About linear elasticity theory in uniform waveguides

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1.1 Fourier-transform

Let H be a separable Hilbert-space. Define

$$\begin{aligned} \mathcal{H} &:= L_2(\mathbb{R}) \otimes H \\ &= L_2(\mathbb{R}, H) \end{aligned}$$

$F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$: Fourier-transform. Set

$$\mathcal{F} := F \otimes \mathrm{Id} \; : \; L_2(\mathbb{R}) \otimes H \to L_2(\mathbb{R}) \otimes H$$

1.2 Decomposable operator

$\mathcal{A}$ self-adjoint in $\mathcal{H}$

$$\Rightarrow \; \mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} \text{ is self-adjoint in } \mathcal{H}.$$

Suppose that

$$\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} = \int_{\mathbb{R}} A(\xi)\, d\xi.$$

$$\left(\begin{array}{l} \left(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} \varphi\right)(\xi) \;=\; A(\xi)\, \varphi(\xi) \\ \text{for } \varphi \in L_2(\mathbb{R}, H) \text{ with } \mathcal{F}^{-1}\varphi \in D(\mathcal{A}). \end{array} \right)$$

1.3 Operator Family

For every fixed $\xi \in \mathbb{R}$:

- $A(\xi)$ is self-adjoint in H and positive.

- $A(\xi)\, v_j^{(\xi)} = \lambda_j^{(\xi)} v_j^{(\xi)}$, $j = 1, 2, \ldots,$ where $\{v_1^{(\xi)}, v_2^{(\xi)}, \ldots\}$ is a complete ON-System.

For varying ξ:

- $\lambda_j^{(\xi)}$ and $v_j^{(\xi)}$ depend analytically on ξ

- $\sigma_{\text{res}} := \bigcup_{j=1}^{\infty} \left\{ \lambda_j^{(\xi)} : \frac{d\lambda_j^{(\xi)}}{d\xi} = 0 \right\}$
has no finite accumulation point.

- For every $\lambda \in \mathbb{R}$, the set $\{(j, \xi) : \lambda_j^{(\xi)} = \lambda\}$ is finite or empty.

1.4 Theorem

Suppose: $\mathcal{A} = \mathcal{F}^{-1} \circ \int_{\mathbb{R}} A(\xi)\, d\xi \circ \mathcal{F}$, where $A(\xi)$ satisfies assumptions made above.

Then: $\mathcal{A}$ self-adjoint in $\mathcal{H} = L_2(\mathbb{R}, H)$ and

$$\frac{dP_\lambda f}{d\lambda} = \frac{1}{\sqrt{2\pi}} \sum_{(j,k)\in M(\lambda)} \left| \frac{dr_{jk}(\lambda)}{d\lambda} \right| \psi(r_{jk}(\lambda))$$

for $\lambda \in \mathbb{R} \setminus \sigma_{\text{res}}$, where

$r_{j1}, r_{j2}, \ldots$: local inverses of $\xi \mapsto \lambda_j^{(\xi)}$,

$$M(\lambda) := \{(j,k) : \lambda \in \text{domain of } r_{jk}\}$$

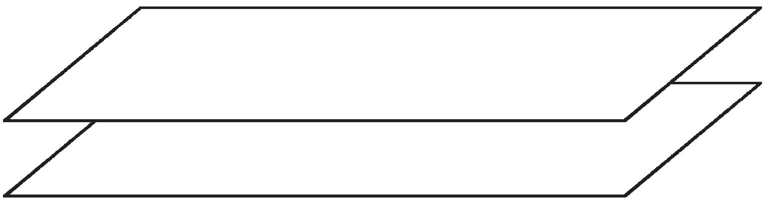
for $\lambda \in \mathbb{R} \setminus \sigma_{\text{res}}$,

$$\psi(r)(x_1, x_2) \;:=\; e^{irx_1} \left\langle (\mathcal{F}f)(r), v_j^{(r)} \right\rangle_H v_j^{(r)}(x_2)$$

