Master's Thesis

On the Universal Abelian Category of an Exact Category

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Preface

Introduction

Exact categories

An exact category is an additive category equipped with a family of short exact sequences satisfying certain properties. This notion of an exact category was introduced by Quillen [12, §2]. As a general reference on exact categories, we use Bühler's article [2].

For example, any abelian category forms an exact category when equipped with all short exact sequences. On the other hand, it also forms an exact category when equipped with the split short exact sequences only. In practice, there also appear exact structures in between these extremal cases. For example, let mod-$\mathbf{Z}$ denote the category of finitely generated abelian groups and let $\mathrm{C}(\text{mod-}\mathbf{Z})$ denote the category of complexes with entries in mod-$\mathbf{Z}$. We may equip the latter with the pointwise split short exact sequences. In this way, $\mathrm{C}(\text{mod-}\mathbf{Z})$ becomes an exact category, which is even a Frobenius category, whose corresponding stable category is the homotopy category $\mathrm{K}(\text{mod-}\mathbf{Z})$.

Given any abelian category $\mathcal{B}$, then an extension-closed full additive subcategory $\mathcal{A} \subseteq \mathcal{B}$ forms an exact category when equipped with the short exact sequences in $\mathcal{B}$ whose three objects lie in $\mathcal{A}$. Somewhat more generally, an immersion of an exact category into an abelian category is a functor from the former to the latter that is full, faithful, exact and detects exactness. Moreover, we call such an immersion closed if its essential image is closed under extensions. So the inclusion functor from $\mathcal{A}$ to $\mathcal{B}$ is an example of a closed immersion.

But for $\mathrm{C}(\text{mod-}\mathbf{Z})$, equipped with the pointwise split short exact sequences, there is no obvious way how to immerse it into an abelian category. So we ask how, in general, an exact category $\mathcal{A}$ can be immersed into an abelian one.

The immersion of Gabriel, Quillen and Laumon

Quillen states in [12, p. 16] that we may closedly immerse $\mathcal{A}$ into the category $\mathrm{GQL}(\mathcal{A})$ of left-exact contravariant functors from $\mathcal{A}$ to abelian groups via the Yoneda functor $\mathrm{Y}\colon \mathcal{A} \to \mathrm{GQL}(\mathcal{A})$

and that one can prove this assertion following ideas of Gabriel [4]. As far as I know, Laumon was the first to publish a proof [8, th. 1.0.3]. His proof relies on the theory of sheaves on sites, originally due to Grothendieck and Verdier [5], cf. also [9]. However, this immersion is not universal.

A universal construction

Adelman [1] gives a construction of the universal abelian category $\operatorname{Adel}(\mathcal{A})$ of the underlying additive category $\mathcal{A}$, cf. also [14]. The objects of $\operatorname{Adel}(\mathcal{A})$ are diagrams in $\mathcal{A}$ of the form $X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2$. Let $\mathrm{I}\colon \mathcal{A} \to \operatorname{Adel}(\mathcal{A})$ denote the inclusion functor that maps $A \in \operatorname{Ob}\mathcal{A}$ to $0 \longrightarrow A \longrightarrow 0$. For each abelian category $\mathcal{B}$ and each additive functor $F\colon \mathcal{A} \to \mathcal{B}$, there exists an exact functor $\hat{F}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{B}$, unique up to isotransformation, such that $\hat{F} \circ \mathrm{I} = F$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\mathrm{I}"'] & \mathcal{B} \\ \operatorname{Adel}(\mathcal{A}) \arrow[ur, "\hat{F}"'] & \end{tikzcd}$$

Let $\mathcal{N}$ denote the intersection of the kernels of the induced functors $\hat{F}$, where F runs through all exact functors from $\mathcal{A}$ into arbitrary abelian categories. He defines the universal abelian category $\mathcal{U}(\mathcal{A})$ of the exact category $\mathcal{A}$ to be the localisation of $\operatorname{Adel}(\mathcal{A})$ by $\mathcal{N}$, where morphisms in $\operatorname{Adel}(\mathcal{A})$ with kernel and cokernel in $\mathcal{N}$ become formally inverted.
Let $\mathrm{L}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{U}(\mathcal{A})$ denote the localisation functor. The universal property of $\mathcal{U}(\mathcal{A})$, together with the exact universal functor $\mathfrak{I} = \mathrm{L} \circ \mathrm{I}\colon \mathcal{A} \to \mathcal{U}(\mathcal{A})$, states that for each abelian category $\mathcal{B}$ and each exact functor $F\colon \mathcal{A} \to \mathcal{B}$, there exists an exact functor $\tilde{F}\colon \mathcal{U}(\mathcal{A}) \to \mathcal{B}$, unique up to isotransformation, such that $\tilde{F} \circ \mathfrak{I} = F$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\mathrm{I}"'] \arrow[dd, bend right, "\mathfrak{I}"'] & \mathcal{B} \\ \operatorname{Adel}(\mathcal{A}) \arrow[ur, "\hat{F}"'] \arrow[d, "\mathrm{L}"'] & \\ \mathcal{U}(\mathcal{A}) \arrow[uur, bend right, "\tilde{F}"'] & \end{tikzcd}$$

In the extremal cases, we obtain the following. If $\mathcal{A}$ is abelian and equipped with all short exact sequences, then $\mathcal{A}$ is its own universal abelian category. More precisely, $\mathfrak{I}$ is an equivalence in this case. On the other hand, if $\mathcal{A}$ is abelian and equipped with the split short exact sequences only, then L is an equivalence.

Adelman also shows that for a relative projective object P in $\mathcal{A}$, the image $\mathfrak{I}(P)$ is projective in $\mathcal{U}(\mathcal{A})$ [1, th. 2.5].

Keller gives a similar construction in his manuscript [6]. Additionally, he proposes to study the connection between the universal category $\mathcal{U}(\mathcal{A})$ and the Gabriel-Quillen-Laumon category

$\mathrm{GQL}(\mathcal{A})$ via the functor $\tilde{\mathrm{Y}}\colon \mathcal{U}(\mathcal{A}) \to \mathrm{GQL}(\mathcal{A})$ provided by the universal property of $\mathcal{U}(\mathcal{A})$. We follow that idea in this thesis.

Properties of the universal functor

We consider the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{Y}"] \arrow[d, "\mathrm{I}"'] \arrow[dd, bend right=50, "\mathfrak{I}"'] & \mathrm{GQL}(\mathcal{A}) \\
\mathrm{Adel}(\mathcal{A}) \arrow[ur, "\hat{\mathrm{Y}}"'] \arrow[d, "\mathrm{L}"'] & \\
\mathcal{U}(\mathcal{A}) \arrow[uur, bend right=30, "\tilde{\mathrm{Y}}"'] &
\end{tikzcd}$$

to study how properties of Y can be transported to $\mathfrak{I}$ via $\tilde{\mathrm{Y}}$. It follows directly that $\mathfrak{I}$ is faithful and that it detects exactness since Y has these properties. To obtain further results, we need to study $\mathrm{GQL}(\mathcal{A})$ in detail. Following Laumon [8], we equip $\mathcal{A}$ with the structure of a site and interpret $\mathrm{GQL}(\mathcal{A})$ as a certain subcategory of the category of abelian sheaves on $\mathcal{A}$. To explicitly calculate in $\mathrm{GQL}(\mathcal{A})$, we need tools from sheaf theory, which we provide in a self-contained course on the Grothendieck-Verdier theory of sheaves on sites. This allows us to study further properties of $\mathfrak{I}$, $\hat{\mathrm{Y}}$ and $\tilde{\mathrm{Y}}$. We show that $\mathfrak{I}$ is also full and therefore an immersion, cf. theorem 4.4.4. Under the assumption that $\mathcal{A}$ has enough relative projectives (or injectives), we prove that the immersion $\mathfrak{I}$ is closed, cf. proposition 4.4.6. However, I do not know whether the immersion $\mathfrak{I}$ is closed in general.

The universal functor is self-dual, as follows from the universal property. We provide an example where Y is not self-dual and where $\mathcal{N} \subsetneq \mathrm{Ker}(\hat{\mathrm{Y}})$, cf. proposition 4.5.4. This shows that neither $\hat{\mathrm{Y}}$ nor $\tilde{\mathrm{Y}}$ are faithful in general. In particular, $\tilde{\mathrm{Y}}$ is not an equivalence and hence $\mathrm{GQL}(\mathcal{A})$ is not universal. We also see that I is not always left-exact, cf. proposition 4.5.1.

Furthermore, we give an alternative description of the thick subcategory $\mathcal{N}$ of $\mathrm{Adel}(\mathcal{A})$. It is generated by the homologies of the images of the pure short exact sequences under I, which are of the forms $0 \longrightarrow X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z \longrightarrow 0$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$, where $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a pure short exact sequence in $\mathcal{A}$. We denote the corresponding full subcategory of $\mathrm{Adel}(\mathcal{A})$ by $\mathcal{H}$. So $\langle \mathcal{H} \rangle = \mathcal{N}$ and $\mathcal{U}(\mathcal{A}) = \mathrm{Adel}(\mathcal{A}) /\!\!/ \langle \mathcal{H} \rangle$. A larger family of generating objects of $\mathcal{N}$ is given by the pure exact sequences of $\mathcal{A}$.

We present two ways how to obtain the objects of such a thick subcategory from the generating objects in general. In section 3.1.2.2, we successively construct larger subcategories and finally obtain the desired thick subcategory as their union. We start with the subcategory of the generating objects. To obtain the larger subcategory in each step, we add the middle object of a short exact sequence when the outer objects lie in the previous subcategory. Conversely, if the middle object lies in the previous subcategory, then we add the outer objects.

A different approach is described in section 3.1.2.3. At first, we add subfactors of the generating objects to obtain an enlarged subcategory that is closed under subobjects and factor objects and contains the generating objects. Then we take all objects that have a finite filtration with subfactors in this enlarged subcategory and these form the desired thick subcategory.

Results

We summarise. Suppose given an exact category $\mathcal{A}$. Adelman [1, §2] constructs an abelian category $\mathcal{U}(\mathcal{A})$ and an exact universal functor $\mathfrak{I}\colon \mathcal{A} \to \mathcal{U}(\mathcal{A})$ such that the following universal property is fulfilled. For each abelian category $\mathcal{B}$ and each exact functor $F\colon \mathcal{A} \to \mathcal{B}$, there exists an exact functor $\tilde{F}\colon \mathcal{U}(\mathcal{A}) \to \mathcal{B}$, unique up to isotransformation, such that $\tilde{F} \circ \mathfrak{I} = F$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\mathfrak{I}"'] & \mathcal{B} \\ \mathcal{U}(\mathcal{A}) \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

Theorem. The universal functor $\mathfrak{I}$ is an immersion, i.e. it is full, faithful, exact and detects exactness, cf. theorem 4.4.4. Moreover, if $\mathcal{A}$ has enough relative projectives or enough relative injectives, then $\mathfrak{I}$ is a closed immersion, i.e. the essential image of $\mathfrak{I}$ is closed under extensions, cf. proposition 4.4.6.

Let $\mathrm{Y}\colon \mathcal{A} \to \mathrm{GQL}(\mathcal{A})$ denote the closed immersion of Gabriel, Quillen and Laumon.

Proposition. The induced functor $\tilde{\mathrm{Y}}\colon \mathcal{U}(\mathcal{A}) \to \mathrm{GQL}(\mathcal{A})$ is not faithful in general, cf. proposition 4.5.4. In particular, it is not an equivalence and therefore Y is not universal in general.

Outline of the thesis

We summarise the required preliminaries in chapter 1. In particular, section 1.5 contains the needed results on exact categories.

In chapter 2, we present the general theory of sheaves on sites and construct the sheafification functor which is left adjoint to the inclusion functor from the category of sheaves into the category of presheaves. This adjunction is used to show that the category of R-sheaves is abelian for any ring R. Then we explain how an exact category is equipped with the structure of a site and that the additive sheaves turn out to be the left-exact functors. Afterwards, we are able to prove the Gabriel-Quillen-Laumon immersion theorem. We also give a description of cokernels in $\mathrm{GQL}(\mathcal{A})$ without using the language of sheaves, which seems involved and difficult to work with.

The topic of chapter 3 is a recapitulation of the localisation theory of an abelian category at a thick subcategory. At first, we study thick subcategories and how to generate them. For example, we show how subfactors and filtrations can be used to obtain the objects of the resulting thick subcategory from the generating objects. Then we recall the construction of the quotient category and the corresponding localisation functor. We prove that the former is abelian and that the latter is exact. Finally, we present the universal property.

Chapter 4 contains the construction of the universal abelian category of an exact category. We begin with an overview of Adelman's construction of the universal abelian category of an additive category. Then we define the subcategory $\mathcal{H}$ consisting of the homologies of the images of the pure short exact sequences under I. We show that a functor $F\colon \mathcal{A} \to \mathcal{B}$ is exact if and only if the functor $\hat{F}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{B}$ vanishes on $\mathcal{H}$. Moreover, we provide Adelman's characterisation of the thick subcategory $\langle\mathcal{H}\rangle$ in terms of the kernels of the induced functors of exact functors. We also present a larger family of generating objects of $\langle\mathcal{H}\rangle$, namely the pure exact sequences. Next, we define the universal abelian category $\mathcal{U}(\mathcal{A})$ to be the quotient category $\operatorname{Adel}(\mathcal{A}) /\!\!/ \langle\mathcal{H}\rangle$ and prove the universal property. In section 4.4, we use the Gabriel-Quillen-Laumon immersion to show that the universal functor $\mathfrak{I}$ is an immersion and that it is also closed if $\mathcal{A}$ has enough relative projectives. We conclude by disproving some assertions concerning $\mathcal{U}(\mathcal{A})$ and its relationship to $\operatorname{GQL}(\mathcal{A})$ and by formulating some open questions.

I would like to thank my advisor Matthias Künzer for an uncountable amount of very useful suggestions and for his continual support.

Conventions

We assume the reader to be familiar with elementary category theory. An introduction to category theory can be found in [13]. Some basic definitions and notations are given in the conventions below. Cf. [14, conventions].

Suppose given categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$.

1. All categories are supposed to be small with respect to a sufficiently big universe, cf. [13, §3.2 and §3.3].

2. The set of objects in $\mathcal{A}$ is denoted by $\operatorname{Ob}\mathcal{A}$. The set of morphisms in $\mathcal{A}$ is denoted by $\operatorname{Mor}\mathcal{A}$. Given $A, B \in \operatorname{Ob}\mathcal{A}$, the set of morphisms from A to B is denoted by $\operatorname{Hom}_{\mathcal{A}}(A, B)$ and the set of isomorphisms from A to B is denoted by $\operatorname{Hom}^{\text{iso}}_{\mathcal{A}}(A, B)$. The identity morphism of $A \in \operatorname{Ob}\mathcal{A}$ is denoted by 1_A . We write $1 := 1_A$ if unambiguous. Given $f \in \operatorname{Hom}^{\text{iso}}_{\mathcal{A}}(A, B)$, we denote its inverse by $f^{-1} \in \operatorname{Hom}^{\text{iso}}_{\mathcal{A}}(B, A)$.

3. Suppose given $A \xrightarrow{f} B$ in $\mathcal{A}$. We call $A \in \operatorname{Ob}\mathcal{A}$ the *domain* or *source* $\operatorname{dom}(f)$ of f. We call $B \in \operatorname{Ob}\mathcal{A}$ the *codomain* or *target* $\operatorname{cod}(f)$ of f.

4. The composition of morphisms is written naturally:
$\big(A \xrightarrow{f} B \xrightarrow{g} C\big) = \big(A \xrightarrow{fg} C\big) = \big(A \xrightarrow{f \cdot g} C\big)$ in $\mathcal{A}$.

5. The composition of functors is written traditionally: $\big(\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}\big) = \big(\mathcal{A} \xrightarrow{G \circ F} \mathcal{C}\big)$.

6. Given $A, B \in \operatorname{Ob} \mathcal{A}$, we write $A \cong B$ if A and B are isomorphic in $\mathcal{A}$.

7. The opposite category (or dual category) of $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathrm{op}}$. Given $f \in \operatorname{Mor} \mathcal{A}$, the corresponding morphism in $\mathcal{A}^{\mathrm{op}}$ is denoted by f^{op}. Cf. [14, rem 1].

8. We call $\mathcal{A}$ *preadditive* if $\operatorname{Hom}_{\mathcal{A}}(A, B)$ carries the structure of an abelian group for $A, B \in \operatorname{Ob} \mathcal{A}$, written additively, and if $f(g + g')h = fgh + fg'h$ holds for
$A \xrightarrow{f} B \overset{g}{\underset{g'}{\rightrightarrows}} C \xrightarrow{h} D$ in $\mathcal{A}$.

   The zero morphisms in a preadditive category $\mathcal{A}$ are denoted by $0_{A,B} \in \operatorname{Hom}_{\mathcal{A}}(A, B)$ for $A, B \in \operatorname{Ob} \mathcal{A}$. We write $0_A := 0_{A,A}$ and $0 := 0_{A,B}$ if unambiguous.

9. The set of integers is denoted by $\mathbf{Z}$. The set of positive integers is denoted by $\mathbf{N}$ and the set of non-negative integers is denoted by $\mathbf{N}_0$.

10. Given $a, b \in \mathbf{Z}$, we write $[a, b] := \{z \in \mathbf{Z} \colon a \leq z \leq b\}$.

11. For $n \in \mathbf{N}_0$, let Δ_n be the poset category of $[0, n]$ with the partial order inherited from $\mathbf{Z}$. This category has $n + 1$ objects $0, 1, 2, \ldots, n$ and morphisms $i \longrightarrow j$ for $i, j \in \mathbf{Z}$ with $i \leq j$.

12. Suppose that $\mathcal{A}$ is preadditive and suppose given $A, B \in \operatorname{Ob} \mathcal{A}$. A *direct sum* of A and B is a diagram $A \underset{p}{\overset{i}{\rightleftarrows}} C \underset{q}{\overset{j}{\leftrightarrows}} B$ in $\mathcal{A}$ satisfying $ip = 1_A$, $jq = 1_B$ and $pi + qj = 1_C$.
This is generalized to an arbitrary finite number of objects as follows.

