = \lambda_{j_0}(\xi)\binom{u_1(\xi)}{u_D(\xi)}$ and the definition of $A(\xi)$,

$$A(-\xi)\begin{pmatrix} -u_1(\xi) \\ u_D(\xi) \end{pmatrix} = \lambda_{j_0}(\xi)\begin{pmatrix} -u_1(\xi) \\ u_D(\xi) \end{pmatrix}$$

follows. In particular, for every $\xi > 0$, there exists $j \in \mathbb{N}$, such that $\lambda_j(-\xi) = \lambda_{j_0}(\xi)$. This implies, that there is at least one $j \in \mathbb{N}$, such that $\lambda_j(-\xi) = \lambda_{j_0}(\xi)$ for more than countable many values of $\xi \in [0,1]$. Hence $\lambda_j(-\xi) = \lambda_{j_0}(\xi)$ for $\xi \in \mathbb{R}$ by analycity of λ_j and λ_{j_0}. □

Lemma 4.2 *Suppose that ∂D has the segment property and let $\{\lambda_j(\xi)\}_{j\in\mathbb{N}}$ denote the eigenvalues of $A(\xi)$ given by Lemma 4.1. Then*

$$\lambda_{\min} := \min\{\lambda_j(\xi) : j \in \mathbb{N},\ \xi \in \mathbb{R}\} > 0. \tag{4.5}$$

Furthermore, for every $\lambda \geq \lambda_{\min}$, equation $\lambda_j(\xi) = \lambda$ admits only finitely many solutions $(j, \xi) \in \mathbb{N} \times \mathbb{R}$.

Proof: It is sufficient to consider $\lambda_j(\xi)$ with $\xi \geq 0$ according to Item 3 of Lemma 4.1.

Note that

$$2\mathrm{Re}\left\langle A(\xi)v_j(\xi), \frac{d}{d\xi}v_j(\xi)\right\rangle_D = \lambda_j(\xi)\frac{d}{d\xi}\|v_j(\xi)\|_D^2 = \lambda_j(\xi)\frac{d}{d\xi}1 = 0 \quad \text{for } \xi \in \mathbb{R}.$$

From this, one obtains with $v_j(\xi) = \binom{u_1(\xi)}{u_D(\xi)}$ and $\xi \geq 0$ that

$$\begin{aligned}
\frac{d}{d\xi}\lambda_j(\xi) &= \frac{d}{d\xi}\left\langle A(\xi)v_j(\xi), v_j(\xi)\right\rangle_D \\
&= 2\mathrm{Re}\left\langle A(\xi)v_j(\xi), \frac{d}{d\xi}v_j(\xi)\right\rangle_D + 2\xi\left\|v_j(\xi)\right\|_D^2 \\
&\quad -c_0\left\langle \begin{pmatrix} -2\xi u_1(\xi) + \mathrm{i}\,\mathrm{div}_D\, u_D(\xi) \\ \mathrm{i}\,\mathrm{grad}_D\, u_1(\xi) \end{pmatrix}, \begin{pmatrix} u_1(\xi) \\ u_D(\xi) \end{pmatrix}\right\rangle_D \\
&= 2\xi\left(\left\|v_j(\xi)\right\|_D^2 + c_0\left\|u_1(\xi)\right\|_D^2\right) + 2c_0\,\mathrm{Im}\left\langle \mathrm{grad}_D\, u_1(\xi), u_D(\xi)\right\rangle_D \\
&\geq -|c_0|\left(\left\|\mathrm{grad}_D\, u_1(\xi)\right\|_D^2 + \left\|u_D(\xi)\right\|_D^2\right) \\
&\geq -|c_0|\left(\left\|\mathrm{grad}_D\, u_1(\xi)\right\|_D^2 + 1\right),
\end{aligned}$$

since $\|u_D(\xi)\|_D^2 \leq \|v_j(\xi)\|_D^2 = 1$. Inequality (4.4) implies that

$$\left\|\mathrm{grad}_D\, u_1(\xi)\right\|_D^2 \leq \frac{1}{\min\{1, 1+c_0\}}\left\langle A(\xi)v_j(\xi), v_j(\xi)\right\rangle_D = \frac{\lambda_j(\xi)}{\min\{1, 1+c_0\}}.$$