for $\mathcal{F}f \in H^s(\mathbb{R}, L_2(0,1))$ with some $s > \frac{1}{2}$

2.1 Waveguides

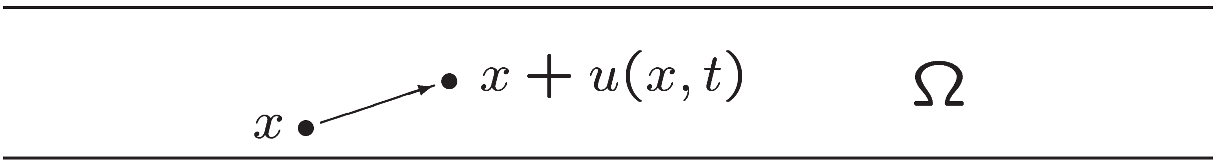
$$\Omega = \mathbb{R} \times (0,1)$$



$$\Omega = \mathbb{R}^2 \times (0,1)$$

$$\Omega = \mathbb{R} \times D$$

2.2 Linear Elasticity



$u(t,x)$: shift of point x at time t

$f(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$: periodic force

$$\begin{aligned}
\partial_t^2 u(t,x) - (\Delta + c_0\,\mathrm{grad}\,\mathrm{div})u(t,x) & \\
&= f(x)\,\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{in } \Omega \\
u(t,x) &= 0 \quad \text{on } \partial\Omega \\
u(0,x) = \partial_t u(0,x) &= 0 \quad \text{in } \Omega
\end{aligned}$$

$c_0 = 1 + \frac{\lambda}{\mu}$, λ, μ: Lamé-constants

2.3 Hilbert-space

$$\begin{aligned}\mathcal{H} &:= L_2(\mathbb{R} \times (0,1)) \\ &= L_2(\mathbb{R}) \otimes L_2(0,1) \\ &= L_2(\mathbb{R}, L_2(0,1))\end{aligned}$$

2.4 Fourier-transform

$$\mathcal{A}\varphi = -(\Delta + c_0 \operatorname{grad}\operatorname{div})\varphi =$$

$$-\begin{pmatrix} (1+c_0)\partial_1^2 + \partial_2^2 & c_0\,\partial_1\,\partial_2 \\ c_0\,\partial_2\,\partial_1 & \partial_1^2 + (1+c_0)\partial_2^2 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$A(\xi) =$$

$$-\begin{pmatrix} -(1+c_0)\xi^2 + \partial_2^2 & \mathrm{i}c_0\,\xi\,\partial_2 \\ \mathrm{i}c_0\,\xi\,\partial_1 & -\xi^2 + (1+c_0)\partial_2^2 \end{pmatrix}$$

$$D(A(\xi)) = H^2(0,1) \cap \mathring{H}^1(0,1).$$

2.5 Operator family, ξ fixed

- $A(\xi)$ is self-adjoint and positive,

- $A(\xi)$ is coercive:

$$\langle A(\xi)\varphi, \varphi\rangle_{L_2(0,1)} \geq c_1 \|\varphi\|^2_{H^1(0,1)} - c_2 \|\varphi\|^2_{L_2(0,1)},$$

- $A(\xi)$ has compact resolvent,

- $A(\xi)$ has a complete (in $L_2(0,1)$) system of orthonormal eigenfunctions $v_1^{(\xi)}, v_2^{(\xi)}, \ldots$ with associated eigenvalues $\lambda_1^{(\xi)}, \lambda_2^{(\xi)}, \ldots,$

- $\lambda_j^{(\xi)} \to \infty$ as $j \to \infty$.

2.6 Operator family, ξ variable

- $A(\xi)$ depends analytically on ξ,

- $D(A(\xi))$ is independent of ξ,

- $\lambda_j^{(\xi)}, v_j^{(\xi)}$ depend analytically on ξ,

- $\lambda_j^{(\xi)} \to \infty$ as $\xi \to \pm\infty$ $(j = 1, 2, \dots)$,

- " $\lambda_j^{(-\xi)} = \lambda_j^{(\xi)}$ ".