   Suppose given $n \in \mathbf{N}_0$ and $A_k \in \operatorname{Ob} \mathcal{A}$ for $k \in [1, n]$. A direct sum of $A_1, \ldots, A_n$ is a tuple $\big(C, (i_k)_{k \in [1,n]}, (p_k)_{k \in [1,n]}\big)$ with $C \in \operatorname{Ob} \mathcal{A}$ and morphisms $A_k \underset{p_k}{\overset{i_k}{\rightleftarrows}} C$ in $\mathcal{A}$ satisfying $i_k p_k = 1_{A_k}$, $i_k p_\ell = 0_{A_k, A_\ell}$ and $\sum_{m=1}^{n} p_m i_m = 1_C$ for $k, \ell \in [1, n]$ with $k \neq \ell$. Cf. [14, rem. 2].
Sometimes we abbreviate $C := \big(C, (i_k)_{k \in [1,n]}, (p_k)_{k \in [1,n]}\big)$ for such a direct sum.

   We often use the following matrix notation for morphisms between direct sums.

   Suppose given $n, m \in \mathbf{N}_0$, a direct sum $\big(C, (i_k)_{k \in [1,n]}, (p_k)_{k \in [1,n]}\big)$ of $A_1, \ldots, A_n$ in $\mathcal{A}$ and a direct sum $\big(D, (j_k)_{k \in [1,m]}, (q_k)_{k \in [1,m]}\big)$ of $B_1, \ldots, B_m$ in $\mathcal{A}$.

   Any morphism $f \colon C \to D$ in $\mathcal{A}$ can be written as $f = \sum_{k=1}^{n} \sum_{\ell=1}^{m} p_k f_{k\ell} j_\ell$ with unique morphisms $f_{k\ell} = i_k f q_\ell \colon A_k \to B_\ell$ for $k \in [1, n]$ and $\ell \in [1, m]$. We write

$$f = \big(f_{k\ell}\big)_{\substack{k \in [1,n], \\ \ell \in [1,m]}} = \begin{pmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm} \end{pmatrix} .$$

   Omitted matrix entries are zero.

13. An object $A \in \operatorname{Ob}\mathcal{A}$ is called a *zero object* if for every $B \in \operatorname{Ob}\mathcal{A}$, there exists a single morphism from A to B and there exists a single morphism from B to A.

    If $\mathcal{A}$ is preadditive, then $A \in \operatorname{Ob}\mathcal{A}$ is a zero object if and only if $1_A = 0_A$ holds.

14. We call $\mathcal{A}$ *additive* if $\mathcal{A}$ is preadditive, if $\mathcal{A}$ has a zero object and if for $A, B \in \operatorname{Ob}\mathcal{A}$, there exists a direct sum of A and B in $\mathcal{A}$. In an additive category direct sums of arbitrary finite length exist.

15. Suppose that $\mathcal{A}$ is additive. We choose a zero object $0_\mathcal{A}$ and write $0 := 0_\mathcal{A}$ if unambiguous. We choose $0_{\mathcal{A}^{\mathrm{op}}} = 0_\mathcal{A}$, cf. [14, rem. 1.(c)].

    For $n \in \mathbf{N}_0$ and $A_1, \dots, A_n \in \operatorname{Ob}\mathcal{A}$, we choose a direct sum

    $$\left(\bigoplus_{k=1}^{n} A_k, \left(i_\ell^{(A_k)_{k\in[1,n]}} \right)_{\ell\in[1,n]}, \left(p_\ell^{(A_k)_{k\in[1,n]}} \right)_{\ell\in[1,n]} \right).$$

    We sometimes write $A_1 \oplus \cdots \oplus A_n := \bigoplus_{k=1}^{n} A_k$.

    Note that $p_\ell^{(A_k)_{k\in[1,n]}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $i_\ell^{(A_k)_{k\in[1,n]}} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$ in matrix notation, where the ones are in the ℓth row resp. column for $\ell \in [1, n]$.

    In functor categories we use the direct sums of the target category to choose the direct sums, cf. [14, rem. 5].

16. Suppose given a full subcategory $\mathcal{N}$ of $\mathcal{A}$. We say that $\mathcal{N}$ is *closed under isomorphisms in* $\mathcal{A}$ if $A \cong B$ in $\mathcal{A}$ for $A \in \operatorname{Ob}\mathcal{N}$ and $B \in \operatorname{Ob}\mathcal{A}$ implies $B \in \operatorname{Ob}\mathcal{N}$.

17. Suppose that $\mathcal{A}$ is additive and suppose given a full subcategory $\mathcal{N}$ of $\mathcal{A}$. We call $\mathcal{N}$ a *full additive subcategory of* $\mathcal{A}$ if $\mathcal{N}$ contains a zero object of $\mathcal{A}$ and if $\mathcal{N}$ contains a direct sum of A and B for objects $A, B \in \operatorname{Ob}\mathcal{N}$.

18. Suppose that $\mathcal{A}$ is additive. Suppose given $M \subseteq \operatorname{Ob}\mathcal{A}$. The full subcategory ${}_{\mathrm{add}}\langle M \rangle$ of $\mathcal{A}$ defined by

    $$\operatorname{Ob}\left({}_{\mathrm{add}}\langle M \rangle\right) := \left\{ Y \in \operatorname{Ob}\mathcal{A} \colon Y \cong \bigoplus_{i=1}^{\ell} X_i \text{ for some } \ell \in \mathbf{N}_0 \text{ and some } X_i \in M \text{ for } i \in [1, \ell] \right\}$$

    shall be the *full additive subcategory of* $\mathcal{A}$ *generated by* M.

19. Suppose that $\mathcal{A}$ is preadditive. Suppose given $f \colon A \to B$ in $\mathcal{A}$. A *kernel* of f is a morphism $k \colon K \to A$ in $\mathcal{A}$ with $kf = 0$ such that the following factorisation property holds.

Given $g\colon X \to A$ with $gf = 0$, there exists a unique morphism $u\colon X \to K$ such that $uk = g$ holds.

Sometimes we refer to K as the kernel of f.

The dual notion of a kernel is called a *cokernel*.

Note that kernels are monomorphic and that cokernels are epimorphic.

20. In diagrams, we often denote monomorphisms by $\rightarrowtail$ and epimorphisms by $\twoheadrightarrow$. In exact categories, we use this notation for pure monomorphisms resp. epimorphisms, cf. definition 1.5.1.

21. Suppose that $\mathcal{A}$ is preadditive. Suppose given $f\colon A \to B$ in $\mathcal{A}$, a kernel $k\colon K \to A$ of f, a cokernel $c\colon B \to C$ of f, a cokernel $p\colon A \to J$ of k and a kernel $i\colon I \to B$ of c.

    There exists a unique morphism $\hat{f}\colon J \to I$ such that the following diagram commutes.

$$\begin{tikzcd}
K \arrow[r, tail, "k"] & A \arrow[rrr, "f"] \arrow[dr, two heads, "p"'] & & & B \arrow[r, two heads, "c"] & C \\
& & J \arrow[r, "\hat{f}"] & I \arrow[ur, tail, "i"'] & &
\end{tikzcd}$$

    We call such a diagram a *kernel-cokernel-factorisation* of f, and we call $\hat{f}$ the induced morphism of the kernel-cokernel-factorisation. Cf. [14, rem 8].

22. We call $\mathcal{A}$ *abelian* if $\mathcal{A}$ is additive, if each morphism in $\mathcal{A}$ has a kernel and a cokernel and if for each morphism f in $\mathcal{A}$, the induced morphism $\hat{f}$ of each kernel-cokernel-factorisation of f is an isomorphism.

23. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are preadditive. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *additive* if $F(f+g) = F(f) + F(g)$ holds for $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$ in $\mathcal{A}$, cf. [14, prop 4].

24. Suppose given functors $F, G\colon \mathcal{A} \to \mathcal{B}$.
    A tuple $\alpha = (\alpha_X)_{X \in \operatorname{Ob}\mathcal{A}}$, where $\alpha_X\colon F(X) \to G(X)$ is a morphism in $\mathcal{B}$ for $X \in \operatorname{Ob}\mathcal{A}$, is called *natural* if $F(f)\,\alpha_Y = \alpha_X\, G(f)$ holds for $X \xrightarrow{f} Y$ in $\mathcal{A}$. A natural tuple is also called a *transformation*. We write $\alpha\colon F \to G$, $\alpha\colon F \Rightarrow G$ or $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$.

    A transformation $\alpha\colon F \Rightarrow G$ is called an *isotransformation* if and only if α_X is an isomorphism in $\mathcal{B}$ for $X \in \operatorname{Ob}\mathcal{A}$. The isotransformations from F to G are precisely the elements of $\operatorname{Hom}^{\mathrm{iso}}_{\mathcal{B}^{\mathcal{A}}}(F, G)$, cf. convention 25.

    Suppose given $\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{description, name=G}] \arrow[r, bend right=50, "H"'{name=H}] & \mathcal{B} \arrow[r, bend left, "K"{name=K}] \arrow[r, bend right, "L"'{name=L}] & \mathcal{C} \arrow[from=F, to=G, phantom, "\alpha\Downarrow"] \arrow[from=G, to=H, phantom, "\beta\Downarrow"] \arrow[from=K, to=L, phantom, "\gamma\Downarrow"] \end{tikzcd}$.

    We have the transformation $\alpha \cdot \beta = \alpha\beta\colon F \Rightarrow H$ with $(\alpha\beta)_X := \alpha_X \beta_X$ for $X \in \operatorname{Ob}\mathcal{A}$.

We have the transformation $\gamma \star \alpha \colon K \circ F \Rightarrow L \circ G$ with $(\gamma \star \alpha)_X := K(\alpha_X)\,\gamma_{G(X)} = \gamma_{F(X)}\,L(\alpha_X)$ for $X \in \operatorname{Ob}\mathcal{A}$.

$$\begin{tikzcd}
(K \circ F)(X) \arrow[r, "\gamma_{F(X)}"] \arrow[d, "K(\alpha_X)"'] & (L \circ F)(X) \arrow[d, "L(\alpha_X)"] \\
(K \circ G)(X) \arrow[r, "\gamma_{G(X)}"'] & (L \circ G)(X)
\end{tikzcd}$$

We have the transformation $1_F \colon F \Rightarrow F$ with $(1_F)_X := 1_{F(X)}$ for $X \in \operatorname{Ob}\mathcal{A}$.

We set $\gamma \star F := \gamma \star 1_F \colon K \circ F \Rightarrow L \circ F$ and $K \star \alpha := 1_K \star \alpha \colon K \circ F \Rightarrow K \circ G$.

So $(\gamma \star F)_X = \gamma_{F(X)}$ and $(1_K \star \alpha)_X = K(\alpha_X)$ for $X \in \operatorname{Ob}\mathcal{A}$.

Thus we also have

$$\gamma \star \alpha = (\gamma \star F) \cdot (L \star \alpha) = (K \star \alpha) \cdot (\gamma \star G).$$

Cf. [14, lem. 6].

25. Let $\mathcal{A}^{\mathcal{C}}$ denote the *functor category* whose objects are the functors from $\mathcal{C}$ to $\mathcal{A}$ and whose morphisms are the transformations between such functors. Cf. remark 1.1.9.

26. The *identity functor* of $\mathcal{A}$ shall be denoted by $1_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$.

27. A functor $F \colon \mathcal{A} \to \mathcal{B}$ is called an *isomorphism of categories* if there exists a functor $G \colon \mathcal{B} \to \mathcal{A}$ with $F \circ G = 1_{\mathcal{B}}$ and $G \circ F = 1_{\mathcal{A}}$. In this case, we write $F^{-1} := G$.

28. A functor $F \colon \mathcal{A} \to \mathcal{B}$ is called an *equivalence of categories* if there exists a functor $G \colon \mathcal{B} \to \mathcal{A}$ with $F \circ G \cong 1_{\mathcal{B}}$ in $\mathcal{B}^{\mathcal{B}}$ and $G \circ F \cong 1_{\mathcal{A}}$ in $\mathcal{A}^{\mathcal{A}}$. The functor F is an equivalence if and only if it is full, faithful and dense.

29. Suppose given a functor $F \colon \mathcal{A} \to \mathcal{B}$. The full subcategory $\operatorname{Im}_{\mathrm{ess}}(F)$ of $\mathcal{B}$ defined by

$$\operatorname{Ob}(\operatorname{Im}_{\mathrm{ess}}(F)) := \{B \in \operatorname{Ob}\mathcal{B} \colon \text{ there exists } A \in \operatorname{Ob}\mathcal{A} \text{ such that } F(A) \cong B \text{ in } \mathcal{B}\}$$

is called the *essential image of* F.

30. Suppose given $A \xrightarrow{f} B$ in $\mathcal{A}$. The morphism f is called a *retraction* if there exists a morphism $g \colon B \to A$ with $gf = 1_Y$. The dual notion of a retraction is called a *coretraction.*

31. Suppose that $\mathcal{A}$ is abelian. An object $P \in \operatorname{Ob}\mathcal{A}$ is called *projective* if for each morphism $g \colon P \to Y$ and each epimorphism $f \colon X \to Y$, there exists $h \colon P \to X$ with $hf = g$.

$$\begin{tikzcd}
 & P \arrow[dl, "h"'] \arrow[d, "g"] \\
X \arrow[r, "f"', "\mid" marking] & Y
\end{tikzcd}$$

The dual notion of a projective object is called an *injective* object.

Cf. remark 1.2.5.

32. Suppose that $\mathcal{A}$ is abelian. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}$. A diagram $X \xrightarrow{p} I \xrightarrow{i} Y$ is called an *image* of f if p is epimorphic, if i is monomorphic and if $pi = f$ holds.

33. Suppose that $\mathcal{A}$ is preadditive. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is called *left-exact* if f is a kernel of g and *right-exact* if g is a cokernel of f. Such a sequence is called *short exact* if it is left-exact and right-exact.

    Suppose that $\mathcal{A}$ is abelian.

    Suppose given $n \in \mathbf{N}_0$ and a sequence $A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} A_{n-1} \xrightarrow{a_{n-1}} A_n$ in $\mathcal{A}$. The sequence is called *exact* if for each choice of images

$$\left(A_k \xrightarrow{a_k} A_{k+1} \right) = \left(A_k \xrightarrow{p_k} I_k \xrightarrow{i_k} A_{k+1} \right)$$

    for $k \in [0, n-1]$, the sequences $I_k \xrightarrow{i_k} A_{k+1} \xrightarrow{p_{k+1}} I_{k+1}$ are short exact for $k \in [0, n-2]$.

    Cf. [14, rem. 12].

34. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are abelian.
    An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *left-exact* if for each left-exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is also left-exact.

    An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *right-exact* if for each right-exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is also right-exact.

    An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *exact* if it is left-exact and right-exact.

    Cf. also definitions 1.5.3 and 1.5.7.

35. Suppose given a subcategory $\mathcal{A}' \subseteq \mathcal{A}$ and a subcategory $\mathcal{B}' \subseteq \mathcal{B}$. Suppose given a functor $F\colon \mathcal{A} \to \mathcal{B}$ with $F(f) \in \operatorname{Mor} \mathcal{B}'$ for $f \in \operatorname{Mor} \mathcal{A}'$.

    Let $F|_{\mathcal{A}'}^{\mathcal{B}'}\colon \mathcal{A}' \to \mathcal{B}'$ be defined by $F|_{\mathcal{A}'}^{\mathcal{B}'}(A) = F(A)$ for $A \in \operatorname{Ob} \mathcal{A}'$ and $F|_{\mathcal{A}'}^{\mathcal{B}'}(f) = F(f)$ for $f \in \operatorname{Mor} \mathcal{A}'$.

    Let $F|_{\mathcal{A}'} := F|_{\mathcal{A}'}^{\mathcal{B}}$. If $\mathcal{A}' = \mathcal{A}$, let $F|^{\mathcal{B}'} := F|_{\mathcal{A}}^{\mathcal{B}'}$.

36. All rings are supposed to be associative with an identity element and all modules are supposed to be right unitary.
    Suppose given a ring R. The category of (right) R-modules is denoted by Mod-R.

37. Suppose given a ring R. Let ${}_R\Upsilon\colon$ Mod-$R \to$ `Set` denote the forgetful functor.

38. Suppose that $\mathcal{A}$ is additive. A short exact sequence $A \xrightarrow{i} C \xrightarrow{q} B$ is called *split short exact* if there exist $C \xrightarrow{p} A$ and $B \xrightarrow{j} C$ in $\mathcal{A}$ such that $A \underset{p}{\overset{i}{\rightleftarrows}} C \underset{q}{\overset{j}{\leftrightarrows}} B$ is a direct sum of A and B. Cf. lemma 1.1.4.

39. Let $\mathcal{A} \times \mathcal{B}$ denote the *product category* of $\mathcal{A}$ and $\mathcal{B}$.

40. Suppose that $\mathcal{A}$ is abelian and that $\mathcal{N}$ is a full subcategory of $\mathcal{A}$. We say that $\mathcal{N}$ *is closed under subobjects in* $\mathcal{A}$ if for each monomorphism $A \xrightarrow{f} B$ in $\mathcal{A}$ with $B \in \operatorname{Ob} \mathcal{N}$, we have $A \in \operatorname{Ob} \mathcal{N}$.
 We say that $\mathcal{N}$ *is closed under factor objects in* $\mathcal{A}$ if for each epimorphism $A \xrightarrow{f} B$ in $\mathcal{A}$ with $A \in \operatorname{Ob} \mathcal{N}$, we have $B \in \operatorname{Ob} \mathcal{N}$.
 We say that $\mathcal{N}$ *is closed under extensions in* $\mathcal{A}$ if for each short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ with $A, C \in \operatorname{Ob} \mathcal{N}$, we have $B \in \operatorname{Ob} \mathcal{N}$.

41. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are additive and that $F \colon \mathcal{A} \to \mathcal{B}$ is a functor. The *kernel of* F is the full subcategory of $\mathcal{A}$ defined by $\operatorname{Ob}(\operatorname{Ker}(F)) := \{A \in \operatorname{Ob} \mathcal{A} \colon F(A) \cong 0_{\mathcal{B}} \text{ in } \mathcal{B}\}$.

42. Suppose given a functor $F \colon \mathcal{A} \to \mathcal{B}$. The opposite functor $F^{\mathrm{op}} \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is defined by $F^{\mathrm{op}}(A) = F(A)$ for $A \in \operatorname{Ob} \mathcal{A}$ and $F^{\mathrm{op}}(f^{\mathrm{op}}) = F(f)^{\mathrm{op}}$ for $f \in \operatorname{Mor} \mathcal{A}$. Cf. [14, rem. 1.(g)].

