Codierung - was ist das und wozu?

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Mathematik-Tag

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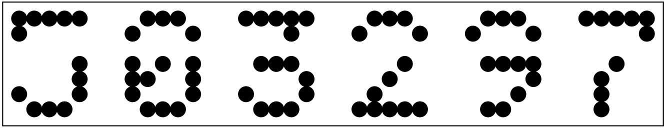
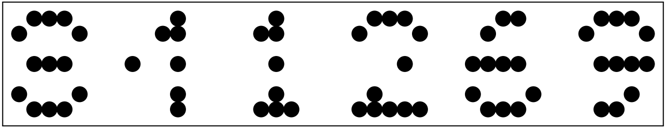
Die Mengenlehre-Uhr
Siehe Wikipedia-Artikel: Berlin-Uhr

TAN-Generator

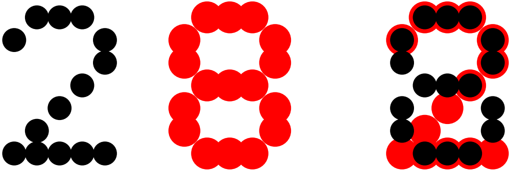
123456

Fehlerhafte Pixel



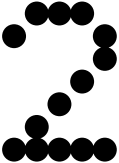


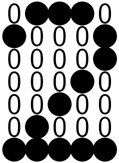
Der Abstand von Ziffern



In der Ziffer 8:	7	Punkte, die nicht in der 2 enthalten
In der Ziffer 2:	4	Punkte, die nicht in der 8 enthalten
Abstand 8 zu 2:	11	

Mathematisch





$$\begin{matrix} 0, 1, 1, 1, 0 \\ 1, 0, 0, 0, 1 \\ 0, 0, 0, 0, 1 \\ 0, 0, 0, 1, 0 \\ 0, 0, 1, 0, 0 \\ 0, 1, 0, 0, 0 \\ 1, 1, 1, 1, 1 \end{matrix}$$

$$0,1,1,1,0|1,0,0,0,1|0,0,0,0,1|0,0,0,1,0|0,0,1,0,0|0,1,0,0,0|1,1,1,1,1$$

$$(0,1,1,1,0,1,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,1,1,1,1)$$

Mathematisch

Abstand von 2 und 8:

(0,1,1,1,0,1,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,1,1,1,1)
(0,1,1,1,0,1,0,0,0,1,1,0,0,0,1,0,1,1,1,0,1,0,0,0,1,1,0,0,0,1,0,1,1,1,0)
Differenz: 1 1 1 1 1 1 1 1 1 1 1
Abstand: 11

Abstandstabelle

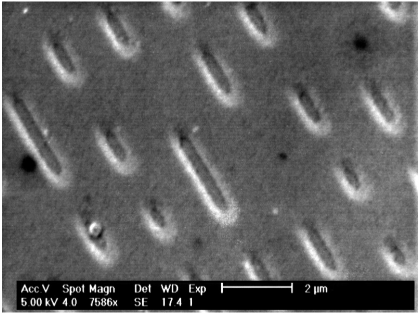
	0	1	2	3	4	5	6	7	8	9
0	0	19	15	15	19	10	10	14	6	10
1	19	0	14	14	18	17	13	15	17	17
2	15	14	0	16	20	17	17	13	11	11
3	15	14	16	0	20	8	9	13	9	13
4	19	18	20	20	0	21	17	19	21	21
5	10	17	17	8	21	0	14	14	12	14
6	10	13	17	9	17	14	0	18	6	12
7	14	15	13	13	19	14	18	0	16	14
8	6	17	11	9	21	12	6	16	0	6
9	10	17	11	13	21	14	12	14	6	0