Hence

$$\frac{d}{d\xi}\lambda_j(\xi) \geq -|c_0| - \frac{|c_0|}{\min\{1, 1+c_0\}}\lambda_j(\xi) =: -|c_0| - c_1\lambda_j(\xi)$$

follows. Application of Gronwall's Lemma yields

$$\lambda_j(\xi) \geq \lambda_j(0)\mathrm{e}^{-c_1\xi} - |c_0|\int_0^\xi \mathrm{e}^{-c_1(\xi-s)}\,ds = \left(\lambda_j(0) + \frac{|c_0|}{c_1}\right)\mathrm{e}^{-c_1\xi} - \frac{|c_0|}{c_1} \tag{4.6}$$

for $\xi \geq 0$. (Set $\varphi(\xi) := \mathrm{e}^{c_1\xi}\lambda_j(\xi)$ and integrate the resulting differential inequality.)

In order to prove Lemma 4.2, it is convenient to search for eigenvalues $\lambda_j(\xi)$ being smaller than some given $\lambda \geq 0$. From (4.4) one obtains that

$$\lambda_j(\xi) = \left\langle A(\xi)v_j(\xi), v_j(\xi)\right\rangle_D \geq \min\{1, 1+c_0\}\,\xi^2\left\|v_j(\xi)\right\|_D^2 = \min\{1, 1+c_0\}\,\xi^2 \tag{4.7}$$

and hence

$$\lambda_j(\xi) > \lambda \qquad \text{for } j \in \mathbb{N},\ \xi \geq \sqrt{\frac{\lambda}{\min\{1, 1+c_0\}}} =: c_2.$$

If $\xi \in [0, c_2]$, then (4.6) implies that

$$\lambda_j(\xi) \geq \left(\lambda_j(0) + \frac{|c_0|}{c_1}\right)\mathrm{e}^{-c_1c_2} - \frac{|c_0|}{c_1} > \lambda \qquad \text{for } j > J_0$$

with suitable chosen $J_0 \in \mathbb{N}$, since $\lambda_j(0) \to \infty$ as $j \to \infty$. Hence equation $\lambda_j(\xi) = \lambda$ admits only solutions $(j, \xi) \in \mathbb{N} \times \mathbb{R}$ with $j \leq J_0$ and $\xi \in [0, c_2]$. By the analycity of $\lambda_j(.)$, both assertions of Lemma 4.2 follow. $\square$

Lemma 4.3 *Let $\Omega \subset \mathbb{R}^n$ be given by (1.3), where D is supposed to have the segment property. Set $\mathcal{H} := L_2(\Omega)^n = L_2(\mathbb{R}, H)$ with $H = L_2(D)^n$) and denote by $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ the Fourier-transform (see (1.8)). Then*

$$\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} \;=\; \int_{\mathbb{R}}^{\oplus} A(\xi)\, d\xi \;=:\; \widehat{\mathcal{A}} \tag{4.8}$$

with $\mathcal{A}$ and $A(\xi)$ being defined by respectively (4.1) and (4.3).

Proof: For every $\varphi, \psi \in L_2(D)^n$, the mapping $\xi \mapsto \langle \varphi, (A(\xi) + \mathrm{Id})^{-1}\psi \rangle_D$ is continuous on $\mathbb{R}$ by Lemma 4.1, and hence measurable. This shows that $\int_{\mathbb{R}}^{\oplus} A(\xi)\, d\xi$ is defined. It is sufficient to prove $\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} \subset \widehat{\mathcal{A}}$. This implies that $\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} = \widehat{\mathcal{A}}$, since both operators are self-adjoint. (For the case of $\widehat{\mathcal{A}}$ see Theorem XIII.85 in [14].) Suppose that $f \in D(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1})$, which means that $\mathcal{F}^{-1} f \in D(\mathcal{A})$. Since $D(\mathcal{A}) \subset \mathring{H}^1(\Omega)^n$, there exists a sequence $\{\varphi_j\}$ in $C_0^\infty(\Omega)^n$ with

$$\left\| \varphi_j - \mathcal{F}^{-1} f \right\|_{H^1(\Omega)^n} \;\to\; 0 \qquad \text{as } j \to \infty.$$