43. A diagram $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ with transformations $\varepsilon \colon 1_{\mathcal{A}} \Rightarrow G \circ F$ and $\eta \colon F \circ G \Rightarrow 1_{\mathcal{B}}$ such that $(F \star \varepsilon) \cdot (\eta \star F) = 1_F$ and $(\varepsilon \star G) \cdot (G \star \eta) = 1_G$ is called an adjunction, written $F \dashv G$ or
 $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\star\varepsilon"] \arrow[dr, Rightarrow, "1_F"'] & F\circ G\circ F \arrow[d, Rightarrow, "\eta\star F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\varepsilon\star G"] \arrow[dr, Rightarrow, "1_G"'] & G\circ F\circ G \arrow[d, Rightarrow, "G\star\eta"] \\ & G \end{tikzcd}$$

 In this case, the functor F is called *left adjoint* to G, the functor G is called *right adjoint* to F, the transformation ε is called a *unit* of the adjunction and the transformation η is called a *counit* of the adjunction.

 Note that if $(F, G, \varepsilon, \eta)$ is an adjunction, then $(G^{\mathrm{op}}, F^{\mathrm{op}}, \eta^{\mathrm{op}}, \varepsilon^{\mathrm{op}})$ is an adjunction as well. In particular, we have $F \dashv G$ if and only if $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$.

 Now suppose given an adjunction $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$ with unit $\varepsilon \colon 1_{\mathcal{A}} \Rightarrow G \circ F$ and counit $\eta \colon F \circ G \Rightarrow 1_{\mathcal{B}}$.

 For $X \in \operatorname{Ob} \mathcal{A}$ and $Y \in \operatorname{Ob} \mathcal{B}$, let

$$\Phi^{\varepsilon,\eta}_{X,Y} \colon \operatorname{Hom}_{\mathcal{B}}(F(X), Y) \to \operatorname{Hom}_{\mathcal{A}}(X, G(Y)) \colon s \mapsto \varepsilon_X \cdot G(s).$$

This is a bijection with inverse

$$\left(\Phi^{\varepsilon,\eta}_{X,Y}\right)^{-1}\colon\ \mathrm{Hom}_{\mathcal{A}}(X,G(Y)) \to \mathrm{Hom}_{\mathcal{B}}(F(X),Y)\colon t \mapsto F(t)\cdot \eta_Y\,.$$

For $X' \xrightarrow{f} X$ in $\mathcal{A}$, $Y \xrightarrow{g} Y'$ in $\mathcal{B}$ and $F(X) \xrightarrow{s} Y$ in $\mathcal{B}$, we have

$$\begin{aligned}\big(F(f)\cdot s\cdot g\big)\,\Phi^{\varepsilon,\eta}_{X',Y'} &= \varepsilon_{X'}\cdot (G\circ F)(f)\cdot G(s)\cdot G(g) = f\cdot \varepsilon_X \cdot G(s)\cdot G(g)\\ &= f\cdot (s)\,\Phi^{\varepsilon,\eta}_{X,Y}\cdot G(g).\end{aligned}$$

44. A diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

in $\mathcal{A}$ is called a *pullback* if for $T \xrightarrow{s} B$ and $T \xrightarrow{t} C$ in $\mathcal{A}$ with $sg' = tf'$, there exists $T \xrightarrow{u} A$ in $\mathcal{A}$ such that $uf = s$ and $ug = t$.

The dual notion of a pullback is called a *pushout*.

In diagrams, we often denote pullbacks by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

and pushouts by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"'] \arrow[ur, phantom, very near end, "\lrcorner"] & D. \end{tikzcd}$$

Chapter 1

Preliminaries

1.1 Miscellaneous

Lemma 1.1.1. Suppose given an abelian category $\mathcal{A}$ and $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$.

Suppose given a kernel $K \xrightarrow{k} A$ of f, a kernel $L \xrightarrow{\ell} A$ of fg and a kernel $M \xrightarrow{m} B$ of g.

Suppose given a cokernel $B \xrightarrow{p} P$ of f, a cokernel $C \xrightarrow{q} Q$ of fg and a cokernel $C \xrightarrow{r} R$ of g. Then there exist unique morphisms a, b, c, d, e such that the diagram

$$
\begin{tikzcd}
 & & M \arrow[rr, "c"] \arrow[dr, "m"'] & & P \arrow[ddrr, "d"] & & \\
 & & & B \arrow[ur, "p"'] \arrow[dr, "g"] & & & \\
L \arrow[uurr, "b"] \arrow[rr, "\ell"] & & A \arrow[ur, "f"] \arrow[rr, "fg"'] & & C \arrow[rr, "q"] \arrow[dr, "r"'] & & Q \arrow[dl, "e"] \\
 & K \arrow[ul, "a"] \arrow[ur, "k"'] & & & & R &
\end{tikzcd}
$$

commutes. Moreover, the sequence

$$0 \longrightarrow K \xrightarrow{a} L \xrightarrow{b} M \xrightarrow{c} P \xrightarrow{d} Q \xrightarrow{e} R \longrightarrow 0$$

is exact.

Proof. See [2, prop. 8.11] and [13, prob. 13.6.8]. □

Lemma 1.1.2. Suppose given an abelian category $\mathcal{A}$ and a commutative diagram

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[r, "c"] \arrow[d, "g'"] & C \arrow[d, "v"] \\
K' \arrow[r, "k'"] & D \arrow[r, "f'"] & E \arrow[r, "c'"] & C'
\end{tikzcd}
$$

in $\mathcal{A}$ such that k is a kernel of f, k' is a kernel of f', c is a cokernel of f and c' is a cokernel of f'. Note that u is the induced morphism between the kernels and that v is the induced morphism between the cokernels.

(a) The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

is a pullback if and only if u is an isomorphism and v is a monomorphism.

(b) The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

is a pushout if and only if u is an epimorphism and v is a isomorphism.

Proof. See [7, lem. 134]. $\square$

Lemma 1.1.3. Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$ and a functor $F\colon \mathcal{A} \to \mathcal{B}$. The following four statements are equivalent.

(a) The functor F is additive.

(b) For a direct sum $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$ in $\mathcal{A}$, the morphism

$$\left(\begin{smallmatrix} F(p) & F(q) \end{smallmatrix} \right) : F(A \oplus B) \to F(A) \oplus F(B)$$

is an isomorphism with inverse

$$\left(\begin{smallmatrix} F(i) \\ F(j) \end{smallmatrix} \right) : F(A) \oplus F(B) \to F(A \oplus B).$$

(c) We have $F(0_{\mathcal{A}}) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$ and for a direct sum $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$ in $\mathcal{A}$, the morphism $\left(\begin{smallmatrix} F(p) & F(q) \end{smallmatrix} \right) : F(A \oplus B) \to F(A) \oplus F(B)$ is a monomorphism.

(d) We have $F(0_{\mathcal{A}}) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$ and for a direct sum $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$ in $\mathcal{A}$, the morphism $\left(\begin{smallmatrix} F(i) \\ F(j) \end{smallmatrix} \right) : F(A) \oplus F(B) \to F(A \oplus B)$ is an epimorphism.

Cf. [3, th. 3.11].

Proof. Ad (b)⇒(a). Suppose given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{A}$. Choose a direct sum $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$ in $\mathcal{A}$.

We obtain

$$\begin{aligned} F(f+g) &= F\big((i+gj)(pf+q)\big) = F(i+gj)\,F(pf+q) \\ &= F(i+gj)\,\begin{pmatrix} F(p) & F(q) \end{pmatrix} \begin{pmatrix} F(i) \\ F(j) \end{pmatrix} F(pf+q) \\ &= \begin{pmatrix} F(1) & F(g) \end{pmatrix} \begin{pmatrix} F(f) \\ F(1) \end{pmatrix} \\ &= F(f) + F(g). \end{aligned}$$

$$\begin{tikzcd} F(A) \arrow[r, "F(i+gj)"] \arrow[dr, bend right, "{\begin{pmatrix} F(1) & F(g) \end{pmatrix}}"'] & F(A \oplus B) \arrow[r, "F(pf+q)"] \arrow[d, shift right, "{\begin{pmatrix} F(p) & F(q) \end{pmatrix}}"'] & F(B) \\ & F(A) \oplus F(B) \arrow[u, shift right, "{\begin{pmatrix} F(i) \\ F(j) \end{pmatrix}}"'] \arrow[ur, bend right, "{\begin{pmatrix} F(f) \\ F(1) \end{pmatrix}}"'] & \end{tikzcd}$$

Ad (a)⇒(b). See [14, prop. 4].

Moreover, we have $F(0_{\mathcal{A}}) \cong 0_{\mathcal{B}}$ in case (a) or (b) are true by loc. cit.

Now suppose that $F(0_{\mathcal{A}}) \cong 0_{\mathcal{B}}$. Note that for a direct sum $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$ in $\mathcal{A}$, we always have

$$\begin{pmatrix} F(i) \\ F(j) \end{pmatrix} \begin{pmatrix} F(p) & F(q) \end{pmatrix} = \begin{pmatrix} F(i)F(p) & F(i)F(q) \\ F(j)F(p) & F(j)F(q) \end{pmatrix} = \begin{pmatrix} F(ip) & F(iq) \\ F(jp) & F(jq) \end{pmatrix} = \begin{pmatrix} F(1) & F(0) \\ F(0) & F(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the statements (b), (c) and (d) are equivalent as well. □

Lemma 1.1.4.
Suppose given an additive category $\mathcal{A}$ and a short exact sequence $A \overset{i}{\rightarrowtail} C \overset{q}{\twoheadrightarrow} B$ in $\mathcal{A}$. The following three statements are equivalent.

(a) The sequence $A \overset{i}{\rightarrowtail} C \overset{q}{\twoheadrightarrow} B$ is split short exact, cf. convention 38.

(b) The morphism q is a retraction.

(c) The morphism i is a coretraction.

Proof. By definition, (a) implies (b) and (c).

Ad (b)⇒(a). So there exists $C \xrightarrow{p} A$ in $\mathcal{A}$ such that $ip = 1$. We conclude that $i(1-pi) = 0$. Since q is a cokernel of i, there exists $B \xrightarrow{j} C$ in $\mathcal{A}$ such that $qj = 1 - pi$. Moreover, we have $qjq = (1 - pi)q = 1 = q1$ and thus $jq = 1$ since q is epimorphic. We conclude that $A \xrightarrow{i} C \xrightarrow{q} B$ is split short exact.

$$\begin{tikzcd} A \arrow[r, shift left, "i"] & C \arrow[l, shift left, "p"] \arrow[r, "q"] \arrow[d, "1-pi"'] & B \arrow[dl, "j"] \\ & C & \end{tikzcd}$$

Ad (c)⇒(a). This is dual to (b)⇒(a). □

Lemma 1.1.5. Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$ and functors $F, G\colon \mathcal{A} \to \mathcal{B}$. If $F \cong G$ in $\mathcal{B}^{\mathcal{A}}$, then $\operatorname{Ker}(F) = \operatorname{Ker}(G)$, cf. convention 41.

Proof. Since $F \cong G$ in $\mathcal{B}^{\mathcal{A}}$, there exists an isotransformation $\alpha\colon F \Rightarrow G$.
In particular, $F(A) \xrightarrow{\alpha_A} G(A)$ is an isomorphism in $\mathcal{B}$ for $A \in \operatorname{Ob}\mathcal{A}$. Thus $F(A) \cong 0_{\mathcal{B}}$ if and only if $G(A) \cong 0_{\mathcal{B}}$ for $A \in \operatorname{Ob}\mathcal{A}$. □

Lemma 1.1.6. Suppose given an abelian category $\mathcal{A}$. A sequence $X \xrightarrow{0} Y \xrightarrow{0} Z$ in $\mathcal{A}$ is short exact if and only if $X, Y, Z \cong 0_{\mathcal{A}}$ in $\mathcal{A}$.

Proof. If $X, Y, Z \cong 0_{\mathcal{A}}$ in $\mathcal{A}$, then $X \xrightarrow{0} Y \xrightarrow{0} Z$ is short exact.

Conversely, if $X \xrightarrow{0} Y \xrightarrow{0} Z$ is short exact, then $Z \cong 0_{\mathcal{A}}$ since $Y \xrightarrow{0} Z$ is epimorphic and $X, Y \cong 0_{\mathcal{A}}$ since $X \xrightarrow{0} Y$ has to be an isomorphism as kernel of $Y \xrightarrow{0} Z$. □

Lemma 1.1.7.
Suppose given an abelian category $\mathcal{A}$ and a short exact sequence $X \to Y \to Z$ in $\mathcal{A}$. We have $Y \cong 0_{\mathcal{A}}$ if and only if $X, Z \cong 0_{\mathcal{A}}$ in $\mathcal{A}$.

Proof. This is a consequence of lemma 1.1.6. □

Lemma 1.1.8. Suppose given an additive category $\mathcal{A}$ and a full additive subcategory $\mathcal{B}$ of $\mathcal{A}$. Suppose given $K \xrightarrow{k} A \xrightarrow{f} B$ in $\mathcal{B}$ such that k is a kernel of f in $\mathcal{A}$. Then k is a kernel of f in $\mathcal{B}$.

Proof. Suppose given $t\colon T \to B$ in $\mathcal{B}$ with $tf = 0$. Since $tf = 0$ also in $\mathcal{A}$, there exists $u\colon T \to A$ in $\mathcal{A}$ such that $uk = t$. Since $\mathcal{B}$ is a full subcategory, we have $u \in \operatorname{Mor}\mathcal{B}$. The induced morphism is unique since it is even unique in $\mathcal{A}$. □

Remark 1.1.9. Suppose given a category $\mathcal{C}$ and an additive category $\mathcal{A}$.

Suppose given $F \xrightarrow{\alpha} G$ in $\mathcal{A}^{\mathcal{C}}$.

(a) Suppose that $\mathcal{C}$ is an additive category. Suppose that α is an isotransformation. If F is an additive functor, then G is an additive functor as well.

(b) For $X \in \operatorname{Ob}\mathcal{C}$, suppose given a cokernel $G(X) \xrightarrow{c_X} C(X)$ of α_X in $\mathcal{A}$.

For $X \xrightarrow{f} Y$ in $\mathcal{C}$, we have the commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

in $\mathcal{A}$. Let $C(f)$ be the induced morphism between the cokernels c_X and c_Y.

This yields a functor $C \in \operatorname{Ob}(\mathcal{A}^{\mathcal{C}})$ and a transformation $c = (c_X)_{X \in \operatorname{Ob}\mathcal{C}} \colon G \Rightarrow C$.

The morphism c is a cokernel of α in $\mathcal{A}^{\mathcal{C}}$.

Consequently, the following four statements are equivalent.

- α is an epimorphism in $\mathcal{A}^{\mathcal{C}}$.
- $c = 0$ in $\mathcal{A}^{\mathcal{C}}$.
- $c_X = 0$ in $\mathcal{A}$ for $X \in \operatorname{Ob}\mathcal{C}$.
- α_X is an epimorphism in $\mathcal{A}$ for $X \in \operatorname{Ob}\mathcal{C}$.

Suppose that $\mathcal{C}$ is an additive category. If G is an additive functor, then C is additive as well.

As usual, the dual statements for kernels are also true.

(c) Suppose that $\mathcal{A}$ is an abelian category. Then $\mathcal{A}^{\mathcal{C}}$ is abelian as well.

(d) Suppose that $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is an additive category. Suppose given a cokernel $G \xrightarrow{\beta} H$ of α in $\mathcal{A}^{\mathcal{C}}$. If G is an additive functor, then H is additive as well.

As usual, the dual statements for kernels are also true.

(e) Suppose that $\mathcal{A}$ is an additive category. The zero objects of $\mathcal{A}^{\mathcal{C}}$ are additive functors. If F and G are additive functors, then $F \oplus G$ is additive as well.

(f) Suppose that $\mathcal{A}$ is an additive category. Let $\operatorname{Add}(\mathcal{C}, \mathcal{A})$ denote the full subcategory of $\mathcal{A}^{\mathcal{C}}$ whose objects are the additive functors from $\mathcal{C}$ to $\mathcal{A}$. The full subcategory $\operatorname{Add}(\mathcal{C}, \mathcal{A})$ is a full additive subcategory which is closed under isomorphisms.

(g) Suppose that $\mathcal{A}$ is an additive category. Suppose given $K \xrightarrow{k} F \xrightarrow{\alpha} G \xrightarrow{c} C$ in $\mathcal{A}^{\mathcal{C}}$ such that k is a kernel of α and c is a cokernel of α. If α is additive, then k and c are additive as well.

(h) Suppose that $\mathcal{A}$ is an abelian category. Kernels and cokernels of morphisms in $\operatorname{Add}(\mathcal{C}, \mathcal{A})$ formed in $\mathcal{A}^{\mathcal{C}}$ lie again in $\operatorname{Add}(\mathcal{C}, \mathcal{A})$ by (g). In particular, the category $\operatorname{Add}(\mathcal{C}, \mathcal{A})$ is abelian.

Cf. [14, rem. 5].

Definition 1.1.10. Suppose given a category $\mathcal{C}$ and a functor $F\colon \mathcal{C} \to \mathtt{Set}$. A tuple $(G(A))_{A\in\operatorname{Ob}\mathcal{C}}$ with $G(A) \subseteq F(A)$ for $A \in \operatorname{Ob}\mathcal{C}$ such that $G(f) := F(f)|_{G(A)}^{G(B)}$ exists for all $A \xrightarrow{f} B$ defines a functor $G\colon \mathcal{C} \to \mathtt{Set}$. Such a functor is called a *subfunctor* of F.

Definition 1.1.11. Suppose given a category $\mathcal{C}$. Let $\hat{\mathcal{C}} := \mathtt{Set}^{(\mathcal{C}^{\mathrm{op}})}$ denote the functor category with objects being functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathtt{Set}$.

Suppose given a ring R. Let ${}_R\hat{\mathcal{C}} := (\text{Mod-}R)^{(\mathcal{C}^{\mathrm{op}})}$ denote the functor category with objects being functors from $\mathcal{C}^{\mathrm{op}}$ to Mod-R.

Definition 1.1.12. Suppose given categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ and a functor $L\colon \mathcal{A} \to \mathcal{B}$.