2.7 Dispersion relation

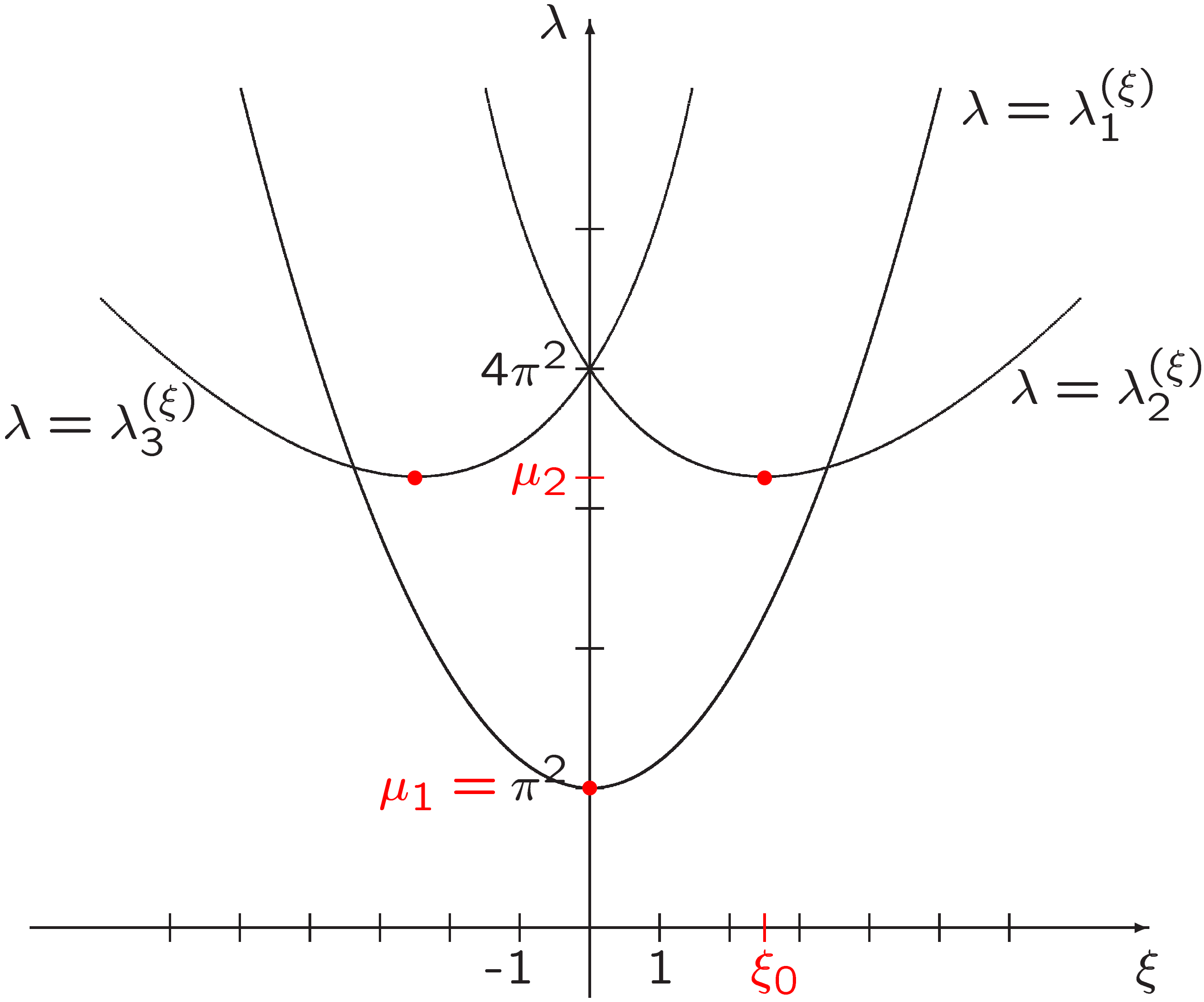
$\lambda \in \mathbb{R}$ is eigenvalue of $A(\xi)$ $\Leftrightarrow$

$$0 = 2\xi^2 \left(1 - \cos\sqrt{\lambda - \xi^2} \cdot \cos\sqrt{\frac{\lambda}{1+c_0} - \xi^2}\right)$$

$$+ \left(\xi^4 + (\xi^2 - \lambda)\left(\xi^2 - \frac{\lambda}{1+c_0}\right)\right)$$

$$\cdot \frac{\sin\sqrt{\lambda - \xi^2} \cdot \sin\sqrt{\frac{\lambda}{1+c_0} - \xi^2}}{\sqrt{\lambda - \xi^2}\sqrt{\frac{\lambda}{1+c_0} - \xi^2}}$$

2.8 Eigenvalues, $c_0 = 3$



2.9 Spectral family, $\lambda \leq \mu_1$

$$P_\lambda f = 0 \qquad \text{for } \lambda \leq \mu_1.$$

2.10 Spectral family, $\lambda \downarrow \mu_1$

$$\lambda_1^{(\xi)} = \mu_1 + \sum_{j=2}^{\infty} d_j \, \xi^j \qquad (|\xi| < R),$$

$$\Rightarrow \quad r_{11}(\lambda) = \sum_{j=1}^{\infty} e_j (\lambda - \mu_1)^{j/2}$$

for $\mu_1 \leq \lambda < \mu_1 + \delta$ with $e_1 = \frac{1}{\sqrt{d_2}}$.