Since $\mathcal{F}$ is norm-invariant and commutes with derivatives with respect to $x_2, \ldots, x_n$, this implies that

$$\int_{\mathbb{R}} \|(\mathcal{F}\varphi_j)(\xi) - f(\xi)\|^2_{H^1(D)^n}\, d\xi \;\to\; 0 \qquad \text{as } j \to \infty.$$

There exists a subsequence of $\{\varphi_j\}$, again denoted by $\{\varphi_j\}$, such that

$$\|(\mathcal{F}\varphi_j)(\xi) - f(\xi)\|^2_{H^1(D)^n} \;\to\; 0 \qquad \text{as } j \to \infty \text{ a.e. on } \mathbb{R}.$$

Note that $(\mathcal{F}f)(\xi) \in C_0^\infty(D)^n$ for every $\xi \in \mathbb{R}$. Hence $f(\xi) \in \mathring{H}^1(D)^n$ a.e. for $\xi \in \mathbb{R}$.

If $\varphi \in C_0^\infty(\Omega)^n$, then $(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1}\varphi)(\xi) = A(\xi)\varphi(\xi, .)$ and

$$\begin{aligned}\left\langle \mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} f, \varphi \right\rangle_\Omega \;&=\; \left\langle f, \mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1}\varphi \right\rangle_\Omega \;=\; \int_{\mathbb{R}} \langle f(\xi), A(\xi)\varphi(\xi, .) \rangle_D\, d\xi \\ =\; &\int_{\mathbb{R}} \left\langle \left(-\Delta_D + \xi^2 - c_0 \begin{pmatrix} -\xi^2 & \mathrm{i}\xi \mathrm{div}_D \\ \mathrm{i}\xi \mathrm{grad}_D & \mathrm{grad}_D \mathrm{div}_D \end{pmatrix} \right) f(\xi), \varphi(\xi, .) \right\rangle_D d\xi\end{aligned}$$

by the symmetry of $\mathcal{F}\circ\mathcal{A}\circ\mathcal{F}^{-1}$, Fubini's theorem and definition of weak derivative. Again using Fubini's theorem, one obtains that

$$\left(-\Delta_D+\xi^2-c_0\begin{pmatrix}-\xi^2 & \mathrm{i}\xi\mathrm{div}_D\\ \mathrm{i}\xi\mathrm{grad}_D & \mathrm{grad}_D\,\mathrm{div}_D\end{pmatrix}\right)f(\xi)\;=\;(\mathcal{F}\circ\mathcal{A}\circ\mathcal{F}^{-1}f)(\xi)\;\in\;L_2(D)^n$$

a.e. for $\xi\in\mathbb{R}$, and hence that $f(\xi)\in D(A(\xi))$ and $A(\xi)f(\xi)=(\mathcal{F}\circ\mathcal{A}\circ\mathcal{F}^{-1}f)(\xi)$ a.e. for $\xi\in\mathbb{R}$. Furthermore,

$$\int_{\mathbb{R}}\|A(\xi)f(\xi)\|_D^2\,d\xi\;=\;\int_{\mathbb{R}}\left\|(\mathcal{F}\circ\mathcal{A}\circ\mathcal{F}^{-1}f)(\xi)\right\|_D^2\,d\xi\;<\;\infty$$

follows. This implies that $f\in D(\widehat{\mathcal{A}})$ and

$$(\widehat{\mathcal{A}}f)(\xi)\;=\;A(\xi)f(\xi)\;=\;(\mathcal{F}\circ\mathcal{A}\circ\mathcal{F}^{-1}f)(\xi)\qquad\text{a.e. for }\xi\in\mathbb{R},$$

which concludes the proof of $\mathcal{F}\circ\mathcal{A}\circ\mathcal{F}^{-1}\subset\widehat{\mathcal{A}}$. □