We obtain a functor $L^{\mathcal{C}}\colon \mathcal{A}^{\mathcal{C}} \to \mathcal{B}^{\mathcal{C}}$ by setting

$$L^{\mathcal{C}}\big(F \overset{\alpha}{\Longrightarrow} G\big) := \big(L \circ F \overset{L\star\alpha}{\Longrightarrow} L \circ G\big)$$

for $F \overset{\alpha}{\Longrightarrow} G$ in $\mathcal{A}^{\mathcal{C}}$.

We also write $L_{\mathcal{C}^{\mathrm{op}}} := L^{\mathcal{C}}$.

In this way, we obtain the functor ${}_R\Upsilon_{\mathcal{C}}\colon {}_R\hat{\mathcal{C}} \to \hat{\mathcal{C}}$, cf. convention 37 and definition 1.1.11.

Remark 1.1.13. Suppose given a ring R, a set S and an R-module M.
The set $\operatorname{Hom}_{\mathtt{Set}}(S, {}_R\Upsilon(M))$ carries an R-module structure, where addition and multiplication are defined pointwise using the R-module structure on M. Note that M and ${}_R\Upsilon(M)$ are the same, viewed as sets.

So for $f, g \in \operatorname{Hom}_{\mathtt{Set}}(S, {}_R\Upsilon(M))$, $r \in R$ and $a \in S$, we have $(s)(f+g) := (s)f + (s)g$ and $(s)(fr) := (s)f \cdot r$.

We obtain the functor $\mathscr{H}\colon \mathtt{Set}^{\mathrm{op}} \times \text{Mod-}R \to \text{Mod-}R$ by setting

$$\mathscr{H}(S, M) := \operatorname{Hom}_{\mathtt{Set}}(S, {}_R\Upsilon(M)) \qquad \text{and} \qquad (f)\,\mathscr{H}(\sigma^{\mathrm{op}}, \mu) := \sigma f \mu$$

for $(S, M) \xrightarrow{(\sigma^{\mathrm{op}},\mu)} (T, N)$ in $\mathtt{Set}^{\mathrm{op}} \times \text{Mod-}R$ and $f \in \operatorname{Hom}_{\mathtt{Set}}(S, {}_R\Upsilon(M))$.

1.2 Yoneda functors

Lemma/Definition 1.2.1.

(a) Suppose given a category $\mathcal{C}$ and an object $A \in \operatorname{Ob}\mathcal{C}$.

For $C \xrightarrow{f} B$ in $\mathcal{C}$, let

$$\mathrm{y}_{\mathcal{C},A}(B) := \operatorname{Hom}_{\mathcal{C}}(B, A)$$

and

$$\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})\colon\ \mathrm{Hom}_{\mathcal{C}}(B,A)\to\mathrm{Hom}_{\mathcal{C}}(C,A)\colon\ h\mapsto fh.$$

This defines a functor $\mathrm{y}_{\mathcal{C},A}\colon\mathcal{C}^{\mathrm{op}}\to\mathtt{Set}$. So $\mathrm{y}_{\mathcal{C},A}\in\mathrm{Ob}\,\hat{\mathcal{C}}$. We call $\mathrm{y}_{\mathcal{C},A}$ the *Yoneda functor* of A in $\mathcal{C}$.

(b) Suppose given a preadditive category $\mathcal{C}$ and an object $A\in\mathrm{Ob}\,\mathcal{C}$. Recall that for $B\in\mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(B,A)$ is endowed with a $\mathbf{Z}$-module structure, cf. convention 8.

For $C\xrightarrow{f}B$ in $\mathcal{C}$, let

$$_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(B):=\mathrm{Hom}_{\mathcal{C}}(B,A)$$

and

$$_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})\colon\ \mathrm{Hom}_{\mathcal{C}}(B,A)\to\mathrm{Hom}_{\mathcal{C}}(C,A)\colon\ h\mapsto fh.$$

This defines an additive functor $_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}\colon\mathcal{C}^{\mathrm{op}}\to\text{Mod-}\mathbf{Z}$. So $_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}\in\mathrm{Ob}\,\hat{\mathcal{C}}$. We call $_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}$ the $\mathbf{Z}$-*Yoneda functor* of A in $\mathcal{C}$.

Moreover, $_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}$ preserves kernels.

By construction, we have $_{\mathbf{Z}}\Upsilon_{\mathcal{C}}({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A})={}_{\mathbf{Z}}\Upsilon\circ{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}=\mathrm{y}_{\mathcal{C},A}$, cf. definition 1.1.12.

Proof. Ad (a). Suppose given $D\xrightarrow{g}C\xrightarrow{f}B$ in $\mathcal{C}$. For $h\in\mathrm{Hom}_{\mathcal{C}}(B,A)$, we have

$$(h)\big(\mathrm{y}_{\mathcal{C},A}(1_B^{\mathrm{op}})\big)=1_B\,h=h$$

and

$$(h)\ \mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})\cdot\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})=(fh)\ \mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})=gfh=(h)\ \mathrm{y}_{\mathcal{C},A}((gf)^{\mathrm{op}})=(h)\ \mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}g^{\mathrm{op}}),$$

so $\mathrm{y}_{\mathcal{C},A}(1_B^{\mathrm{op}})=1_{\mathrm{y}_{\mathcal{C},A}(B)}$ and $\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})\cdot\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})=\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}g^{\mathrm{op}})$.

Ad (b). Suppose given $C\xrightarrow{f}B$ in $\mathcal{C}$ and $h,k\in\mathrm{Hom}_{\mathcal{C}}(B,A)$. We have

$$(h+k)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})=f(h+k)=fh+fk=(h)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})+(k)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}),$$

so $_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})\in\mathrm{Mor}(\text{Mod-}\mathbf{Z})$.

The same calculation as in (a) now shows that $_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}$ is a functor. We show that it is additive.

Suppose given $C\overset{f}{\underset{g}{\rightrightarrows}}B$ in $\mathcal{C}$. For $h\in\mathrm{Hom}_{\mathcal{C}}(B,A)$, we have

$$\begin{aligned}(h)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}+g^{\mathrm{op}})&=(h)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}((f+g)^{\mathrm{op}})=(f+g)h=fh+gh\\&=(h)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})+(h)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})=(h)({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})+{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})).\end{aligned}$$

Thus ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}} + g^{\mathrm{op}}) = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}) + {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})$.

Suppose given a right-exact sequence $Z \xrightarrow{g} Y \xrightarrow{f} X$ in $\mathcal{C}$. We have to show that the sequence

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})} \mathrm{Hom}_{\mathcal{A}}(Y, A) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})} \mathrm{Hom}_{\mathcal{A}}(Z, A)$$

is left-exact in Mod-$\mathbf{Z}$. The morphism ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})$ is monomorphic in Mod-$\mathbf{Z}$ since f is epimorphic in $\mathcal{A}$. Moreover, for $h \in \mathrm{Hom}_{\mathcal{A}}(Y, A)$, we have $(h)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}}) = gh = 0$ if and only if there exists $u \in \mathrm{Hom}_{\mathcal{A}}(X, A)$ such that $(u)\,{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}) = uf = h$ since f is a cokernel of g. □

Lemma 1.2.2 (Yoneda)**.**

(a) Suppose given a category $\mathcal{C}$, an object $A \in \mathrm{Ob}\,\mathcal{C}$ and a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathtt{Set}$. Recall that $\mathrm{Hom}_{\hat{\mathcal{C}}}(\mathrm{y}_{\mathcal{C},A}, F)$ denotes the set of transformations from $\mathrm{y}_{\mathcal{C},A}$ to F.

The map

$$\begin{aligned} \gamma_{F,A}\colon\ \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathrm{y}_{\mathcal{C},A}, F) &\to F(A) \\ \alpha &\mapsto (1_A)\alpha_A \end{aligned}$$

is a bijection with inverse

$$\begin{aligned} F(A) &\to \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathrm{y}_{\mathcal{C},A}, F) \\ x &\mapsto \big(\alpha^x_B\colon\ \mathrm{Hom}_{\mathcal{C}}(B, A) \to F(B)\colon f \mapsto (x)\,F(f^{\mathrm{op}})\big)_{B \in \mathrm{Ob}\,\mathcal{C}} . \end{aligned}$$

(b) Suppose given a preadditive category $\mathcal{C}$, an object $A \in \mathrm{Ob}\,\mathcal{C}$ and an additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to$ Mod-$\mathbf{Z}$.

The map

$$\begin{aligned} {}_{\mathbf{Z}}\gamma_{F,A}\colon\ \mathrm{Hom}_{{}_{\mathbf{Z}}\hat{\mathcal{C}}}({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}, F) &\to F(A) \\ \alpha &\mapsto (1_A)\alpha_A \end{aligned}$$

is an isomorphism in Mod-$\mathbf{Z}$ with inverse

$$\begin{aligned} F(A) &\to \mathrm{Hom}_{{}_{\mathbf{Z}}\hat{\mathcal{C}}}({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}, F) \\ x &\mapsto \big(\alpha^x_B\colon\ \mathrm{Hom}_{\mathcal{C}}(B, A) \to F(B)\colon f \mapsto (x)\,F(f^{\mathrm{op}})\big)_{B \in \mathrm{Ob}\,\mathcal{C}} . \end{aligned}$$

Proof. Ad (a). Suppose given $x \in F(A)$. We show that $\alpha^x := (\alpha^x_B)_{B \in \mathrm{Ob}\,\mathcal{C}}$ is natural.

Suppose given $D \xrightarrow{f} B$ in $\mathcal{C}$. For $h \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, we have

$$(h)\alpha^x_B \cdot F(f^{\mathrm{op}}) = (x)\,F(h^{\mathrm{op}}) \cdot F(f^{\mathrm{op}}) = (x)\,F((fh)^{\mathrm{op}}) = (fh)\alpha^x_D = (h)\ \mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}) \cdot \alpha^x_D ,$$

so $\alpha^x_B \cdot F(f^{\mathrm{op}}) = \mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}}) \cdot \alpha^x_D$.

It remains to show that the maps are mutually inverse.

Suppose given $\alpha \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathrm{y}_{\mathcal{C},A}, F)$, $B \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(B, A)$.

We have

$$(f)\,\alpha_B^{(1_A)\alpha_A} = (1_A)\alpha_A F(f^{\mathrm{op}}) = (1_A)\;\mathrm{y}_{\mathcal{C},A}(f^{\mathrm{op}})\,\alpha_B = (f)\alpha_B$$

and therefore $\alpha^{(1_A)\alpha_A} = \alpha$.

Conversely, suppose given $x \in F(A)$. We have $(1_A)\alpha_A^x = (x)\,F(1_A^{\mathrm{op}}) = x$.

Ad (b). Suppose given $x \in F(A)$ and $B \in \mathrm{Ob}\,\mathcal{C}$. We show that α_B^x is $\mathbf{Z}$-linear.

For $f, g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, we have

$$\begin{aligned}(f+g)\alpha_B^x &= (x)\,F((f+g)^{\mathrm{op}}) = (x)\,F(f^{\mathrm{op}} + g^{\mathrm{op}}) = (x)\,(F(f^{\mathrm{op}}) + F(g^{\mathrm{op}})) \\ &= (x)\,F(f^{\mathrm{op}}) + (x)\,F(g^{\mathrm{op}}) = (f)\alpha_B^x + (g)\alpha_B^x.\end{aligned}$$

The same calculation as in (a) now shows that ${}_{\mathbf{Z}}\gamma_{F,A}$ is a bijection. We show that it is $\mathbf{Z}$-linear. For $\alpha, \beta \in \mathrm{Hom}_{{}_{\mathbf{Z}}\hat{\mathcal{C}}}({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}, F)$, we have

$$(\alpha+\beta)\,{}_{\mathbf{Z}}\gamma_{F,A} = (1_A)(\alpha+\beta)_A = (1_A)(\alpha_A + \beta_A) = (1_A)\alpha_A + (1_A)\beta_A = (\alpha)\,{}_{\mathbf{Z}}\gamma_{F,A} + (\beta)\,{}_{\mathbf{Z}}\gamma_{F,A}\,.$$

□

Lemma/Definition 1.2.3.

(a) Suppose given a category $\mathcal{C}$. For $A \in \mathrm{Ob}\,\mathcal{C}$, let $\mathrm{y}_{\mathcal{C}}(A) := \mathrm{y}_{\mathcal{C},A}$.

For $A \xrightarrow{f} B$ in $\mathcal{C}$ and $D \in \mathrm{Ob}\,\mathcal{C}$, let $(\mathrm{y}_{\mathcal{C}}(f))_D \colon \mathrm{Hom}_{\mathcal{C}}(D, A) \to \mathrm{Hom}_{\mathcal{C}}(D, B) \colon h \mapsto hf$.

Let $\mathrm{y}_{\mathcal{C}}(f) := \big((\mathrm{y}_{\mathcal{C}}(f))_D\big)_{D \in \mathrm{Ob}\,\mathcal{C}} \colon \mathrm{y}_{\mathcal{C},A} \Rightarrow \mathrm{y}_{\mathcal{C},B}$.

This defines a full and faithful functor $\mathrm{y}_{\mathcal{C}} \colon \mathcal{C} \to \hat{\mathcal{C}}$.

(b) Suppose given a preadditive category $\mathcal{C}$. For $A \in \mathrm{Ob}\,\mathcal{C}$, let ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(A) := {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A}$.

For $A \xrightarrow{f} B$ in $\mathcal{C}$ and $D \in \mathrm{Ob}\,\mathcal{C}$, let $({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_D \colon \mathrm{Hom}_{\mathcal{C}}(D, A) \to \mathrm{Hom}_{\mathcal{C}}(D, B) \colon h \mapsto hf$.

Let ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f) := \big(({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_D\big)_{D \in \mathrm{Ob}\,\mathcal{C}} \colon {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},A} \Rightarrow {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C},B}$.

This defines a full, faithful and additive functor ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}} \colon \mathcal{C} \to {}_{\mathbf{Z}}\hat{\mathcal{C}}$.

By construction, we have ${}_{\mathbf{Z}}\Upsilon_{\mathcal{C}} \circ {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}} = \mathrm{y}_{\mathcal{C}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{y}_{\mathcal{C}}"] \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}"'] & \hat{\mathcal{C}} \\ {}_{\mathbf{Z}}\hat{\mathcal{C}} \arrow[ur, "{}_{\mathbf{Z}}\Upsilon_{\mathcal{C}}"'] & \end{tikzcd}$$

Suppose given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$.

The sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is left-exact in $\mathcal{C}$ if and only if the sequence ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(A) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(B) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(C)$ is left-exact in ${}_{\mathbf{Z}}\hat{\mathcal{C}}$.

Proof. Ad (a). Suppose given $A \xrightarrow{f} B$ in $\mathcal{C}$. We show that $\mathrm{y}_{\mathcal{C}}(f)$ is natural.

Suppose given $E \xrightarrow{g} D$ in $\mathcal{C}$. For $h \in \mathrm{y}_{\mathcal{C},A}(D) = \mathrm{Hom}_{\mathcal{C}}(D, A)$, we have

$$\begin{aligned}(h)\,(\mathrm{y}_{\mathcal{C}}(f))_D \cdot \mathrm{y}_{\mathcal{C},B}(g^{\mathrm{op}}) = (hf)\ \mathrm{y}_{\mathcal{C},B}(g^{\mathrm{op}}) = ghf = (gh)\,(\mathrm{y}_{\mathcal{C}}(f))_E \\ = (h)\,\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}}) \cdot (\mathrm{y}_{\mathcal{C}}(f))_E\end{aligned}$$

and therefore $(\mathrm{y}_{\mathcal{C}}(f))_D \cdot \mathrm{y}_{\mathcal{C},B}(g^{\mathrm{op}}) = \mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}}) \cdot (\mathrm{y}_{\mathcal{C}}(f))_E$.

$$\begin{tikzcd}
\mathrm{y}_{\mathcal{C},A}(D) \arrow[r, "(\mathrm{y}_{\mathcal{C}}(f))_D"] \arrow[d, "\mathrm{y}_{\mathcal{C},A}(g^{\mathrm{op}})"'] & \mathrm{y}_{\mathcal{C},B}(D) \arrow[d, "\mathrm{y}_{\mathcal{C},B}(g^{\mathrm{op}})"] \\
\mathrm{y}_{\mathcal{C},A}(E) \arrow[r, "(\mathrm{y}_{\mathcal{C}}(f))_E"] & \mathrm{y}_{\mathcal{C},B}(E)
\end{tikzcd}$$

The functor $\mathrm{y}_{\mathcal{C}}$ is faithful:

For $A \xrightarrow{f} B$ and $A \xrightarrow{g} B$ in $\mathcal{C}$ with $\mathrm{y}_{\mathcal{C}}(f) = \mathrm{y}_{\mathcal{C}}(g)$, we obtain

$$f = (1_A)\,\big(\mathrm{y}_{\mathcal{C}}(f)\big)_A = (1_A)\,\big(\mathrm{y}_{\mathcal{C}}(g)\big)_A = g.$$

The functor $\mathrm{y}_{\mathcal{C}}$ is full:

Suppose given $\alpha\colon\ \mathrm{y}_{\mathcal{C},A} \Rightarrow \mathrm{y}_{\mathcal{C},B}$ for $A, B \in \mathrm{Ob}\,\mathcal{C}$. Write $f := (1_A)\alpha_A \in \mathrm{y}_{\mathcal{C},B}(A) = \mathrm{Hom}_{\mathcal{C}}(A, B)$.

We *claim* that $\alpha = \mathrm{y}_{\mathcal{C}}(f)$.

For $D \in \mathrm{Ob}\,\mathcal{C}$ and $h \in \mathrm{y}_{\mathcal{C},A}(D) = \mathrm{Hom}_{\mathcal{C}}(D, A)$ we have

$$(h)\big(\mathrm{y}_{\mathcal{C}}(f))_D = hf = h \cdot (1_A)\alpha_A = (1_A)(\alpha_A) \cdot \mathrm{y}_{\mathcal{C},B}(h^{\mathrm{op}}) = (1_A)\ \mathrm{y}_{\mathcal{C},A}(h^{\mathrm{op}}) \cdot \alpha_D = (h)\alpha_D\,,$$

so $\alpha_D = \big(\mathrm{y}_{\mathcal{C}}(f)\big)_D$. This proves the *claim*.

$$\begin{tikzcd}
\mathrm{y}_{\mathcal{C},A}(A) \arrow[r, "\alpha_A"] \arrow[d, "\mathrm{y}_{\mathcal{C},A}(h^{\mathrm{op}})"'] & \mathrm{y}_{\mathcal{C},B}(A) \arrow[d, "\mathrm{y}_{\mathcal{C},B}(h^{\mathrm{op}})"] \\
\mathrm{y}_{\mathcal{C},A}(D) \arrow[r, "\alpha_D"] & \mathrm{y}_{\mathcal{C},B}(D)
\end{tikzcd}$$

Ad (b). Suppose given $A \xrightarrow{f} B$ in $\mathcal{C}$ and $D \in \mathrm{Ob}\,\mathcal{C}$. We show that $({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_D$ is $\mathbf{Z}$-linear. For $h, k \in \mathrm{Hom}_{\mathcal{C}}(D, A)$, we have

$$(h + k)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_D = (h + k)f = hf + kf = (h)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_D + (k)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_D\,.$$

The same calculation as in (a) now shows that ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}$ is a full and faithful functor. We show that it is additive.