Abstandstabelle

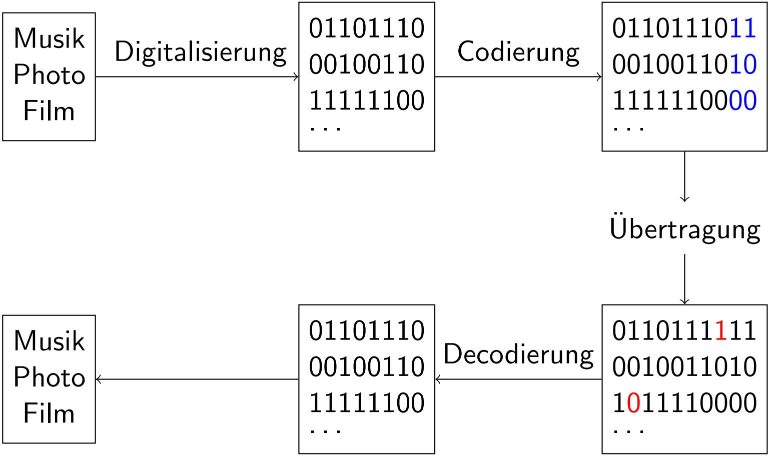
	0	1	2	3	4	5	6	7	8	9
0	0	19	15	15	19	10	10	14	6	10
1	...	0	14	14	18	17	13	15	17	17
2	...	...	0	16	20	17	17	13	11	11
3	...	...	...	0	20	8	9	13	9	13
4	...	...	...	...	0	21	17	19	21	21
5	...	...	...	...	...	0	14	14	12	14
6	...	...	...	...	...	...	0	18	6	12
7	...	...	...	...	...	...	...	0	16	14
8	...	...	...	...	...	...	...	...	0	6
9	...	...	...	...	...	...	...	...	...	0

Codierungstheorie: Wie kann Information möglichst sparsam in 0-1-Pakete gepackt werden, so dass auch bei Lesefehlern die richtige Information herausgeholt werden kann?

Anwendungen:
Photos von Churyumov-Gerasimenko zur Erde schicken
Musik auf einer CD speichern



Quelle: https://de.wikipedia.org/wiki/Compact_Disc


Musik
Photo
Film
Digitalisierung
01101110
00100110
11111100
...
Codierung
0110111011
0010011010
1111110000
...
Übertragung
0110111111
0010011010
1011110000
...
Decodierung
01101110
00100110
11111100
...
Musik
Photo
Film

Codes, die einen Fehler korrigieren können

$$(1,1,0,1) \xrightarrow{\text{Codierung}} (1,1,0,1,1,0,0)$$

$$\left.\begin{array}{c}(1,1,0,1,1,0,0)\\(0,1,0,1,1,0,0)\\(1,0,0,1,1,0,0)\\(1,1,1,1,1,0,0)\\(1,1,0,0,1,0,0)\\(1,1,0,1,0,0,0)\\(1,1,0,1,1,1,0)\\(1,1,0,1,1,0,1)\end{array}\right\} \xrightarrow{\text{Decodierung}} (1,1,0,1)$$

4 Bit $\xrightarrow{\text{Codierung}}$ 7 Bit

4 Bit entsprechen $2^4 = 16$ Möglichkeiten

7 Bit entsprechen $2^7 = 128$ Möglichkeiten

1-Fehler korrigierender perfekter Code

$$\begin{array}{rccc}
1 & (0,0,0,0) & \rightarrow & (0,0,0,0,0,0,0) \\
2 & (0,0,0,1) & \rightarrow & (0,0,0,1,1,1,1) \\
3 & (0,0,1,0) & \rightarrow & (0,0,1,0,0,1,1) \\
4 & (0,0,1,1) & \rightarrow & (0,0,1,1,1,0,0) \\
5 & (0,1,0,0) & \rightarrow & (0,1,0,0,1,0,1) \\
6 & (0,1,0,1) & \rightarrow & (0,1,0,1,0,1,0) \\
7 & (0,1,1,0) & \rightarrow & (0,1,1,0,1,1,0) \\
8 & (0,1,1,1) & \rightarrow & (0,1,1,1,0,0,1) \\
9 & (1,0,0,0) & \rightarrow & (1,0,0,0,1,1,0) \\
10 & (1,0,0,1) & \rightarrow & (1,0,0,1,0,0,1) \\
11 & (1,0,1,0) & \rightarrow & (1,0,1,0,1,0,1) \\
12 & (1,0,1,1) & \rightarrow & (1,0,1,1,0,1,0) \\
13 & (1,1,0,0) & \rightarrow & (1,1,0,0,0,1,1) \\
14 & (1,1,0,1) & \rightarrow & (1,1,0,1,1,0,0) \\
15 & (1,1,1,0) & \rightarrow & (1,1,1,0,0,0,0) \\
16 & (1,1,1,1) & \rightarrow & (1,1,1,1,1,1,1)
\end{array}$$

Konstruktion fehlerkorrigierender Codes

- Fasse jeweils 8 Bit als ein Zeichen auf, das richtig oder falsch übertragen wird: 256 Zeichen.
- Zur Vereinfachung nehmen wir an, es wären nur 31 Zeichen: $0, 1, \dots, 30$.