The following corollary is obtained from Lemmata 4.1, 4.2, 4.3 and from (4.4), (4.7):

Corollary 4.4 *Suppose that $D\subset\mathbb{R}^{n-1}$ is bounded and has the segment property. Then:*

1) *The operator family $\{A(\xi)\}_{\xi\in\mathbb{R}}$ given by (4.3) satisfies Assumptions (A1) – (A4) (see page 4).*

2) *All assertions of Theorem 2.3 are valid for spectrum and spectral family of the operator $\mathcal{A}$ given by (4.1).*

5 Elastic wave equation

We study the long-time behaviour of u solving (1.2). L_2-estimates are given by Theorem 5.1, if u is a strong solution. Pointwise estimates of the classical solution are presented in Theorem 5.2.

In Section 2, Banach-spaces $\mathcal{H}_s = L_{2,s}(\mathbb{R}, H)$ with assiciated norm $\|.\|_s$ $(s \in \mathbb{R})$ were defined (see (2.10)). In the following application, $H := L_2(D)^n$ has to be set, since $\Omega = \mathbb{R} \times D$. Note that

$$\mathcal{H}_s \;=\; H^0_s(\Omega)^n, \qquad \|.\|_s \text{ is equivalent to } \|.\|_{0,s},$$

where

$$\left.\begin{aligned} H^k_s(\Omega)^n &:= \left\{\varphi \in H^k_{\text{loc}}(\Omega)^n : \|\varphi\|_{k,s} < \infty\right\}, \\ \|\varphi\|_{k,s} &:= \left(\int_\Omega (1+|x|^2)^s \left(\sum_{|\alpha|\le k} |D^\alpha \varphi(x)|^2\right) dx\right)^{1/2} \end{aligned}\right\} \tag{5.1}$$

$(k \in \mathbb{N}_0,\ s \geq 0)$.

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^n$ be given by (1.3) whith D having the segment property, and let $\mathcal{A}$ be the elastic spatial operator defined by (4.1). Denote by u the solution of (1.11) (with $\mathcal{H} := L_2(\Omega)^n$), where*

$$\omega \geq 0, \quad u_0 \in D(\mathcal{A}), \quad u_1 \in D(\mathcal{A}^{1/2}), \quad f \in L_2(\Omega)^n \tag{5.2}$$

is supposed.

1) *Principle of limiting amplitude: If, in addition to (5.2), $\omega^2 \notin \sigma_{\text{res}}(\mathcal{A})$ (see (2.2)) and $u_0, u_1, f \in H^0_s(\Omega)^n$ for some $s > \frac{1}{2}$, then*

$$u(t) \;=\; \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega + r(t), \tag{5.3}$$

where

$$\|r(t)\|_{0,-s'} \;=\; o(1) \qquad \text{as } t \to \infty \tag{5.4}$$

for every $s' > \frac{1}{2}$ and $u_\omega := u^{(0)}_\omega$ is given by (3.21) (with $\mathcal{B} := H^0_{-s'}(\Omega)^n$).

2) *Resonance case: If, in addition to (5.2), $\omega^2 = \sigma_p \in \sigma_{\text{res}}(\mathcal{A})$ and $u_0, u_1, f \in H^0_s(\Omega)^n$ for some $s > N_p + \frac{1}{2}$ (see (2.14)), then resonance of order t^{1-1/N_p}*