Suppose given $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{C}$. For $D \in \operatorname{Ob}\mathcal{C}$ and $h \in \operatorname{Hom}_{\mathcal{C}}(D, A)$, we have

$$\begin{aligned}(h)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f+g))_D = h(f+g) = hf + hg &= (h)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_D + (h)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_D \\ &= (h)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f) + {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_D\,.\end{aligned}$$

Thus ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f+g) = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f) + {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g)$.

Suppose given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$.

Note that the sequence ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(A) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(B) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(C)$ is left-exact in ${}_{\mathbf{Z}}\hat{\mathcal{C}}$ if and only if the sequence

$$\begin{aligned}&\Big({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(A)(X) \xrightarrow{({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_X} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(B)(X) \xrightarrow{({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_X} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(C)(X)\Big) \\ = &\Big(\operatorname{Hom}_{\mathcal{C}}(X, A) \xrightarrow{({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_X} \operatorname{Hom}_{\mathcal{C}}(X, B) \xrightarrow{({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_X} \operatorname{Hom}_{\mathcal{C}}(X, C)\Big)\end{aligned}$$

is left-exact in Mod-$\mathbf{Z}$ for all $X \in \operatorname{Ob}\mathcal{C}$.

Suppose that the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is left-exact in $\mathcal{C}$. Suppose given $X \in \operatorname{Ob}\mathcal{C}$ and $h \in \operatorname{Hom}_{\mathcal{C}}(X, B)$ such that $hg = (h)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_X = 0$. Since f is a kernel of g in $\mathcal{C}$, there exists a unique $k \in \operatorname{Hom}_{\mathcal{C}}(X, A)$ such that $(k)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_X = kf = 0$.

Thus the sequence $\operatorname{Hom}_{\mathcal{C}}(X, A) \xrightarrow{({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_X} \operatorname{Hom}_{\mathcal{C}}(X, B) \xrightarrow{({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_X} \operatorname{Hom}_{\mathcal{C}}(X, C)$ is left-exact in Mod-$\mathbf{Z}$.

Conversely, suppose that the sequence ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(A) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(B) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(C)$ is left-exact in ${}_{\mathbf{Z}}\hat{\mathcal{C}}$. Suppose given $X \xrightarrow{h} B$ in $\mathcal{C}$ such that $(h)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_X = hg = 0$. Since $({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_X$ is a kernel of $({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(g))_X$ in Mod-$\mathbf{Z}$, there exists a unique $k \in \operatorname{Hom}_{\mathcal{C}}(X, A)$ such that $kf = (k)\,({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{C}}(f))_X = h$.
Thus the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is left-exact in $\mathcal{C}$. □

Lemma/Definition 1.2.4.

(a) Suppose given a category $\mathcal{C}$ and an object $A \in \operatorname{Ob}\mathcal{C}$.

For $B \xrightarrow{f} C$ in $\mathcal{C}$, let

$$\mathrm{y}^{\mathcal{C},A}(B) := \operatorname{Hom}_{\mathcal{C}}(A, B)$$

and

$$\mathrm{y}^{\mathcal{C},A}(f)\colon \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{C}}(A, C)\colon h \mapsto hf.$$

This defines a functor $\mathrm{y}^{\mathcal{C},A}\colon \mathcal{C} \to \mathtt{Set}$. We call $\mathrm{y}^{\mathcal{C},A}$ the *hom-functor* of A in $\mathcal{C}$.

(b) Suppose given a preadditive category $\mathcal{C}$ and an object $A \in \operatorname{Ob}\mathcal{C}$. Recall that for $B \in \operatorname{Ob}\mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is endowed with a $\mathbf{Z}$-module structure, cf. convention 8.

For $B \xrightarrow{f} C$ in $\mathcal{C}$, let

$$ {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(B) := \mathrm{Hom}_{\mathcal{C}}(A,B) $$

and

$$ {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f)\colon\ \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C)\colon h \mapsto hf. $$

This defines an additive functor ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}\colon \mathcal{C} \to \text{Mod-}\mathbf{Z}$. We call ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}$ the $\mathbf{Z}$-*hom-functor* of A in $\mathcal{C}$.

Moreover, ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}$ preserves kernels.

By construction, we have ${}_{\mathbf{Z}}\Upsilon_{\mathcal{C}}({}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}) = {}_{\mathbf{Z}}\Upsilon \circ {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A} = \mathrm{y}^{\mathcal{C},A}$, cf. definition 1.1.12.

Proof. Ad (a). Suppose given $B \xrightarrow{f} C \xrightarrow{g} C$ in $\mathcal{C}$. For $h \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, we have

$$ (h)\big(\,\mathrm{y}^{\mathcal{C},A}(1_B)\big) = 1_B\, h = h $$

and

$$ (h)\ \mathrm{y}^{\mathcal{C},A}(f) \cdot \mathrm{y}^{\mathcal{C},A}(g) = (hf)\ \mathrm{y}^{\mathcal{C},A}(g) = hfg = (h)\ \mathrm{y}^{\mathcal{C},A}(fg), $$

so $\mathrm{y}^{\mathcal{C},A}(1_B) = 1_{\mathrm{y}^{\mathcal{C},A}(B)}$ and $\mathrm{y}^{\mathcal{C},A}(f) \cdot \mathrm{y}^{\mathcal{C},A}(g) = \mathrm{y}^{\mathcal{C},A}(fg)$.

Ad (b). Suppose given $B \xrightarrow{f} C$ in $\mathcal{C}$ and $h, k \in \mathrm{Hom}_{\mathcal{C}}(A,B)$. We have

$$ (h+k)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f) = (h+k)f = hf + kf = (h)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f) + (k)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f), $$

so ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f) \in \mathrm{Mor}(\text{Mod-}\mathbf{Z})$.

The same calculation as in (a) now shows that ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}$ is a functor. We show that it is additive.

Suppose given $B \underset{g}{\overset{f}{\rightrightarrows}} C$ in $\mathcal{C}$. For $h \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, we have

$$ \begin{aligned} (h)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f+g) &= h(f+g) = hf + hg = (h)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f) + (h)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(g) \\ &= (h)({}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f) + {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(g)). \end{aligned} $$

Thus ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f+g) = {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f) + {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(g)$.

Suppose given a left-exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. We have to show that the sequence

$$ \mathrm{Hom}_{\mathcal{A}}(A,X) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f)} \mathrm{Hom}_{\mathcal{A}}(A,Y) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(g)} \mathrm{Hom}_{\mathcal{A}}(A,Z) $$

is left-exact in Mod-$\mathbf{Z}$. The morphism ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f)$ is monomorphic in Mod-$\mathbf{Z}$ since f is monomorphic in $\mathcal{A}$. Moreover, for $h \in \mathrm{Hom}_{\mathcal{A}}(A,Y)$, we have $(h)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(g) = hg = 0$ if and only if there exists $u \in \mathrm{Hom}_{\mathcal{A}}(A,X)$ such that $(u)\, {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{C},A}(f) = uf = h$ since f is a kernel of g. $\square$

Remark 1.2.5. Suppose given an abelian category $\mathcal{A}$ and $P \in \mathrm{Ob}\,\mathcal{A}$. The functor ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}$ is exact if and only if P is projective in $\mathcal{A}$, cf. convention 31.

1.3 Adjoint functors

Lemma 1.3.1. Suppose given an additive category $\mathcal{A}$.

Suppose given diagrams $A = \big(A_0 \xrightarrow{a} A_1\big), B = \big(B_0 \xrightarrow{b} B_1\big) \in \mathrm{Ob}(\mathcal{A}^{\Delta_1})$ and an isomorphism $\eta = (\eta_0, \eta_1) \in \mathrm{Hom}_{\mathcal{A}^{\Delta_1}}(A, B)$.

(a) Suppose that $K \xrightarrow{k} A_0$ is a kernel of a in $\mathcal{A}$. Then $k\eta_0$ is a kernel of b in $\mathcal{A}$.

$$\begin{tikzcd}
K \arrow[r, "k"] & A_0 \arrow[r, "a"] \arrow[d, "\eta_0"', "\sim"] & A_1 \arrow[d, "\eta_1"', "\sim"] \\
K \arrow[r, "k\eta_0"] & B_0 \arrow[r, "b"] & B_1
\end{tikzcd}$$

(b) Suppose that $A_1 \xrightarrow{c} C$ is a cokernel of a in $\mathcal{A}$. Then $\eta_1^{-1}c$ is a cokernel of b in $\mathcal{A}$.

$$\begin{tikzcd}
A_0 \arrow[r, "a"] \arrow[d, "\eta_0"', "\sim"] & A_1 \arrow[r, "c"] \arrow[d, "\eta_1"', "\sim"] & C \\
B_0 \arrow[r, "b"] & B_1 \arrow[r, "\eta_1^{-1}c"] & C
\end{tikzcd}$$

(c) Suppose that the diagram

$$\begin{tikzcd}
K \arrow[r, "k"] & A_0 \arrow[rrr, "a"] \arrow[dr, "p"'] & & & A_1 \arrow[r, "c"] & C \\
& & I \arrow[r, "u"'] & J \arrow[ur, "i"'] & &
\end{tikzcd}$$

is a kernel-cokernel-factorisation of a in $\mathcal{A}$. Then the diagram

$$\begin{tikzcd}
K \arrow[r, "k\eta_0"] & B_0 \arrow[rrr, "b"] \arrow[dr, "\eta_0^{-1}p"'] & & & B_1 \arrow[r, "\eta_0^{-1}c"] & C \\
& & I \arrow[r, "u"'] & J \arrow[ur, "i\eta_1"'] & &
\end{tikzcd}$$

is a kernel-cokernel-factorisation of b in $\mathcal{A}$.

Lemma 1.3.2. Suppose given an adjunction $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$ with unit $\varepsilon\colon 1_{\mathcal{A}} \Rightarrow G \circ F$ and counit $\eta\colon F \circ G \Rightarrow 1_{\mathcal{B}}$, cf. convention 43.

(a) If $\mathcal{A}$ and $\mathcal{B}$ are additive categories, then F and G are additive functors.

(b) If $\mathcal{A}$ and $\mathcal{B}$ are additive categories, then F preserves cokernels and G preserves kernels. So if $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, then F is right-exact and G is left-exact.

(c) Suppose that $\mathcal{A}$ is abelian and $\mathcal{B}$ is additive. Suppose that F preserves kernels and G is full and faithful. Then $\mathcal{B}$ is abelian and, consequently, F is exact and G is left-exact.

Suppose given $Y \xrightarrow{f} Y'$ in $\mathcal{B}$. Suppose given a kernel k of $G(f)$ and a cokernel c of $G(f)$. A kernel of f is given by $F(k) \cdot \eta_Y$ and a cokernel of f is given by $\eta_{Y'}^{-1} \cdot F(c)$.

Moreover, the counit η is an isotransformation with $G(\eta_Y^{-1}) = \varepsilon_{G(Y)}$ for $Y \in \operatorname{Ob}\mathcal{B}$.

Cf. [10, prop. V.5.3] and [13, cor. 16.6.2].

Proof. Write $\Phi_{X,Y} := \Phi^{\varepsilon,\eta}_{X,Y}$ for $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$, cf. convention 43.

Ad (a). We want to show that F is additive.

We have $\operatorname{Hom}_{\mathcal{B}}(F(0_{\mathcal{A}}), F(0_{\mathcal{A}})) \cong \operatorname{Hom}_{\mathcal{A}}(0_{\mathcal{A}}, (G \circ F)(0_{\mathcal{A}}))$ in $\texttt{Set}$ via $\Phi_{0_{\mathcal{A}},F(0_{\mathcal{A}})}$.

Since $0_{\mathcal{A}}$ is a zero object in $\mathcal{A}$, both sets contain precisely one element. So $1_{F(0_{\mathcal{A}})} = 0_{F(0_{\mathcal{A}})}$ and therefore $F(0_{\mathcal{A}}) \cong 0_{\mathcal{B}}$.

For $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$, we have

$$\begin{aligned}(0_{F(X),Y})\,\Phi_{X,Y} &= (0_{F(X)} \cdot 0_{F(X),Y})\,\Phi_{X,Y} = (F(0_X) \cdot 0_{F(X),Y} \cdot 1_Y)\,\Phi_{X,Y} \\ &= 0_X \cdot (0_{F(X),Y})\,\Phi_{X,Y} \cdot 1_{G(Y)} = 0_{X,G(Y)}\end{aligned}$$

Suppose given $X, X' \in \operatorname{Ob}\mathcal{A}$. Write $i := \iota_1^{X,X'}$, $j := \iota_2^{X,X'}$, $p := p_1^{X,X'}$ and $q := p_2^{X,X'}$. So $X \underset{p}{\overset{i}{\rightleftarrows}} X \oplus X' \underset{q}{\overset{j}{\leftrightarrows}} X'$ is a direct sum in $\mathcal{A}$, cf. convention 15.

It suffices to show that $\left(\begin{smallmatrix} F(i) \\ F(j) \end{smallmatrix}\right) : F(X) \oplus F(X') \to F(X \oplus X')$ is epimorphic, cf. lemma 1.1.3.

Suppose given $F(X \oplus X') \xrightarrow{u} Y$ in $\mathcal{B}$ with $0 = \left(\begin{smallmatrix} F(i) \\ F(j) \end{smallmatrix}\right) u = \left(\begin{smallmatrix} F(i)u \\ F(j)u \end{smallmatrix}\right)$.

We have

$$0 = (0)\,\Phi_{X,Y} = (F(i) \cdot u)\,\Phi_{X,Y} = i \cdot (u)\,\Phi_{X \oplus X',Y}$$

and similarly

$$0 = (0)\,\Phi_{X',Y} = (F(j) \cdot u)\,\Phi_{X',Y} = j \cdot (u)\,\Phi_{X \oplus X',Y}.$$

So $\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right) \cdot (u)\,\Phi_{X \oplus X',Y} = 0$ and thus $(u)\,\Phi_{X \oplus X',Y} = 0 = (0)\,\Phi_{X \oplus X',Y}$. Therefore $u = 0$ since $\Phi_{X \oplus X',Y}$ is a bijection.

We conclude that F is additive. Dually, G is also additive.

Ad (b). We want to show that F preserves cokernels.

Suppose given a morphism $X \xrightarrow{f} X'$ in $\mathcal{A}$ with cokernel $X' \xrightarrow{c} C$.

Suppose given $F(X') \xrightarrow{t} T$ in $\mathcal{B}$ with $F(f) \cdot t = 0$.

We have $f \cdot (t)\,\Phi_{X',T} = (F(f) \cdot t)\,\Phi_{X,T} = (0)\,\Phi_{X,T} = 0$. Since c is a cokernel of f, there exists $C \xrightarrow{u} G(T)$ in $\mathcal{A}$ with $(t)\,\Phi_{X',T} = c \cdot u$.

We have

$$\big(F(c) \cdot (u)\,\Phi_{C,T}^{-1}\big)\,\Phi_{X',T} = c \cdot \big((u)\,\Phi_{C,T}^{-1}\big)\,\Phi_{C,T} = c \cdot u = (t)\,\Phi_{X',T}$$

and therefore $F(c) \cdot (u)\,\Phi_{X,T}^{-1} = t$.

It now suffices to show that $F(c)$ is epimorphic.

Suppose given $F(C) \xrightarrow{v} Y$ in $\mathcal{B}$ with $F(c) \cdot v = 0$.

We have

$$c \cdot (v)\,\Phi_{C,Y} = (F(c) \cdot v)\,\Phi_{X',Y} = (0)\,\Phi_{X',Y} = 0.$$

Since c is epimorphic, we conclude that $(v)\,\Phi_{C,Y} = 0 = (0)\,\Phi_{C,Y}$. So $v = 0$.

Dually, G preserves kernels.

Ad (c).

We *claim* that the counit η is an isotransformation.

Suppose given $Y \in \operatorname{Ob}\mathcal{B}$. Since G is full, we may choose $Y \xrightarrow{f} (F \circ G)(Y)$ in $\mathcal{B}$ such that $G(f) = \varepsilon_{G(Y)} \colon G(Y) \to (G \circ F \circ G)(Y)$.

We have

$$G(f\eta_Y) = \varepsilon_{G(Y)}\,G(\eta_Y) = 1_{G(Y)} = G(1_Y).$$

Since G is faithful, we conclude that $f\eta_Y = 1_Y$.

We have

$$\eta_Y\,f = (F \circ G)(f)\,\eta_{(F\circ G)(Y)} = F(\varepsilon_{G(Y)})\,\eta_{(F\circ G)(Y)} = 1_{(F\circ G)(Y)}.$$

So η_Y is an isomorphism with inverse f. This proves the *claim*.

Suppose given $Y \xrightarrow{f} Y'$ in $\mathcal{B}$.

We will use the fact that $\big(Y \xrightarrow{f} Y'\big)$ and $\big((F \circ G)(Y) \xrightarrow{(F\circ G)(f)} (F \circ G)(Y')\big)$ are isomorphic in $\operatorname{Ob}(\mathcal{B}^{\Delta_1})$ via $(\eta_Y, \eta_{Y'})$.

