Spectral properties of certain differential operators in uniform waveguides

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1 Motivation

This article illustrates the theory developed in [2] at an example. Proofs and precise results can be found in [2].

Let $\Omega \subset \mathbb{R}^3$ denote the domain between the planes $x_3 = 0$ and $x_3 = 1$ in $\mathbb{R}^3$:

$$\Omega \;=\; \mathbb{R}^2 \times (0,1). \tag{1}$$

Suppose that Ω is filled with an homogeneous and isotropic medium. If a force $f(t,x) \in \mathbb{R}^3$ acts in $x \in \Omega$ at time $t \geq 0$, then the deviation $u(t,x) \in \mathbb{R}^3$ of the point x at time t satisfies the initial-boundary value problem

$$\left.\begin{aligned} \partial_t^2 u(t,x) + (-\Delta - c\,\mathrm{grad}\,\mathrm{div})u(t,x) &= f(t,x) && \text{for } x \in \Omega,\ t \geq 0 \\ u|_{\partial\Omega} &= 0 && \text{for } t \geq 0 \\ u(0,x) \;=\; \partial_t u(t,x) &= 0 && \text{for } x \in \Omega, \end{aligned}\right\} \tag{2}$$

where $c = 1 + \frac{\lambda}{\mu} > -1$ (λ, μ: Lamé-constants).

We set

$$\begin{aligned} D(A) &:= \overset{\circ}{H}{}^1(\Omega)^3 \cap H^2(\Omega)^3, \\ A\,u &:= (-\Delta - c\,\mathrm{grad}\,\mathrm{div})u. \end{aligned} \tag{3}$$

Then A is self-adjoint in $L_2(\Omega)^3$ and positive. Denote by $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ the spectral family of A. The solution u to (2) is given by

$$u(t) \;=\; \int_0^\infty \frac{1}{\lambda - \omega^2}\left(\mathrm{e}^{-\mathrm{i}\omega t} - \cos\sqrt{\lambda}t + \frac{\mathrm{i}\omega}{\sqrt{\lambda}}\sin\sqrt{\lambda}t\right) d(P_\lambda f)\,. \tag{4}$$

The behaviour of $u(t)$ as $t \to \infty$ is closely connected to the local behaviour of $P_\lambda f$ near $\lambda = \omega^2$.

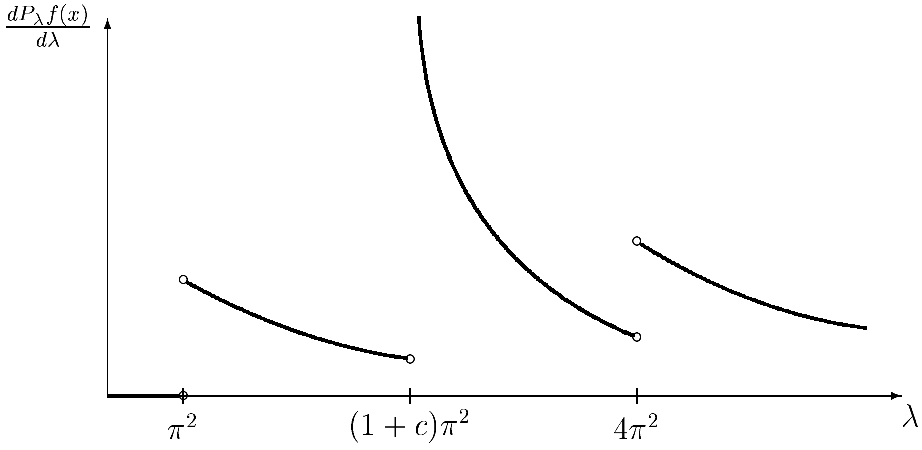

Figure 1: Local behaviour of the spectral family's derivative

2 Typical examples of the local behaviour of $P_\lambda f$

Suppose that $f \in C_0^\infty(\Omega)^3$. Then $P_\lambda f \in C^\infty(\Omega)^3$ for every fixed $\lambda \in \mathbb{R}$. Furthermore $P_\lambda f(x)$ for fixed $x \in \Omega$ depends continuously on $\lambda \in \mathbb{R}$ and vanishes for $\lambda \leq \lambda_1$, where $\lambda_1 > 0$. It can be proved that $P_\lambda f(x)$ is differentiable with respect to λ on the set $\mathbb{R} \setminus \{\lambda_1, \lambda_2, \dots\}$, where $\lambda_j \to \infty$ as $j \to \infty$. If the constant c in the definition of A satisfies $1.5 < c < 3$, then $\lambda_1 = \pi^2$, $\lambda_2 = (1+c)\pi^2$ and $\lambda_3 = 4\pi^2$. A sketch of the local behaviour of $\frac{dP_\lambda f(x)}{d\lambda}$ ($x \in \Omega$ fixed) is given in Figure 1. More precisely:

$$\frac{dP_\lambda f(x)}{d\lambda} \sim \begin{cases} \text{has a finite jump} & \text{at } \lambda = \lambda_j \ (j = 1, 3), \\ \frac{1}{\sqrt{\lambda - \lambda_2}} & \text{as } \lambda \downarrow \lambda_2, \\ \text{is Hölder-continuous} & \text{on } (\lambda_1, \lambda_2) \cup (\lambda_2, \lambda_3). \end{cases}$$

By (4), the following asymptotic behaviour as $t \to \infty$ of the solution u to (2) can be proved:

$$\begin{aligned} &\omega = \sqrt{\lambda_j} \ (j = 1, 3): && u(t) \sim \ln t \cdot \mathrm{e}^{-\mathrm{i}\omega t} v_j, \\ &\omega = \sqrt{\lambda_2}: && u(t) \sim \sqrt{t} \cdot \mathrm{e}^{-\mathrm{i}\omega t} v_2, \\ &\sqrt{\lambda_1} < \omega < \sqrt{\lambda_2}: && u(t) = \mathrm{e}^{-\mathrm{i}\omega t} u_\omega + o(1) \quad \text{(Principle of limiting amplitude);} \end{aligned}$$

here v_j, u_ω denote suitable chosen functions. We denote $\lambda_1, \lambda_2, \dots$ as **resonance points of the spectrum**.