occurs. In particular,

$$
\begin{aligned}
u(t) \;=\; & \mathrm{e}^{-\mathrm{i}\omega t} \sum_{(j,k)\in K^{+}(\sigma_p)} \left(\sum_{l=1}^{N(j,k,p)-1} t^{1-l/N(j,k,p)} \times \right. \\
& \times \frac{D_1(1-\frac{l}{N(j,k,p)}) + \mathrm{i}D_2(1-\frac{l}{N(j,k,p)})}{(2\omega)^{1-l/N(j,k,p)}} Q_{jkpl}(f) \\
& \left. + \ln t \cdot Q_{jkpN(j,k,p)}(f) \right) \\
& - \mathrm{e}^{-\mathrm{i}\omega t} \sum_{(j,k)\in K^{-}(\sigma_p)} \left(\sum_{l=1}^{N(j,k,p)-1} t^{1-l/N(j,k,p)} \times \right. \\
& \times \frac{D_1(1-\frac{l}{N(j,k,p)}) - \mathrm{i}D_2(1-\frac{l}{N(j,k,p)})}{(2\omega)^{1-l/N(j,k,p)}} Q_{jkpl}(f) \\
& \left. + \ln t \cdot Q_{jkpN(j,k,p)}(f) \right) + \mathrm{e}^{-\mathrm{i}\omega t} u_\omega + r(t), \qquad (5.5)
\end{aligned}
$$

where r satisfies (5.4) for every $s' > N_p + \frac{1}{2}$ and where at least one term of order $t^{1-1/N(j,k,p)}$ with $N(j,k,p) = N_p$ does not vanish and is not cancelled out by other terms, if f is chosen suitable. Furthermore, u_ω is given by

$$
\begin{aligned}
u_\omega \;=\; & \lim_{\varepsilon\downarrow 0} \left(\int_{|\lambda-\sigma_p|\geq\varepsilon} \frac{1}{\lambda-\sigma_p}\, d(P_\lambda f) \right. \\
& - \sum_{(j,k)\in K^{+}(\sigma_p)} \left(\sum_{l=1}^{N(j,k,p)-1} \frac{1}{\left(1-\frac{l}{N(j,k,p)}\right)\varepsilon^{1-l/N(j,k,p)}} Q_{jkpl}(f) \right. \\
& \left. + \left(\frac{\mathrm{i}\pi}{2} + C_e - \ln(2\omega)\right) Q_{jkpN(j,k,p)}(f) \right) \\
& + \sum_{(j,k)\in K^{-}(\sigma_p)} \left(\sum_{l=1}^{N(j,k,p)-1} \frac{1}{\left(1-\frac{l}{N(j,k,p)}\right)\varepsilon^{1-l/N(j,k,p)}} Q_{jkpl}(f) \right. \\
& \left.\left. + \left(\frac{\mathrm{i}\pi}{2} - C_e + \ln(2\omega)\right) Q_{jkpN(j,k,p)}(f) \right) \right), \qquad (5.6)
\end{aligned}
$$

where the limit has to be taken in $H^0_{-s'}(\Omega)^n$ for $s' > N_p + \frac{1}{2}$.

Proof: According to Corollary 4.4, the spectral family of $\mathcal{A}$ and its properties are given by (2.12), (2.13) and (2.15). Application of Corollary 3.3, Part 2 (with $k = 1$ and $\tilde{\mathcal{B}} := \mathcal{H}_s = H_s^0(\Omega)^n$, $\mathcal{B} := \mathcal{H}_{-s'} = H_{-s'}^0(\Omega)^n$) yields (5.3) and (5.4). From Item 1 of Theorem 3.2, (5.5) with (5.4) and (5.6) is obtained.

It remains to prove, that resonance of order t^{1-1/N_p} occurs, if f is chosen suitable. Note that every term on the right-hand side of (5.5) with $N(j,k,p) < N_p$ or with $l > 1$ grows slower than t^{1-1/N_p}. Thus only summands with $N(j,k,p) = N_p$ and $l = 1$ have to be studied. Choose one pair $(j_0,k_0) \in K^+(\sigma_p) \cup K^-(\sigma_p)$ such that $N(j_0,k_0,p) = N_p$. Furthermore, choose $f \in H_s^0(\Omega)^n$ such that $\langle(\mathcal{F}f)(r_{j_0k_0}(\sigma_p)), v_{j_0}(r_{j_0k_0}(\sigma_p))\rangle_D \neq 0$ (Note that it is possible to suppose $f \in C_0^\infty(\Omega)^n$). Then $Q_{j_0k_0p1}(f) \neq 0$ according to (2.16). The associated resonance-term is not cancelled out by the other terms on the right-hand side of (5.5), which will be shown in the following.