Since $\mathcal{A}$ is abelian, the morphism $G(f)$ has a kernel-cokernel-factorisation as follows.

$$\begin{tikzcd}
K \arrow[r, tail, "k"] & G(Y) \arrow[rrr, "G(f)"] \arrow[dr, two heads, "p"'] & & & G(Y') \arrow[r, two heads, "c"] & C \\
& & I \arrow[r, "\sim", "\tilde{f}"'] & J \arrow[ur, tail, "i"'] & &
\end{tikzcd}$$

The functor F preserves kernels by assumption and cokernels by part (b). So the diagram

$$\begin{tikzcd} F(K) \arrow[r, "F(k)"] & (F\circ G)(Y) \arrow[rrr, "(F\circ G)(f)"] \arrow[dr, "F(p)"'] & & & (F\circ G)(Y') \arrow[r, "F(c)"] & F(C) \\ & & F(I) \arrow[r, "\sim", "F(\tilde{f})"'] & F(J) \arrow[ur, "F(i)"'] & & \end{tikzcd}$$

is a kernel-cokernel-factorisation of $(F \circ G)(f)$ where the induced morphism $F(\tilde{f})$ is an isomorphism.

By lemma 1.3.1.(c), the morphism f also has a kernel-cokernel-factorisation where the induced morphism is an isomorphism.

Moreover, a kernel of f is given by $F(k) \cdot \eta_Y$ and a cokernel of f is given by $\eta_{Y'}^{-1} \cdot F(c)$. Cf. lemma 1.3.1.(a,b).

We conclude that $\mathcal{B}$ is abelian. $\square$

1.4 Direct limits

Definition 1.4.1. A non-empty category $\mathcal{I}$ is called *filtering* if it satisfies the following two conditions.

(F1) For $i, j \in \operatorname{Ob}\mathcal{I}$, there exist $\left(i \xrightarrow{\sigma} k\right), \left(j \xrightarrow{\rho} k\right) \in \operatorname{Mor}\mathcal{I}$.

(F2) For a diagram

$$\begin{tikzcd} & k \\ i \arrow[ur, "\sigma"] \arrow[dr, "\sigma'"'] & j \arrow[u, "\tau"'] \arrow[d, "\tau'"] \\ & k' \end{tikzcd}$$

in $\mathcal{I}$, there exist $\left(k \xrightarrow{\lambda} \ell\right), \left(k' \xrightarrow{\lambda'} \ell\right) \in \operatorname{Mor}\mathcal{I}$ such that $\sigma\lambda = \sigma'\lambda'$ and $\tau\lambda = \tau'\lambda'$.

$$\begin{tikzcd} & k \arrow[dr, "\lambda"] & \\ i \arrow[ur, "\sigma"] \arrow[dr, "\sigma'"'] & j \arrow[u, "\tau"'] \arrow[d, "\tau'"] & \ell \\ & k' \arrow[ur, "\lambda'"'] & \end{tikzcd}$$

Definition 1.4.2. Suppose given a filtering category $\mathcal{I}$ and a category $\mathcal{C}$. A functor $A\colon \mathcal{I} \to \mathcal{C}$ is called a *system* over $\mathcal{I}$ in $\mathcal{C}$.

We usually write $A_i := A(i)$ for $i \in \operatorname{Ob}\mathcal{I}$ and $a_\sigma := A(\sigma)$ for $\sigma \in \operatorname{Mor}\mathcal{I}$.

Suppose given systems $A, B\colon \mathcal{I} \to \mathcal{C}$. A transformation $\xi\colon A \Rightarrow B$ is called a *morphism of systems.*

Lemma 1.4.3. Suppose given a non-empty poset category I. Then I is filtering if and only if (F1) holds.

Proof. If I is filtering, then (F1) holds by definition.

Conversely, suppose that (F1) holds. For $i, j \in \operatorname{Ob} I$, write $i \le j$ if $(i \to j) \in \operatorname{Mor} I$.

A diagram

$$\begin{tikzcd} & k \\ i \arrow[ur] \arrow[dr] & j \arrow[u] \arrow[d] \\ & k' \end{tikzcd}$$

in I translates to $i \le k$, $j \le k$, $i \le k'$ and $j \le k'$. We may choose $\ell \in \operatorname{Ob} I$ with $k \le \ell$ and $k' \le \ell$ by (F1). So $i \le \ell$ and $j \le \ell$ by transitivity. Since moreover morphisms between fixed objects in I are unique, we conclude that (F2) holds. $\square$

Definition 1.4.4. Suppose given categories $\mathcal{I}$ and $\mathcal{C}$. Suppose given $M \in \operatorname{Ob}\mathcal{C}$. We obtain a *constant functor* $\mathrm{C}_M^{\mathcal{I},\mathcal{C}}\colon \mathcal{I} \to \mathcal{C}$ by setting $\mathrm{C}_M^{\mathcal{I},\mathcal{C}}(i) := M$ for $i \in \operatorname{Ob}\mathcal{I}$ and $\mathrm{C}_M^{\mathcal{I},\mathcal{C}}(\sigma) := 1_M$ for $\sigma \in \operatorname{Mor}\mathcal{I}$.

Definition 1.4.5. Suppose given a filtering category $\mathcal{I}$ and a category $\mathcal{C}$. Suppose given a system A over $\mathcal{I}$ in $\mathcal{C}$ and $M \in \operatorname{Ob}\mathcal{C}$. A morphism of systems $\xi\colon A \Rightarrow \mathrm{C}_M^{\mathcal{I},\mathcal{C}}$ is called a *compatible family* for A. So $\xi = (\xi_i)_{i \in \operatorname{Ob}\mathcal{I}}$ such that $a_\sigma \xi_j = \xi_i$ for $i \xrightarrow{\sigma} j$ in $\mathcal{I}$.

Definition 1.4.6. Suppose given a filtering category $\mathcal{I}$ and a category $\mathcal{C}$. Suppose given a system A over $\mathcal{I}$ in $\mathcal{C}$. Suppose given $L \in \operatorname{Ob}\mathcal{C}$ and a compatible family $\omega\colon A \Rightarrow \mathrm{C}_L^{\mathcal{I},\mathcal{C}}$ for A.

The pair (L, ω) is called a *direct limit* of A if for each $M \in \operatorname{Ob}\mathcal{C}$ and each compatible family $\xi\colon A \Rightarrow \mathrm{C}_M^{\mathcal{I},\mathcal{C}}$ for A, there exists a unique morphism $\big(L \xrightarrow{u} M\big) \in \operatorname{Mor}\mathcal{C}$ such that $\omega_i u = \xi_i$ for $i \in \mathcal{I}$.

$$\begin{tikzcd} A_i \arrow[dr, "\omega_i"'] \arrow[drr, bend left, "\xi_i"] & & \\ & L \arrow[r, "u"] & M \\ A_j \arrow[ur, "\omega_j"] \arrow[urr, bend right, "\xi_j"'] & & \end{tikzcd}$$

Lemma/Definition 1.4.7. Suppose given a filtering category $\mathcal{I}$.

(a) Suppose given a system A over $\mathcal{I}$ in $\texttt{Set}$.

For $(x,i),(y,j) \in \bigsqcup_{\ell\in\operatorname{Ob}\mathcal{I}} A_\ell := \bigcup_{\ell\in\operatorname{Ob}\mathcal{I}}(A_\ell \times \{\ell\})$, let $(x,i) \sim (y,j)$ if there exist $\big(i\xrightarrow{\sigma}k\big), \big(j\xrightarrow{\rho}k\big) \in \operatorname{Mor}\mathcal{I}$ such that $xa_\sigma = ya_\rho$. This defines an equivalence relation on $\bigsqcup_{\ell\in\operatorname{Ob}\mathcal{I}} A_\ell$. Let $\varinjlim A := \big(\bigsqcup_{\ell\in\operatorname{Ob}\mathcal{I}} A_\ell\big)/\!\sim$ denote the factor set. Let $[x,i]$ denote the equivalence class of $(x,i) \in \bigsqcup_{\ell\in\operatorname{Ob}\mathcal{I}} A_\ell$.

Let $\omega_i^A \colon A_i \to \varinjlim A \colon x \mapsto [x,i]$ for $i \in \operatorname{Ob}\mathcal{I}$.

This defines a compatible family $\omega^A := (\omega_i^A)_{i\in\operatorname{Ob}\mathcal{I}} \colon A \Rightarrow \mathrm{C}^{\mathcal{I},\texttt{Set}}_{\varinjlim A}$ for A. Moreover, $(\varinjlim A, \omega^A)$ is a direct limit of A.

(b) Suppose given a ring R. Recall that ${}_R\Upsilon \colon \text{Mod-}R \to \texttt{Set}$ denotes the forgetful functor, cf. convention 37.

Suppose given a system A over $\mathcal{I}$ in Mod-R. We use the construction in (a) and define an R-module structure on the set $\varinjlim({}_R\Upsilon \circ A)$ as follows and denote the resulting R-module by $\varinjlim A$. So $\varinjlim({}_R\Upsilon \circ A)$ and $\varinjlim A$ are the same, viewed as sets, i.e. ${}_R\Upsilon(\varinjlim A) = \varinjlim({}_R\Upsilon \circ A)$.

For $[x,i],[y,j] \in \varinjlim A$, we may choose $\big(i\xrightarrow{\sigma}k\big), \big(j\xrightarrow{\rho}k\big) \in \operatorname{Mor}\mathcal{I}$ by (F1). Let $[x,i]+[y,j] := [xa_\sigma + ya_\rho, k]$. This definition is independent of the choice of representatives and of the choice of σ and ρ.

For $r \in R$ and $[x,i] \in \varinjlim A$, let $[x,i]r := [xr,i]$. This definition is independent of the choice of representatives.

The maps $\omega_i^A := \omega_i^{{}_R\Upsilon\circ A} \colon A_i \to \varinjlim A$ are R-linear for $i \in \operatorname{Ob}\mathcal{I}$ and we obtain a compatible family $\omega^A = (\omega_i^A)_{i\in\operatorname{Ob}\mathcal{I}} \colon A \Rightarrow \mathrm{C}^{\mathcal{I},\text{Mod-}R}_{\varinjlim A}$ for A. Moreover, $(\varinjlim A, \omega^A)$ is a direct limit of A.

Proof. Ad (a). Write $U := \bigsqcup_{\ell\in\operatorname{Ob}\mathcal{I}} A_\ell$.

We show that $\sim$ defines an equivalence relation.

Suppose given $(x,i) \in U$. The identity on i shows that $\sim$ is reflexive.

By definition, $\sim$ is symmetric.

Suppose given $(x,i),(y,j),(z,p) \in U$ with $(x,i) \sim (y,j)$ and $(y,j) \sim (z,p)$. We may choose $\big(i\xrightarrow{\sigma}k\big), \big(j\xrightarrow{\rho}k\big), \big(j\xrightarrow{\eta}\ell\big), \big(p\xrightarrow{\vartheta}\ell\big) \in \operatorname{Mor}\mathcal{I}$ such that $xa_\sigma = ya_\rho$ and $ya_\eta = za_\vartheta$.

For the diagram

$$\begin{tikzcd}
 & k \\
j \arrow[ur, "\rho"] \arrow[dr, "\eta"'] & j \arrow[u, "\rho"'] \arrow[d, "\eta"] \\
 & \ell
\end{tikzcd}$$

in $\mathcal{I}$, we may choose $\big(k \xrightarrow{\lambda} m\big), \big(\ell \xrightarrow{\mu} m\big) \in \operatorname{Mor} \mathcal{I}$ such that $\rho\lambda = \eta\mu$ by (F2).

We obtain

$$xa_{\sigma\lambda} = xa_\sigma a_\lambda = ya_\rho a_\lambda = ya_{\rho\lambda} = ya_{\eta\mu} = ya_\eta a_\mu = za_\vartheta a_\mu = za_{\vartheta\mu},$$

so $(x,i) \sim (z,p)$ and therefore $\sim$ is transitive.

We show that ω^A is a morphism of systems and therefore a compatible family for A.

Suppose given $i \xrightarrow{\sigma} j$ in $\mathcal{I}$ and $x \in A_i$. We have $(xa_\sigma, j) \sim (x,i)$ since $(xa_\sigma)a_{1_j} = (xa_\sigma)1_{A_j} = xa_\sigma$. Therefore $xa_\sigma\omega_j = [xa_\sigma, j] = [x,i] = x\omega_i$. So $a_\sigma\omega_j = \omega_i$.

We conclude that ω^A is a morphism of systems.

It remains to show that $(\varinjlim A, \omega^A)$ is a direct limit of A.

Suppose given $M \in \operatorname{Ob}(\mathtt{Set})$ and a compatible family $\xi\colon A \Rightarrow \mathrm{C}_M^{\mathcal{I},\mathtt{Set}}$.

For $u\colon \varinjlim A \to M$ with $\omega_i u = \xi_i$ for $i \in \operatorname{Ob}\mathcal{I}$, we necessarily have $x\xi_i = x\omega_i u = [x,i]u$ for $x \in A_i$.

Therefore, let $u\colon \varinjlim A \to M\colon [x,i] \mapsto x\xi_i$. This is a well-defined map since in case $[x,i] = [y,j] \in \varinjlim A$, we may choose $\big(i \xrightarrow{\sigma} k\big), \big(j \xrightarrow{\rho} k\big) \in \operatorname{Mor}\mathcal{I}$ such that $xa_\sigma = ya_\rho$ and obtain $x\xi_i = xa_\sigma\xi_k = ya_\rho\xi_k = y\xi_j$.

Suppose given $i \in \operatorname{Ob}\mathcal{I}$ and $x \in A_i$. We have $x\omega_i u = [x,i]u = x\xi_i$, so $\omega_i u = \xi_i$.

We conclude that $(\varinjlim A, \omega^A)$ is a direct limit of A.

Ad (b). Suppose given $[x,i] = [x',i'], [y,j] = [y',j'] \in \varinjlim A$ and $r \in R$.

So there exist $\big(i \xrightarrow{\alpha} \ell\big)$, $\big(i' \xrightarrow{\alpha'} \ell\big)$, $\big(j \xrightarrow{\beta} \ell'\big)$, $\big(j' \xrightarrow{\beta'} \ell'\big) \in \operatorname{Mor}\mathcal{I}$ with $xa_\alpha = x'a_{\alpha'}$ and $ya_\beta = y'a_{\beta'}$.

Suppose given $\big(i \xrightarrow{\sigma} k\big)$, $\big(i' \xrightarrow{\sigma'} k'\big)$, $\big(j \xrightarrow{\rho} k\big)$, $\big(j' \xrightarrow{\rho'} k'\big) \in \operatorname{Mor}\mathcal{I}$.

We want to show that $[xa_\sigma + ya_\rho, k] = [xa_{\sigma'} + ya_{\rho'}, k']$ and that $[xr, i] = [x'r, i']$.

By (F1), we may choose $\big(\ell \xrightarrow{\lambda} m\big)$, $\big(\ell' \xrightarrow{\lambda'} m\big) \in \operatorname{Mor}\mathcal{I}$.

For the diagram

$$\begin{tikzcd} & k \\ i \arrow[ur, "\sigma"] \arrow[dr, "\alpha\lambda"'] & j \arrow[u, "\rho"'] \arrow[d, "\beta\lambda'"] \\ & m \end{tikzcd}$$

in $\mathcal{I}$, we may choose $\big(k \xrightarrow{\gamma} n\big), \big(m \xrightarrow{\delta} n\big) \in \operatorname{Mor}\mathcal{I}$ such that $\sigma\gamma = \alpha\lambda\delta$ and $\rho\gamma = \beta\lambda'\delta$ by (F2).

For the diagram

$$\begin{tikzcd}
 & k' \\
i' \arrow[ur, "\sigma'"] \arrow[dr, "\alpha'\lambda\delta"'] & j' \arrow[u, "\rho'"'] \arrow[d, "\beta'\lambda'\delta"] \\
 & n
\end{tikzcd}$$

in $\mathcal{I}$, we may choose $\big(k' \xrightarrow{\varepsilon} p\big), \big(n \xrightarrow{\zeta} p\big) \in \operatorname{Mor} \mathcal{I}$ such that $\sigma'\varepsilon = \alpha'\lambda\delta\zeta$ and $\rho'\varepsilon = \beta'\lambda'\delta\zeta$ by (F2).

Using $k \xrightarrow{\gamma\zeta} p$ and $k' \xrightarrow{\varepsilon} p$ in $\mathcal{I}$, we obtain

$$\begin{aligned}
(xa_\sigma + ya_\rho)a_{\gamma\zeta} &= xa_{\sigma\gamma\zeta} + ya_{\rho\gamma\zeta} \\
&= xa_{\alpha\lambda\delta\rho} + ya_{\beta\lambda'\delta\zeta} \\
&= xa_\alpha a_{\lambda\delta\rho} + ya_\beta a_{\lambda'\delta\zeta} \\
&= x'a_{\alpha'} a_{\lambda\delta\rho} + y'a_{\beta'} a_{\lambda'\delta\zeta} \\
&= x'a_{\alpha'\lambda\delta\rho} + y'a_{\beta'\lambda'\delta\zeta} \\
&= x'a_{\alpha'\lambda\delta\rho} + y'a_{\beta'\lambda'\delta\zeta} \\
&= x'a_{\sigma'\varepsilon} + y'a_{\rho'\varepsilon} \\
&= (x'a_{\sigma'} + y'a_{\rho'})a_\varepsilon.
\end{aligned}$$

So $[xa_\sigma + ya_\rho, k] = [xa_{\sigma'} + ya_{\rho'}, k']$.

$$\begin{tikzcd}
i \arrow[r, "\sigma"] \arrow[dr, "\alpha"' near end] & k \arrow[drr, "\gamma"] & & & \\
j \arrow[ur, "\rho" near start] \arrow[dr, "\beta" near end] & \ell \arrow[r, "\lambda"] & m \arrow[r, "\delta"'] & n \arrow[dr, "\zeta"] & \\
i' \arrow[ur, "\alpha'" near start] \arrow[dr, "\sigma'" near end] & \ell' \arrow[ur, "\lambda'"'] & & & p \\
j' \arrow[ur, "\beta'" near start] \arrow[r, "\rho'"'] & k' \arrow[urrr, "\varepsilon"'] & & &
\end{tikzcd}$$

Using $k \xrightarrow{\gamma\zeta} p$ and $k' \xrightarrow{\varepsilon} p$ in $\mathcal{I}$, we obtain

$$(xr)a_\alpha = xa_\alpha \cdot r = x'a_{\alpha'} \cdot r = (x'r)a_{\alpha'}$$

So $[xr, i] = [x'r, i']$.

We verify the axioms for R-modules.