$$\frac{dP_\lambda f}{d\lambda} = \frac{1}{\sqrt{2\pi}} \left(\left| \frac{dr_{11}(\lambda)}{d\lambda} \right| \psi(r_{11}(\lambda)) + \left| \frac{d(-r_{11}(\lambda))}{d\lambda} \right| \psi(-r_{11}(\lambda)) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e_1}{\sqrt{\lambda - \mu_1}} \psi(0) + O(1)$$

as $\lambda \downarrow \mu_1$.

2.11 Spectral family, $\mu_1 < \lambda < \mu_2$

$\frac{dP_\lambda f}{d\lambda}$ is Hölder-continuous for $\mu_1 < \lambda < \mu_2$,

if $f \in H^s(\mathbb{R}, L_2(0,1))$ for some $s > \frac{1}{2}$.

2.12 Spectral family, $\lambda \to \mu_2$

$$\frac{dP_\lambda f}{d\lambda} = \begin{cases} \lim_{\lambda \uparrow \mu_2} \dfrac{dP_\lambda f}{d\lambda} + O(\sqrt{\mu_2 - \lambda}) & \text{as } \lambda \uparrow \mu_2 \\ \dfrac{1}{\sqrt{2\pi}} \dfrac{e_1}{\sqrt{\lambda - \mu_2}} \psi(\xi_0) + O(1) & \text{as } \lambda \downarrow \mu_2 \end{cases}$$

3.1 Limiting amplitude

Suppose: 1) $\omega^2 \in (0, \mu_1) \cup (\mu_1, \mu_2)$,

2) $\mathcal{F} f \in H^s(\mathbb{R}, L_2(0,1))$ for some $s > \frac{1}{2}$.

Then: The principle of limiting amplitude holds:

$$u(t,x) = \mathrm{e}^{-\mathrm{i}\omega t}\, u_\omega(x) + o(1)$$

as $t \to \infty$, where

$$\begin{aligned} \left(-(\Delta + (1+c_0)\mathrm{grad}\,\mathrm{div}) - \omega^2\right) u_\omega(x) & \\ = f(x) & \quad \text{in } \Omega, \\ u(x) = 0 & \quad \text{on } \partial\Omega. \end{aligned}$$

3.2 Resonance

Suppose: $\omega = \sqrt{\mu_1}$ and

$$\mathcal{F} f \in H^s(\mathbb{R}, L_2(0,1)) \text{ for some } s > \tfrac{3}{2}.$$

Then: Resonance ocuurs:

$$u(t,x) =$$

$$\mathrm{e}^{-\mathrm{i}\omega t}\, t^{1/2}$$

$$\cdot\, c_1 \int_\Omega \begin{pmatrix} f_1(y_1,y_2) \\ 0 \end{pmatrix} \sin \pi y_2 \, d(y_1,y_2) \ \sin \pi x_2$$

$$+ c_2\, \mathrm{e}^{-\mathrm{i}\omega t} \ln t \cdot (Q_1 f)(x)$$

$$+ \mathrm{e}^{-\mathrm{i}\omega t} u_\omega(x) + o(1)$$

as $t \to \infty$.

4.1 Remarks

- Spectrum $\sigma(\mathcal{A}) = [\mu_1, \infty)$. The whole spectrum is absolutely continuous.

- Suppose that $\omega = \sqrt{\mu_1}$. The set of functions which show no resonance is dense in $L_2(\Omega)$.

- The case $\Omega = \mathbb{R}^{n-1} \times (0,1)$ can be studied in the same way.

- In the case $\Omega = \mathbb{R} \times D$ with bounded $D \subset \mathbb{R}^2$, the numerical treatment is an open problem.

4.2 The method of Briggs

In **R. J. Briggs**, *Electron stream interaction with plasmas*, 1964, a combination of Fourier- and Laplace-transform was developed to study problems of the kind considered here.

He has to exclude the case of convective instabilities.

The method presented here shows, that the condition $\frac{d\lambda_j^{(\xi)}}{d\xi} = 0$ is sufficient to get resonances.