Consider another pair $(j,k) \in K^+(\sigma_p) \cup K^-(\sigma_p)$ with $N(j,k,p) = N_p$. If $r_{jk}(\sigma_p) \neq r_{j_0k_0}(\sigma_p)$, then $Q_{jkp1}(f)$ and $Q_{j_0k_0p1}(f)$ are linearly independent due to the factor $\mathrm{e}^{\mathrm{i}xr_{jk}(\sigma_p)}$ in the definition (2.16) of $Q_{jkp1}(f)$. If $r_{jk}(\sigma_p) = r_{j_0k_0}(\sigma_p)$ and $j \neq j_0$, then $Q_{jkp1}(f)$ and $Q_{j_0k_0p1}(f)$ are again lineartly independent, since $\{v_j(r_{j_0k_0}(\sigma_p))\}_{j\in\mathbb{N}}$ is supposed to be an orthonormal system. It remains to consider the case $r_{jk}(\sigma_p) = r_{j_0k_0}(\sigma_p)$ and $j = j_0$. In this case, $k = k_0 \pm 1$ according to Item 4 of Lemma 2.1 and construction of r_{jk}. Furthermore $Q_{jkp1}(f) = Q_{j_0k_0p1}(f)$ holds (see (2.9) and (2.16)). If $(j,k), (j_0,k_0)$ are contained in the same set $K^+(\sigma_p)$ (or respectively $K^-(\sigma_p)$), then both associated summands in (5.5) lead to the same resonance term, so that the factor of $t^{1-1/N_p}Q_{j_0k_0p1}(f)$ is multiplied by two. If e.g. $(j,k) \in K^+(\sigma_p)$ and $(j_0,k_0) \in K^-(\sigma_p)$, then the factor of $t^{1-1/N_p}Q_{j_0k_0p1}(f)$ is given by

$$\frac{D_1(1-\frac{1}{N_p}) + \mathrm{i}\,D_2(1-\frac{1}{N_p})}{(2\omega)^{1-1/N_p}} - \frac{D_1(1-\frac{1}{N_p}) - \mathrm{i}\,D_2(1-\frac{1}{N_p})}{(2\omega)^{1-1/N_p}} = 2\mathrm{i}\,\frac{D_2(1-\frac{1}{N_p})}{(2\omega)^{1-1/N_p}} \neq 0.$$

The same happens if $(j,k) \in K^-(\sigma_p)$ and $(j_0,k_0) \in K^+(\sigma_p)$. □

Theorem 5.2 *Let Ω and $\mathcal{A}$ be given by respectively (1.3) and (4.1). Suppose that $\omega \geq 0$ and that*

$$\partial D \in C^K, \quad u_0 \in D(\mathcal{A}^{K'}), \quad u_1 \in D(\mathcal{A}^{K'-1/2}), \quad f \in D(\mathcal{A}^{K'-1}) \tag{5.7}$$

for some $K > \frac{n}{2} + 2$, $K' > \frac{n}{4} + 1$. Then Problem (1.2) has a solution $u \in C^2\big([0,\infty) \times \overline{\Omega}\big)$. It is the only solution of (1.2) satisfying

$$u \in C^2([0,\infty), L_2(\Omega)), \quad u(t) \in D(\mathcal{A}) \text{ for } t \geq 0. \tag{5.8}$$

Furthermore the following asymptotic estimates hold:

1) **Principle of limiting amplitude:** *If, in addition to (5.7), $\omega^2 \notin \sigma_{\mathrm{res}}(\mathcal{A})$ and $u_0, u_1, f \in H_s^0(\Omega)^n$ for some $s > \frac{1}{2}$, then*

$$u(t,x) \;=\; \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega(x) + r(t,x), \tag{5.9}$$

where $u_\omega \in C^2\big(\overline{\Omega}\big)$ is solution of (1.6) and, for every $s' > \frac{1}{2}$,

$$\frac{1}{(1+|x|^2)^{s'/2}}\, r(t,x) \;=\; o(1) \qquad \text{as } t \to \infty, \text{ uniformly with respect to } x \in \overline{\Omega}. \tag{5.10}$$