Ein 1-Fehler korrigierender Code

Verfügbarer Zeichensatz: $0, 1, \ldots, 30$.
Zu je vier Zeichen werden zwei Kontrollzeichen ergänzt ([6:4]-Code):

$$
\begin{aligned}
(5, 4, 15, 3) \quad &\rightarrow \quad (5, 4, 15, 3, 1, 3) \\
&\qquad 5 + 4 + 15 + 3 + 1 + 3 = 31 \equiv 0 \mod 31 \\
&\qquad 5 * 5 + 4 * 4 + 3 * 15 + 2 * 3 + 1 = 93 \equiv 0 \mod 31
\end{aligned}
$$

Allgemein:

$$
\begin{aligned}
(x_1, x_2, x_3, x_4) \rightarrow &\quad (x_1, x_2, x_3, x_4, x_5, x_6) \\
&\quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \equiv 0 \mod 31 \\
&\quad 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \equiv 0 \mod 31
\end{aligned}
$$

Informationsdichte: $I = \frac{4}{6} = \frac{2}{3}$.

Ein 2-Fehler korrigierender Code

Verfügbarer Zeichensatz: $0, 1, \ldots, 30$.
Zu je 8 Zeichen werden 4 Kontrollzeichen ergänzt ([12:8]-Code)

$$(x_1, x_2, \ldots, x_8) \;\rightarrow\; (x_1, x_2, \ldots, x_8, x_9, x_{10}, x_{11}, x_{12})$$

$$\begin{aligned}
x_1 + x_2 + \ldots + x_8 + x_9 + x_{10} + x_{11} + x_{12} &\equiv 0\\
11x_1 + 10x_2 + 9x_3 + 8x_4 + \ldots + 4x_8 + 3x_9 + 2x_{10} + x_{11} &\equiv 0\\
11^2x_1 + 10^2x_2 + 9^2x_3 + 8^2x_4 + \ldots + 4^2x_8 + 3^2x_9 + 2^2x_{10} + x_{11} &\equiv 0\\
11^3x_1 + 10^3x_2 + 9^3x_3 + 8^3x_4 + \ldots + 4^3x_8 + 3^3x_9 + 2^3x_{10} + x_{11} &\equiv 0
\end{aligned}$$

Informationsdichte: $I = \dfrac{8}{12} = \dfrac{2}{3}$.

Konkret

$$(1,2,0,0,0,0,0,0) \;\rightarrow\; (1,2,0,0,0,0,0,0,x_9,x_{10},x_{11},x_{12})$$

$$\begin{array}{rcl}
1+2 \quad + x_9 \; + x_{10} + x_{11} + x_{12} & \equiv & 0 \\
\underbrace{11+20}_{\equiv 0} + 3x_9 + 2x_{10} + x_{11} & \equiv & 0 \\
\underbrace{121+100}_{\equiv 4} + 9x_9 + 4x_{10} + x_{11} & \equiv & 0 \\
\underbrace{1331+1000}_{\equiv 6} + 27x_9 + 8x_{10} + x_{11} & \equiv & 0
\end{array}$$

Nach Rechnung:

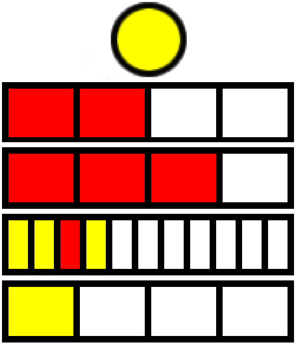
$$(1,2,0,0,0,0,0,0) \;\rightarrow\; (1,2,0,0,0,0,0,0,1,26,7,25)$$

Kombination zweier Codes

$$
\begin{array}{cccc}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{array}
\xrightarrow{\text{Codierung}}
\begin{array}{cccccc}
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
* & * & * & * & \color{blue}{*} & \color{blue}{*} \\
\color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} \\
\color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} \\
\color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} \\
\color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*} & \color{blue}{*}
\end{array}
$$

Reed-Solomon-Codes

$$
\begin{aligned}
(a, b, c, d) &\rightarrow p(x) = ax^3 + bx^2 + cx + d \\
&\rightarrow \big(p(x_1), p(x_2), p(x_3), p(x_4), p(x_5), p(x_6)\big)
\end{aligned}
$$



Quelle: https://de.wikipedia.org/wiki/Berlin-Uhr