The resolvent $R_z := (A - z\,\mathrm{Id})^{-1}$ of A is given by

$$R_z f \;=\; \int_0^\infty \frac{1}{\lambda - z}\, d(P_\lambda f) \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{5}$$

Hence the above results about the local behaviour of $P_\lambda f$ can be used to prove that $R_{\rho+\mathrm{i}\tau} f \to u_\rho$ as $\tau \downarrow 0$ (limiting absorption principle) if $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_2, \lambda_3)$ and that $R_{\rho+\mathrm{i}\tau} f$ is divergent as $\tau \downarrow 0$ if $\rho = \lambda_1, \lambda_2, \lambda_3$.

3 How to find the resonance points λ_j

Let $\mathcal{F}$ denote the (partial) Fourier-transform with respect to x_1, x_2 and set

$$A(\xi) \;:=\; \mathcal{F}\, A\, \mathcal{F}^{-1} \qquad \text{for } \xi \in \mathbb{R}^2.$$

Then

$$\begin{aligned} D(A(\xi)) &= \overset{\circ}{H}{}^1(0,1)^3 \cap H^2(0,1)^3, \\ A(\xi)\, v &= \left(\xi_1^2 + \xi_2^2 - \partial_3^2 - c \begin{pmatrix} -\xi_1^2 & -\xi_1\xi_2 & \mathrm{i}\xi_1\partial_3 \\ -\xi_1\xi_2 & -\xi_2^2 & \mathrm{i}\xi_2\partial_3 \\ \mathrm{i}\xi_1\partial_3 & \mathrm{i}\xi_2\partial_3 & \partial_3^2 \end{pmatrix} \right) v, \end{aligned} \tag{6}$$

$A(\xi)$ is self-adjoint in $L_2(0,1)^3$, has pure point spectrum, and the eigenvalues $\lambda_j(|\xi|)$ $(j = 1, 2, \ldots)$ depend analytically on $|\xi|$. The eigenvalues can be computed from the dispersion relation

$$\begin{aligned} &2|\xi|^2 \left(1 - \cosh\sqrt{|\xi|^2 - \lambda}\, \cosh\sqrt{|\xi|^2 - \frac{\lambda}{1+c}} \right) \\ &+ \left(|\xi|^4 + \left(|\xi|^2 - \lambda\right) \left(|\xi|^2 - \frac{\lambda}{1+c} \right) \right) \frac{\sinh\sqrt{|\xi|^2 - \lambda}\, \sinh\sqrt{|\xi|^2 - \frac{\lambda}{1+c}}}{\sqrt{|\xi|^2 - \lambda}\sqrt{|\xi|^2 - \frac{\lambda}{1+c}}} \;=\; 0. \end{aligned}$$

or from $\lambda_j(|\xi|) = (k\pi)^2 + |\xi|^2$ $(k \in \mathbb{N})$. The following can be proved:

Theorem: 1) $\lambda \in \mathbb{R}$ is resonance point iff

$$\exists r \geq 0,\ j \in \mathbb{N} : \lambda = \lambda_j(r) \ \wedge\ \frac{d\lambda_j}{dr}(r) = 0.$$

2) If $\lambda_0 \in \mathbb{R}$ is resonance point and

$$\lambda_j(r) \;=\; (r - r_0)^m\, \varphi(r) + \lambda_0,$$

where $m \in \mathbb{N}$ and φ is analytical with $\varphi(r_0) \neq 0$, then:

- If $r_0 > 0$: $\quad \dfrac{dP_\lambda f}{d\lambda} \sim \dfrac{1}{|\lambda - \lambda_0|^{1-1/(m+1)}}$ as $\lambda \to \lambda_0$.
- If $r_0 = 0$: $\quad \dfrac{dP_\lambda f}{d\lambda} \sim \begin{cases} \dfrac{1}{|\lambda - \lambda_0|^{1-1/m}} & \text{as } \lambda \to \lambda_0 \quad (m > 1), \\ \text{has a finite jump} & \text{at } \lambda = \lambda_0 \quad (m = 1). \end{cases}$

4 Further applications

The same method was applied in [2] to problem (2) in a more general domain $\Omega \subset \mathbb{R}^n$ given by

$$\Omega = \Omega_b \times \mathbb{R}^l, \quad \Omega_b \subset \mathbb{R}^{n-l} \text{ bounded, } \partial\Omega_b \text{ smooth.}$$

Furthermore this method was applied on the polyharmonic operator $(-\Delta)^m$, self-adjoint extended in $L_2(\Omega)$ ($\Omega \subset \mathbb{R}^n$ as above) with respect to Dirichlet boundary condition.

Extensions of the described method are possible to problems, where the eigenvalues of the transformed Operator $A(\xi)$ depend on ξ and not only on $|\xi|$. Furthermore a modification of this method was used in [1], where the transformed Operator has purely continuously spectrum.

References

[1] J. Giannoulis, *Über das zeitasymptotische Verhalten von Wellen im mehrdimensionalen freien Raum bei eindimensional periodischer Materialverteilung*, Thesis at the University of Stuttgart, Stuttgart, 2001.

[2] P. H. Lesky, *Zur Langzeitasymptotik periodisch angeregter polyharmonischer und elastischer Wellen,* Shaker Verlag, Aachen, 1999.