2) **Resonance case:** *If, in addition to (5.8), $\omega^2 \in \sigma_{\mathrm{res}}(\mathcal{A})$ and $u_0, u_1, f \in H_s^0(\Omega)^n$ for some $s > N_p + \frac{1}{2}$, then u satisfies (5.5) pointwise with respect to $x \in \overline{\Omega}$, where $r(t,x)$ obeyes (5.10) for every $s' > N_p + \frac{1}{2}$ and $u_\omega \in C^2\big(\overline{\Omega}\big)$ solves (1.6).*

For the proof, the following lemma is needed. It can be found in [7], but without proof. For the idea of the proof see e.g. Hilfssatz 1.15 in [12].

Lemma 5.3 *Set $\mathcal{D} := -(\Delta + c_0 \operatorname{grad}\operatorname{div})$ (in distributional sense) and $\mathring{H}^1_{-s}(\Omega)^n :=$ completition of $C_0^\infty(\Omega)^n$ in $H^1_{-s}(\Omega)^n$ ($s \geq 0$). Suppose that Ω is given by (1.3) with $\partial D \in C^K$ for some $K \geq 2$ and that $\varphi \in H^0_{-s}(\Omega)^n$ satisfies*

$$\mathcal{D}^j \varphi \in \mathring{H}^1_{-s}(\Omega)^n \text{ for } j = 0, 1, \ldots, K'-1, \qquad \mathcal{D}^{K'} \varphi \in \mathcal{H}^0_{-s}(\Omega)^n$$

for some $K' \in \mathbb{N}$ and some $s \geq 0$. Then $\varphi \in H^{\min\{K,2K'\}}_{-s}(\Omega)^n$ and

$$\|\varphi\|_{\min\{K,2K'\},-s} \;\leq\; c(K,K',s,\Omega)\left(\left\|\mathcal{D}^{K'}\varphi\right\|_{0,-s} + \|\varphi\|_{0,-s}\right) \tag{5.11}$$

with $c(K,K',s,\Omega) > 0$ not depending on φ.

Proof of Theorem 5.2: Denote by u the solution of (1.11) given by (3.3). From (5.7) and Theorem 3.1, one obtains that $\mathcal{D}^{K'-j}\frac{d^{2j}u}{dt^{2j}} \in C([0,\infty), L_2(\Omega)^n)$, $j \le K'$. Application of Lemma 5.3 (with $s = 0$) and Sobolev's imbedding theorem yields $u \in C^2\big([0,\infty)\times\overline{\Omega}\big)$. By definition of $\mathcal{A}$ (see (4.1)), u solves (1.2). Uniqueness of the solution of (1.2) satisfying (5.8) follows from the uniqueness of the solution of (1.11) asserted by Theorem 3.1.

Suppose that all assumptions of Part 2 are satisfied and that $s' > N_p + \frac{1}{2}$. Note that $\omega^2 = \sigma_p > 0$ by (4.5) and (2.3). According to (5.5) and Corollary 3.3, there exist constants $N \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_N \in [0,1)$, $\alpha_1 < \ldots < \alpha_N$ and operators $Q_0, \ldots, Q_N$, such that

$$\left\| \mathcal{D}^j u(t) - \mathrm{e}^{-\mathrm{i}\omega t}\,\omega^{2j}\left(\sum_{k=1}^{N} t^{\alpha_k}\,Q_k(f) + \ln t\cdot Q_0(f)\right) - \mathrm{e}^{-\mathrm{i}\omega t}u_\omega^{(j)}\right\|_{0,-s'} \;\to\; 0$$

as $t \to \infty$ $(j = 0, 1, \ldots, K')$. Set

$$v(t) \;:=\; t^{-\alpha_N}\,\mathrm{e}^{\mathrm{i}\omega t}\,u(t) \qquad \text{for } t \ge 0.$$