Note that since $\mathcal{I}$ is non-empty, there exists an object $i \in \mathcal{I}$. We have $[0,i] = [0,j]$ for all $j \in \operatorname{Ob}\mathcal{I}$ since we may choose $\big(i \xrightarrow{\sigma} k\big), \big(j \xrightarrow{\rho} k\big) \in \operatorname{Mor}\mathcal{I}$ by (F1) and obtain $0a_\sigma = 0 = 0a_\rho$.

Suppose given $[x,i],[y,j],[z,\ell] \in \varinjlim A$. We may choose

$$\begin{tikzcd}
i \arrow[r, "\sigma"] & k \arrow[dr, "\lambda"] & \\
j \arrow[ur, "\rho"'] & & m \\
\ell \arrow[urr, "\mu"] & &
\end{tikzcd}$$

in $\mathcal{I}$ by (F1).

Suppose given $r, s \in R$.

We have

$$[x,i] + [y,j] = [xa_\sigma + ya_\rho, k] = [ya_\rho + xa_\sigma, k] = [y,j] + [x,i],$$

$$[x,i] + [0,i] = [xa_{1_i} + 0a_{1_i}, i] = [x,i],$$

$$[x,i] + [-x,i] = [xa_{1_i} - xa_{1_i}, i] = [0,i],$$

$$\begin{aligned}([x,i] + [y,j]) + [z,\ell] &= [xa_\sigma + ya_\rho, k] = [(xa_\sigma + ya_\rho)a_\lambda + za_\mu, m] = [xa_{\sigma\lambda} + ya_{\rho\lambda} + za_\mu, m] \\ &= [xa_{\sigma\lambda} + (ya_{\rho\lambda} + za_\mu)a_{1_m}, m] = [x,i] + [ya_{\rho\lambda} + za_\mu, m] \\ &= [x,i] + ([y,j] + [z,\ell]),\end{aligned}$$

$$[x,i]1_R = [x1_R, i] = [x,i],$$

$$([x,i]r)s = [xr,i]s = [(xr)s,i] = [x(rs),i] = [x,i](rs),$$

$$([x,i] + [y,j])r = [xa_\sigma + ya_\rho, k]r = [(xa_\sigma + ya_\rho)r, k] = [(xr)a_\sigma + (yr)a_\sigma, k] = [xr,i] + [yr,j]$$

and

$$\begin{aligned}[x,i](r+s) &= [x(r+s),i] = [xr + xs, i] = [(xr)a_{1_i} + (xs)a_{1_i}, i] = [xr,i] + [xs,i] \\ &= [x,i]r + [x,i]s.\end{aligned}$$

The maps ω_i^A are R-linear by definition. The calculation in (a) shows that ω^A is a compatible family for A.

It remains to show that $(\varinjlim A, \omega^A)$ is a direct limit of A. We adapt the proof in (a) for given $M \in \mathrm{Ob}(\text{Mod-}R)$ and a compatible family $\xi\colon A \Rightarrow \mathrm{C}_M^{\mathcal{I},\text{Mod-}R}$. It remains to verify that $u\colon \varinjlim A \to M\colon [x,i] \mapsto x\xi_i$ is an R-linear.

Suppose given $[x,i],[y,j] \in \varinjlim A$ and $r \in R$. We may choose $\big(i \xrightarrow{\sigma} k\big), \big(j \xrightarrow{\rho} k\big) \in \mathrm{Mor}\,\mathcal{I}$ by (F1).

We have

$$([x,i]r)u = [xr,i]u = (xr)\xi_i = (x\xi_i)r = ([x,i]u)r$$

and

$$\begin{aligned}([x,i]+[y,j])u = [xa_\sigma + ya_\rho, k]u = (xa_\sigma + ya_rho)\xi_k = xa_\sigma\xi_k + ya_\rho\xi_k = x\xi_i + y\xi_j \\ = [x,i]u + [y,j]u.\end{aligned}$$

□

1.5 Exact categories

Definition 1.5.1. A pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is an additive category and $\mathcal{E} \subseteq \mathrm{Ob}(\mathcal{A}^{\Delta_2})$ is a set consisting of short exact sequences of $\mathcal{A}$, is called an *exact category* if the following axioms (E1-7) are satisfied.

Let

$$\begin{aligned}\mathcal{E}_\mathrm{m} := \big\{\big(A \xrightarrow{f} B\big) \in \mathrm{Mor}\,\mathcal{A} \colon \text{there exists } \big(B \xrightarrow{g} C\big) \in \mathrm{Mor}\,\mathcal{A} \\ \text{such that } \big(A \xrightarrow{f} B \xrightarrow{g} C\big) \in \mathcal{E}\big\}\end{aligned}$$

and let

$$\begin{aligned}\mathcal{E}_\mathrm{e} := \big\{\big(B \xrightarrow{g} C\big) \in \mathrm{Mor}\,\mathcal{A} \colon \text{there exists } \big(A \xrightarrow{f} B\big) \in \mathrm{Mor}\,\mathcal{A} \\ \text{such that } \big(A \xrightarrow{f} B \xrightarrow{g} C\big) \in \mathcal{E}\big\}.\end{aligned}$$

(E1) Suppose given $S, T \in \mathrm{Ob}(\mathcal{A}^{\Delta_2})$. If $S \in \mathcal{E}$ and $S \cong T$ in $\mathcal{A}^{\Delta_2}$, then $T \in \mathcal{E}$.

(E2) For $A \in \mathrm{Ob}\,\mathcal{A}$, we have $1_A \in \mathcal{E}_\mathrm{m}$.

(E3) For $A \in \mathrm{Ob}\,\mathcal{A}$, we have $1_A \in \mathcal{E}_\mathrm{e}$.

(E4) For $f, f' \in \mathcal{E}_\mathrm{m}$, we have $ff' \in \mathcal{E}_\mathrm{m}$.

(E5) For $g, g' \in \mathcal{E}_\mathrm{e}$, we have $gg' \in \mathcal{E}_\mathrm{e}$.

(E6) Suppose given a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

in $\mathcal{A}$ with $f \in \mathcal{E}_\mathrm{m}$. Then there exists a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"'] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

in $\mathcal{A}$ with $f' \in \mathcal{E}_\mathrm{m}$.

(E7) Suppose given a diagram

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

in $\mathcal{A}$ with $f \in \mathcal{E}_\mathrm{e}$. Then there exists a pullback

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

in $\mathcal{A}$ with $f' \in \mathcal{E}_\mathrm{e}$.

A short exact sequence $S \in \mathcal{E}$ is called $\mathcal{E}$*-pure* or just *pure* if unambiguous.

A monomorphism $f \in \mathcal{E}_\mathrm{m}$ is called $\mathcal{E}$*-pure* or just *pure* if unambiguous. In diagrams, we often denote pure monomorphisms by $A \overset{f}{\longrightarrow\!\!\!\bullet\!\!\!\to} B$.

An epimorphism $g \in \mathcal{E}_\mathrm{e}$ is called $\mathcal{E}$*-pure* or just *pure* if unambiguous. In diagrams, we often denote pure epimorphisms by $B \overset{g}{\longrightarrow\!\!\!\!+\!\!\!\to} C$.

We also say that $\mathcal{A}$ is equipped with the exact structure $\mathcal{E}$ if $(\mathcal{A}, \mathcal{E})$ is an exact category.

We write $\mathcal{A} = (\mathcal{A}, \mathcal{E})$ if unambiguous.

Remark 1.5.2. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$. We obtain an exact category $(\mathcal{A}^\mathrm{op}, \mathcal{E}^\mathrm{op})$ if we let $\mathcal{E}^\mathrm{op}$ consist of the short exact sequences $C \xrightarrow{g^\mathrm{op}} B \xrightarrow{f^\mathrm{op}} A$ in $\mathcal{A}^\mathrm{op}$, where $\big(A \xrightarrow{f} B \xrightarrow{g} C\big) \in \mathcal{E}$.

Definition 1.5.3. Suppose given exact categories $(\mathcal{A}, \mathcal{E})$ and $(\mathcal{B}, \mathcal{F})$. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *$(\mathcal{E}, \mathcal{F})$-exact* if we have $F(S) \in \mathcal{F}$ for $S \in \mathcal{E}$. We shortly say that F is exact instead of $(\mathcal{E}, \mathcal{F})$-exact if unambiguous.

We say that F *detects $(\mathcal{E}, \mathcal{F})$-exactness* if $F(S) \in \mathcal{F}$ implies $S \in \mathcal{E}$ for a sequence $S \in \operatorname{Ob}(\mathcal{A}^{\Delta_2})$. We shortly say that F detects exactness instead of $(\mathcal{E}, \mathcal{F})$-exactness if unambiguous.

Definition 1.5.4. Suppose given an additive category $\mathcal{A}$. Let $\mathcal{E}_{\mathcal{A}}^{\text{split}}$ consist of the split short exact sequences in $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{E}_{\mathcal{A}}^{\text{split}})$ is an exact category and we call $\mathcal{E}_{\mathcal{A}}^{\text{split}}$ the *split exact structure on $\mathcal{A}$*.

Remark 1.5.5. Suppose given an additive category $\mathcal{A}$ and an exact category $(\mathcal{B}, \mathcal{F})$. Every additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is $(\mathcal{E}_{\mathcal{A}}^{\text{split}}, \mathcal{F})$-exact.

Definition 1.5.6. Suppose given an abelian category $\mathcal{A}$. Let $\mathcal{E}_{\mathcal{A}}^{\text{all}}$ consist of all short exact sequences in $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{E}_{\mathcal{A}}^{\text{all}})$ is an exact category and we call $\mathcal{E}_{\mathcal{A}}^{\text{all}}$ the *natural exact structure on $\mathcal{A}$*. We equip an abelian category with the natural exact structure unless otherwise stated.

Definition 1.5.7. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$, an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$.

The functor F is called *$\mathcal{E}$-exact* or just exact if it is $(\mathcal{E}, \mathcal{E}_{\mathcal{B}}^{\text{all}})$-exact.

The functor F is called *$\mathcal{E}$-left-exact* or just left-exact if $F(S)$ is left-exact for $S \in \mathcal{E}$.

The functor F is called *$\mathcal{E}$-right-exact* or just right-exact if $F(S)$ is right-exact for $S \in \mathcal{E}$.

The functor F is said to *detect $\mathcal{E}$-exactness* or just exactness if it detects $(\mathcal{E}, \mathcal{E}_{\mathcal{B}}^{\text{all}})$-exactness.

Definition 1.5.8. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$, an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$.

The functor F is called an *$\mathcal{E}$-immersion* if it is full, faithful, $\mathcal{E}$-exact and detects $\mathcal{E}$-exactness. Moreover, we call an $\mathcal{E}$-immersion F *closed* if the essential image $\operatorname{Im}_{\text{ess}}(F)$ of F is closed under extensions.

We shortly say that F is an immersion instead of an $\mathcal{E}$-immersion if unambiguous.

Definition 1.5.9. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$ and an object $P \in \operatorname{Ob} \mathcal{A}$. We say that P is *$\mathcal{E}$-relative projective* if the functor ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}\colon \mathcal{A} \to \text{Mod-}\mathbf{Z}$ is $\mathcal{E}$-exact, cf. definition 1.2.4.(b). We shortly say that P is relative projective instead of $\mathcal{E}$-relative projective if unambiguous.

The dual of an ($\mathcal{E}$-)relative projective object is called an *($\mathcal{E}$-)relative injective* object.

Definition 1.5.10. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$.

(a) We say that $\mathcal{A}$ has *enough $\mathcal{E}$-relative projectives* or just enough projectives if for each object $A \in \operatorname{Ob} \mathcal{A}$, there exists an $\mathcal{E}$-relative projective object $P \in \operatorname{Ob} \mathcal{A}$ and an $\mathcal{E}$-pure epimorphism $P \overset{p}{\twoheadrightarrow} A$ in $\mathcal{A}$.

(b) We say that $\mathcal{A}$ has *enough $\mathcal{E}$-relative injectives* or just enough injectives if for each object $A \in \operatorname{Ob}\mathcal{A}$, there exists an $\mathcal{E}$-relative injective object $M \in \operatorname{Ob}\mathcal{A}$ and an $\mathcal{E}$-pure monomorphism $A \xrightarrow{m} M$ in $\mathcal{A}$.

Lemma 1.5.11. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$.

(a) Suppose given pure epimorphisms $P \xrightarrow{p} A$ and $Q \xrightarrow{q} B$ in $\mathcal{A}$.
Then $\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) : P \oplus Q \to A \oplus B$ is purely epimorphic as well.

(b) Suppose given pure monomorphisms $A \xrightarrow{m} M$ and $B \xrightarrow{n} N$.
Then $\left(\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right) : A \oplus B \to M \oplus N$ is purely monomorphic as well.

Proof. Ad (a). The diagram

$$\begin{tikzcd} P \oplus Q \arrow[r, "\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & A \oplus Q \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \\ P \arrow[r, "p"] & A \end{tikzcd}$$

is a pullback by the kernel-cokernel-criterion lemma 1.1.2.(a). By (E7), the morphism $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is purely epimorphic.

Similarly, the diagram

$$\begin{tikzcd} A \oplus Q \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & A \oplus B \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \\ Q \arrow[r, "q"] & B \end{tikzcd}$$

is a pullback by the kernel-cokernel-criterion lemma 1.1.2.(a). By (E7), the morphism $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ is purely epimorphic.

Therefore the composite $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$ is purely epimorphic.

Ad (b). This is dual to (a). $\square$

Proposition 1.5.12. Suppose given an abelian category $\mathcal{B}$. Suppose given a full additive subcategory $\mathcal{A}$ of $\mathcal{B}$ that is closed under extensions.

Let $\mathcal{E} \subseteq \operatorname{Ob}(\mathcal{A}^{\Delta_2})$ be the set of short exact sequences in $\mathcal{B}$ whose objects lie in $\mathcal{A}$. So elements of $\mathcal{E}$ are short exact sequences $\left(A \xrightarrow{f} B \xrightarrow{g} C\right)$ in $\mathcal{B}$ with $A, B, C \in \operatorname{Ob}\mathcal{A}$.

Then $(\mathcal{A}, \mathcal{E})$ is an exact category.

Proof. Note that if a sequence $\left(A \xrightarrow{f} B \xrightarrow{g} C\right)$ in $\mathcal{A}$ is short exact in $\mathcal{B}$, then it is short exact in $\mathcal{A}$ as well.

Note that $\mathcal{A}$ contains the zero objects of $\mathcal{B}$ since it is a full additive subcategory closed under isomophisms.

We verify the axioms (E1-7).

(E1) Suppose given $S, T \in \mathrm{Ob}(\mathcal{A}^{\Delta_2})$ with $S \cong T$. If S is short exact in $\mathcal{B}$, then T is short exact in $\mathcal{B}$ as well.

(E2) The sequence $\big(A \xrightarrow{1_A} A \longrightarrow 0\big)$ is short exact in $\mathcal{B}$, so $1_A \in \mathcal{E}_{\mathrm{m}}$.

(E3) The sequence $\big(0 \longrightarrow A \xrightarrow{1_A} A\big)$ is short exact in $\mathcal{B}$, so $1_A \in \mathcal{E}_{\mathrm{e}}$.

(E4) Suppose given $f, f' \in \mathcal{E}_{\mathrm{m}}$. Cokernels of ff' lie in $\mathcal{A}$ since $\mathcal{A}$ is closed under extensions, cf. lemma 1.1.1.

(E5) Suppose given $g, g' \in \mathcal{E}_{\mathrm{m}}$. Kernels of gg' lie in $\mathcal{A}$ since $\mathcal{A}$ is closed under extensions, cf. lemma 1.1.1.

(E6) Suppose given a diagram

$$\begin{tikzcd} A \arrow[r, "f", "\bullet" marking] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

in $\mathcal{A}$ with $f \in \mathcal{E}_{\mathrm{m}}$. Choose a pushout

$$\begin{tikzcd} A \arrow[r, "f", "\bullet" marking] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner" very near end] & D \end{tikzcd}$$

in $\mathcal{A}$. Then f' is a monomorphism with cokernel in $\mathcal{A}$ by lemma 1.1.2.(b). Since $\mathcal{A}$ is closed under extensions, we conclude that $f' \in \mathcal{E}_{\mathrm{m}}$.

(E7) Suppose given a diagram

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f", "+" marking] & B \end{tikzcd}$$

in $\mathcal{A}$ with $f \in \mathcal{E}_{\mathrm{e}}$. Choose a pullback

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner" very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f", "+" marking] & B \end{tikzcd}$$

in $\mathcal{A}$. Then f' is an epimorphism with kernel in $\mathcal{A}$ by lemma 1.1.2.(a). Since $\mathcal{A}$ is closed under extensions, we conclude that $f' \in \mathcal{E}_{\mathrm{e}}$.

□

Proposition 1.5.13 (Obscure axiom)**.** Suppose given an exact category $(\mathcal{A}, \mathcal{E})$.

(a) Suppose given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. If f has a cokernel and fg is purely monomorphic, then f is purely monomorphic.

(b) Suppose given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. If g has a kernel and fg is purely epimophic, then g is purely epimorphic.

Proof. See [2, prop. 2.16]. □

1.6 Homologies and exact functors

Suppose given an abelian category $\mathcal{B}$.

Definition 1.6.1. Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$. A commutative diagram

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
 & & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

in $\mathcal{B}$ is called a *homology diagram* of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$ if k is a kernel of f, ℓ is a kernel of g, d is a cokernel of f, c is a cokernel of g and $L \xrightarrow{p} H \xrightarrow{j} D$ is an image of ℓd in $\mathcal{B}$.

Remark 1.6.2. Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$ and homology diagrams

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
 & & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d'"] & & Z \arrow[dr, "c'"] & \\
K' \arrow[ur, "k'"] & & L' \arrow[ur, "\ell"'] \arrow[dr, "p'"'] & & D' & & C' \\
 & & & H' \arrow[ur, "j'"'] & & &
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$. By the universal properties of kernels, cokernels and images, the diagrams are isomorphic. In particular, we obtain $K \cong K'$, $H \cong H'$ and $C \cong C'$ in $\mathcal{B}$.

Also note that there exists a homology diagram of $X \xrightarrow{f} Y \xrightarrow{g} Z$ since $\mathcal{B}$ is an abelian category.

Definition 1.6.3. Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$.

The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called *semi-exact* in $\mathcal{B}$ if there exists a homology diagram

$$\begin{tikzcd}
& X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"] \arrow[dr, "p"] & & D & & C \\
& & & H \arrow[ur, "j"] & & &
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $H \cong 0_{\mathcal{B}}$ in $\mathcal{B}$.