Then $v(t), \mathcal{D}v(t), \ldots, \mathcal{D}^{K'}v(t)$ converge as $t \to \infty$ with respect to $\|.\|_{0,-s'}$. Furthermore $v(t), \mathcal{D}v(t), \ldots, \mathcal{D}^{K'-1}v(t) \in \mathring{H}^1_{-s'}(\Omega)^n$ for every $t \ge 0$ (since $v(t) \in D(\mathcal{A}^{K'})$). Application of Lemma 5.3 yields

$$\|v(t) - v(t')\|_{\min\{K,2K'\},-s'} \;\to\; 0 \qquad \text{as } t, t' \to \infty.$$

This implies that $Q_N(f) = \lim_{t\to\infty} v(t) \in H^{\min\{K,2K'\}}_{-s'}(\Omega)^n$, $\mathcal{D}^j Q_N(f) = \omega^{2j} Q_N(f)$ $(j = 1, \ldots, k)$ and $\mathcal{D}^j Q_N(f) \in \mathring{H}^1_{-s'}(\Omega)^n$ for $j = 0, 1, \ldots, k-1$. Repeat the same argument with $v(t) := t^{-\alpha_{N-1}}\,\mathrm{e}^{\mathrm{i}\omega t}\,(u(t) - t^{\alpha_N} Q_N(f))$ and so on. This proves that

$$\begin{aligned}
&\|r(t)\|_{\min\{K,2K'\},-s'} \\
&\quad= \left\| u(t) - \mathrm{e}^{-\mathrm{i}\omega t}\left(\sum_{k=1}^{N} t^{\alpha_k} Q_k(f) + \ln t\cdot Q_0(f)\right) - \mathrm{e}^{-\mathrm{i}\omega t}u_\omega^{(0)}\right\|_{\min\{K,2K'\},-s'} \\
&\quad\to\; 0 \qquad \text{as } t \to \infty.
\end{aligned}$$

Now use Sobolev's inequality to obtain

$$\begin{aligned}\left|\frac{r(t,x)}{(1+|x|^2)^{s'/2}}\right| &\leq c_1 \left\|\frac{r(t)}{(1+|.|^2)^{s'/2}}\right\|_{\min\{K,2K'\},0} \\ &\leq c_2 \|r(t)\|_{\min\{K,2K'\},-s'} \\ &\to 0 \qquad \text{as } t\to\infty\end{aligned}$$

uniformly with repect to $x \in \overline{\Omega}$. Thus (5.10) is proved in the resonance case. Finally note that $u_\omega^{(0)} \in H_{-s'}^{\min\{K,2K'\}} \subset C^2\big(\overline{\Omega}\big)$ and

$$\begin{aligned}\big\langle(\mathcal{D}-\omega^2)u_\omega^{(0)},\varphi\big\rangle &= \big\langle u_\omega^{(0)},(\mathcal{D}-\omega^2)\varphi\big\rangle \\ &= \lim_{t\to\infty}\left\langle \mathrm{e}^{\mathrm{i}\omega t}\,u(t)-\sum_{k=1}^{N} t^{\alpha_k}\,Q_k(f)-\ln t\cdot Q_0(f),(\mathcal{D}-\omega^2)\varphi\right\rangle \\ &= \lim_{t\to\infty}\big\langle(\mathcal{D}-\omega^2)\mathrm{e}^{\mathrm{i}\omega t}\,u(t),\varphi\big\rangle \\ &= \langle f,\varphi\rangle\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)^n$, since $(\mathcal{D}-\omega^2)Q_k(f)=0$ and

$$\big\|\mathrm{e}^{\mathrm{i}\omega t}\,(\mathcal{D}-\omega^2)u(t)-f\big\|_{0,-s'} = \big\|\mathrm{e}^{\mathrm{i}\omega t}\,(\mathcal{A}-\omega^2)u(t)-f\big\|_{0,-s'} \to 0 \qquad \text{as } t\to\infty,$$

which follows by the same arguments used in the proof of Theorem 3.2. This shows that $u_\omega = u_\omega^{(0)}$ solves (1.6). Part 1 of the theorem is proved in the same way. □

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