Lemma 1.6.4. Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$.
The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if it is semi-exact with $fg = 0$.

Some use the term *exact in the middle* for what we call exact, cf. convention 33.

Proof. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact. Choose a kernel $K \xrightarrow{k} X$ of f and a cokernel $Z \xrightarrow{c} C$ of g. Choose an image $X \xrightarrow{u} L \xrightarrow{\ell} Y$ of f and an image $Y \xrightarrow{d} D \xrightarrow{v} Z$ of g. The sequence $L \xrightarrow{\ell} Y \xrightarrow{d} D$ is short exact and thus the diagram

$$\begin{tikzcd}
& X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"] \arrow[dr] & & D & & C \\
& & & 0_{\mathcal{B}} \arrow[ur] & & &
\end{tikzcd}$$

is a homology diagram of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$. So $X \xrightarrow{f} Y \xrightarrow{g} Z$ is semi-exact. Moreover, we have $fg = u\ell dv = 0$.

Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is semi-exact with $fg = 0$.

So we may choose a homology diagram

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"'] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr] & & D & & C \\
 & & & 0_{\mathcal{B}} \arrow[ur] & & &
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$. Note that $\ell d = 0$.

Lemma 1.1.1 yields morphisms $X \xrightarrow{u} L$ and $D \xrightarrow{v} Z$ in $\mathcal{B}$ such that the diagram

$$\begin{tikzcd}
 & & L \arrow[rr, "0"] \arrow[dr, "\ell"'] & & D \arrow[ddrr, "v"] & & \\
 & & & Y \arrow[ur, "d"'] \arrow[dr, "g"] & & & \\
X \arrow[uurr, "u"] \arrow[rr, "1"] & & X \arrow[ur, "f"] \arrow[rr, "0"'] & & Z \arrow[rr, "1"] \arrow[dr, "c"'] & & Z \arrow[dl, "c"] \\
 & K \arrow[ul, "k"] \arrow[ur, "k"'] & & & & C &
\end{tikzcd}$$

commutes and such that the sequence $K \xrightarrow{1} K \xrightarrow{u} L \xrightarrow{0} D \xrightarrow{v} Z \xrightarrow{1} Z$ is exact. Thus u is epimorphic and v is monomorphic.

We conclude that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact since $X \xrightarrow{u} L \xrightarrow{\ell} Y$ is an image of f, $Y \xrightarrow{d} D \xrightarrow{v} Z$ is an image of g and $L \xrightarrow{\ell} Y \xrightarrow{d} D$ is short exact. $\square$

Lemma 1.6.5. Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$ such that $fg = 0$ and a homology diagram

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"'] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
 & & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$

The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is short exact if and only if $K \cong 0_{\mathcal{B}}$, $H \cong 0_{\mathcal{B}}$ and $C \cong 0_{\mathcal{B}}$ in $\mathcal{B}$.

Proof. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is short exact. Then $K \cong 0_{\mathcal{B}}$ since f is monomorphic and $C \cong 0_{\mathcal{B}}$ since g is monomorphic. Since $X \xrightarrow{f} Y \xrightarrow{g} Z$ is in particular exact, lemma 1.6.4 yields that $H \cong 0_{\mathcal{B}}$.

Conversely, suppose that $K \cong 0_{\mathcal{B}}$, $H \cong 0_{\mathcal{B}}$ and $C \cong 0_{\mathcal{B}}$.
By loc. cit., the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact. Since $K \cong 0_{\mathcal{B}}$, we may choose the image $X \xrightarrow{1} X \xrightarrow{f} Y$ of f. Since $C \cong 0_{\mathcal{B}}$, we may choose the image $Y \xrightarrow{g} Z \xrightarrow{1} Z$ of g. We conclude that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is indeed short exact. □

Corollary 1.6.6. Suppose given an additive category $\mathcal{A}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$.

Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ such that $fg = 0$ and a homology diagram

$$\begin{tikzcd}
 & F(X) \arrow[rr, "F(f)"] & & F(Y) \arrow[rr, "F(g)"] \arrow[dr, "d"'] & & F(Z) \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
 & & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

of $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$.

The sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is short exact if and only if $K \cong 0_{\mathcal{B}}$, $H \cong 0_{\mathcal{B}}$ and $C \cong 0_{\mathcal{B}}$ in $\mathcal{B}$.

Corollary 1.6.7. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$.

The functor F is $\mathcal{E}$-exact if and only if for all pure short exact sequences $\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \in \mathcal{E}$, there exists a homology diagram

$$\begin{tikzcd}
 & F(X) \arrow[rr, "F(f)"] & & F(Y) \arrow[rr, "F(g)"] \arrow[dr, "d"'] & & F(Z) \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
 & & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

of $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$ such that $K \cong 0_{\mathcal{B}}$, $H \cong 0_{\mathcal{B}}$ and $C \cong 0_{\mathcal{B}}$.

Lemma 1.6.8. Suppose given an abelian category $\mathcal{A}$ and an exact functor $F\colon \mathcal{A} \to \mathcal{B}$.
The functor F preserves homology diagrams. More precisely, we have the following statement.

Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ and a homology diagram

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"'] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
 & & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

Then the diagram

$$\begin{tikzcd}
& F(X) \arrow[rr, "F(f)"] & & F(Y) \arrow[rr, "F(g)"] \arrow[dr, "F(d)"] & & F(Z) \arrow[dr, "F(c)"] & \\
F(K) \arrow[ur, "F(k)"] & & F(L) \arrow[ur, "F(\ell)"'] \arrow[dr, "F(p)"'] & & F(D) & & F(C) \\
& & & F(H) \arrow[ur, "F(j)"'] & & &
\end{tikzcd}$$

is a homology diagram of $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$.

Proof. This follows from the fact that an exact functor between abelian categories preserves kernels, cokernels and images. $\square$

Corollary 1.6.9. Suppose given an abelian category $\mathcal{A}$ and an exact functor $F \colon \mathcal{A} \to \mathcal{B}$.

Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ and a homology diagram

$$\begin{tikzcd}
& X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"'] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
& & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

(a) The sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is semi-exact if and only if $F(H) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$.

(b) Suppose that $fg = 0$ in $\mathcal{A}$. The sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is short exact if and only if $F(K) \cong 0_{\mathcal{B}}$, $F(H) \cong 0_{\mathcal{B}}$ and $F(C) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$.

Definition 1.6.10. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$ and a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

A commutative diagram

$$\begin{tikzcd}
& X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \arrow[rr, "g"] \arrow[dr, "d"'] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] & & D \arrow[ur, "m"'] & & C
\end{tikzcd}$$

in $\mathcal{A}$ is called an $\mathcal{E}$-*purity diagram* of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ if

$$\big(K \overset{k}{\rightarrowtail} X \overset{e}{\twoheadrightarrow} L\big), \big(L \overset{\ell}{\rightarrowtail} Y \overset{d}{\twoheadrightarrow} D\big), \big(D \overset{m}{\rightarrowtail} Z \overset{c}{\twoheadrightarrow} C\big) \in \mathcal{E}.$$

The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called $\mathcal{E}$-*pure exact* if there exists an $\mathcal{E}$-purity diagram of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

We say *pure exact* instead of $\mathcal{E}$-pure exact and *purity diagram* instead of $\mathcal{E}$-purity diagram if unambiguous.

Lemma 1.6.11. Suppose given an exact category $(\mathcal{A}, \mathcal{E})$ and an $\mathcal{E}$-exact functor $F\colon \mathcal{A} \to \mathcal{B}$. Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

Suppose given an $\mathcal{E}$-purity diagram

$$\begin{tikzcd}
 & X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \arrow[rr, "g"] \arrow[dr, two heads, "d"'] & & Z \arrow[dr, two heads, "c"] & \\
K \arrow[ur, tail, "k"] & & L \arrow[ur, tail, "\ell"'] & & D \arrow[ur, tail, "m"'] & & C
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

Then the diagram

$$\begin{tikzcd}
 & F(X) \arrow[rr, "F(f)"] & & F(Y) \arrow[rr, "F(g)"] \arrow[dr, "F(d)"'] & & F(Z) \arrow[dr, "F(c)"] & \\
F(K) \arrow[ur, "F(k)"] & & F(L) \arrow[ur, "F(\ell)"] \arrow[dr, "0"'] & & F(D) & & F(C) \\
 & & & 0_{\mathcal{B}} \arrow[ur, "0"'] & & &
\end{tikzcd}$$

is a homology diagram of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$.

In particular, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $\mathcal{E}$-pure exact, then $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact in $\mathcal{B}$.

Proof. The morphism $F(k)$ is a kernel of $F(e)$ since F is $\mathcal{E}$-exact. Note that $F(e)F(\ell) = F(f)$ and that $F(\ell)$ is monomorphic since F is $\mathcal{E}$-exact. Thus $F(k)$ is also a kernel of $F(f)$. Similarly, $F(\ell)$ is a kernel of g. Dually, $F(c)$ is a cokernel of $F(g)$ and $F(d)$ is a cokernel of $F(f)$.

Moreover, we have $F(\ell)F(d) = F(\ell d) = F(0) = 0$. $\square$

Chapter 2

Sheaves

2.1 Sieves

Suppose given a category $\mathscr{S}$.

Example

For the sake of illustration, we want to show how sheaves on topological spaces fit into the picture of the general theory of sheaves on sites. Throughout this chapter, we translate the general terms into the language of topological spaces in centered paragraphs like this. None of this will be relevant to the general theory. The arguments in this illustration are often only sketched.

Suppose given a topological space X in the centered paragraphs throughout this chapter. We consider the poset $\mathscr{S}_X$ of open subsets of X as a category.

Suppose given open subsets $V \subseteq U \subseteq X$. We write $\mathrm{i}_V^U \colon V \to U$ for the inclusion morphism. Every morphism in $\mathscr{S}_X$ is of this form.

Definition 2.1.1. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$. A subset $S \subseteq \operatorname{Mor} \mathscr{S}$ is called a *sieve on A* if the following two conditions hold.

(S1) We have $\operatorname{cod}(f) = A$ for $f \in S$, cf. convention 3.

(S2) Suppose given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$. If $f \in S$, then $gf \in S$.

The set of sieves on A is denoted by $\operatorname{Sieves}(A)$.

Example (cont.)

Suppose given an open subset $U \subseteq X$. For an open subset $V \subseteq U$ and a sieve S on U, we identify the inclusion i_V^U with V itself, i.e. we often write $V \in S$ instead of $\mathrm{i}_V^U \in S$.

So a sieve on U is a set S of open subsets of U, such that given $V \in S$ and an open subset $W \subseteq V$, we have $W \in S$.

Informally speaking, sieves are closed under taking open subsets.

Lemma/Definition 2.1.2. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$.
We call $\max(A) := \{f \in \operatorname{Mor} \mathscr{S} \colon \operatorname{cod}(f) = A\}$ the *maximal sieve on* A.

The subset $\max(A) \subseteq \operatorname{Mor} \mathscr{S}$ is in fact a sieve on A.

Proof. By definition, we have $\operatorname{cod}(f) = A$ for $f \in \max(A)$. So (S1) holds. Suppose given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$ with $f \in \max(A)$. We have $\operatorname{cod}(gf) = A$ and therefore $gf \in \max(A)$. So (S2) holds. $\square$

Lemma/Definition 2.1.3. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$ and a sieve S on A.
For $B \xrightarrow{f} A$ in $\mathscr{S}$, we set

$$f^*(S) := \{g \in \operatorname{Mor} \mathscr{S} \colon \operatorname{cod}(g) = B \text{ and } gf \in S\}.$$

The subset $f^*(S) \subseteq \operatorname{Mor} \mathscr{S}$ is a sieve on B, called the *inverse image of* S *along* f.

Proof. By definition, we have $\operatorname{cod}(g) = B$ for $g \in f^*(S)$. So (S1) holds.

Suppose given $D \xrightarrow{h} C \xrightarrow{g} B$ in $\mathscr{S}$ with $g \in f^*(S)$. We have $\operatorname{cod}(hg) = B$. Since S is a sieve on A and since $gf \in S$, we conclude that $(hg)f = h(gf) \in S$ and therefore $hg \in f^*(S)$. So (S2) holds. $\square$

Example (cont.)

Suppose given an open subset $U \subseteq X$ and a sieve S on U.

For an open subset $V \subseteq U$, we have $(\mathrm{i}_V^U)^*(S) = \{W \subseteq V \colon W \in S\}$.

Remark 2.1.4. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$ and sieves S, S' on A with $S \subseteq S'$. Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$. Then $f^*(S) \subseteq f^*(S')$.

Proof. For $\big(C \xrightarrow{g} B\big) \in f^*(S)$, we have $gf \in S \subseteq S'$, so $g \in f^*(S')$. $\square$

Remark 2.1.5. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$ and a sieve S on A.
For $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$, we have $(gf)^*(S) = g^*(f^*(S))$.

Proof. For $h \in (gf)^*(S)$, we have $hgf \in S$, so $hg \in f^*(S)$ and, consequently, $h \in g^*(f^*(S))$.

For $h \in g^*(f^*(S))$, we have $hg \in f^*(S)$, so $hgf \in S$ and, consequently, $h \in (gf)^*(S)$. $\square$

Lemma/Definition 2.1.6. Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$. Let

$$\mathrm{S}_f := \big\{g \in \operatorname{Mor} \mathscr{S} \colon \text{there exists } h \in \operatorname{Mor} \mathscr{S} \text{ such that } g = hf\big\}.$$

We call S_f the *sieve generated by* f. The set S_f is in fact a sieve on A.

Proof. By definition, (S1) holds. For $D \xrightarrow{k} C \xrightarrow{h} B$ in $\mathscr{S}$, we have $k(hf) = (kh)f$. So (S2) holds as well. $\square$

Lemma/Definition 2.1.7. Suppose given an object $A \in \operatorname{Ob}\mathscr{S}$ and a sieve S on A.
Let

$$\mathrm{F}_S(B) := S \cap \operatorname{Hom}_{\mathscr{S}}(B, A)$$

and let

$$\mathrm{F}_S(f^{\mathrm{op}})\colon\ \mathrm{F}_S(B) \to \mathrm{F}_S(C)\colon h \mapsto fh$$

for $C \xrightarrow{f} B$ in $\mathscr{S}$.

This defines a subfunctor $\mathrm{F}_S\colon \mathscr{S}^{\mathrm{op}} \to \mathtt{Set}$ of the Yoneda functor $\mathrm{y}_{\mathscr{S},A}$, cf. definitions 1.1.10 and 1.2.1.

Proof. Suppose given $C \xrightarrow{f} B$ in $\mathscr{S}$ and $h \in \mathrm{F}_S(B)$. Then $fh \in S$ since S is a sieve. Therefore $\mathrm{F}_S(f^{\mathrm{op}})$ is well-defined.

So F_S is a subfunctor of $\mathrm{y}_{\mathscr{S},A}$ by definition. $\square$

Remark 2.1.8. Suppose given an object $A \in \operatorname{Ob}\mathscr{S}$. Then $\mathrm{F}_{\max(A)} = \mathrm{y}_{\mathscr{S},A}$.

Lemma/Definition 2.1.9. Suppose given an object $A \in \operatorname{Ob}\mathscr{S}$ and sieves S, S' on A with $S \subseteq S'$.

Let $(\iota_S^{S'})_B\colon\ \mathrm{F}_S(B) \to \mathrm{F}_{S'}(B)\colon h \mapsto h$ for $B \in \operatorname{Ob}\mathscr{S}$.

This yields a transformation $\iota_S^{S'} := \big((\iota_S^{S'})_B\big)_{B \in \operatorname{Ob}\mathscr{S}}\colon\ \mathrm{F}_S \Rightarrow \mathrm{F}_{S'}$.

Moreover, let $\iota_S := \iota_S^{\max(A)}\colon\ \mathrm{F}_S \Rightarrow \mathrm{y}_{\mathscr{S},A}$.

Proof. For $C \xrightarrow{f} B$ in $\mathscr{S}$ and $h \in \mathrm{F}_S(B)$, we have

$$(h)\ \mathrm{F}_S(f^{\mathrm{op}})\,(\iota_S^{S'})_C = (fh)\,(\iota_S^{S'})_C = fh = f \cdot (h)\,(\iota_S^{S'})_B = (h)\,(\iota_S^{S'})_B\ \mathrm{F}_{S'}(f^{\mathrm{op}}).$$ $\square$

Remark 2.1.10. Suppose given $A \in \operatorname{Ob}\mathscr{S}$ and sieves S, S', S'' on A with $S \subseteq S' \subseteq S''$. Then $\iota_S^{S'}\iota_{S'}^{S''} = \iota_S^{S''}$.

2.2 Sites and sheaves

Suppose given a ring R, cf. convention 36.

From after definition 2.2.1 on, we will suppose given a site $\mathscr{S} = (\mathscr{S}, J)$.

2.2.1 Definitions

Definition 2.2.1. Suppose given a category $\mathscr{S}$. A tuple $J = (J_A)_{A \in \operatorname{Ob}\mathscr{S}}$ with $J_A \subseteq \operatorname{Sieves}(A)$ for $A \in \operatorname{Ob}\mathscr{S}$ is called a *Grothendieck topology on* $\mathscr{S}$ if the following three conditions hold.

(G1) We have $\max(A) \in J_A$ for $A \in \operatorname{Ob}\mathscr{S}$.

(G2) Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$. For $S \in J_A$, we have $f^*(S) \in J_B$.

(G3) Suppose given $A \in \operatorname{Ob}\mathscr{S}$ and $S \in J_A$. Suppose given a sieve T on A such that $f^*(T) \in J_B$ for $\big(B \xrightarrow{f} A\big) \in S$. Then $T \in J_A$.

The set J_A is called the *set of coverings of* A for $A \in \operatorname{Ob}\mathscr{S}$. We say that S *covers* A (or that S is a covering of A) if $S \in J_A$ for $A \in \operatorname{Ob}\mathscr{S}$.

A tuple $(\mathscr{S}, J)$ consisting of a category $\mathscr{S}$ and a Grothendieck topology J on $\mathscr{S}$ is called a *site*. We often abbreviate $\mathscr{S} = (\mathscr{S}, J)$ if unambiguous.

For a way to obtain Grothendieck topologies in practice, see section 2.2.4.

Example (cont.)

For an open subset $U \subseteq X$, let the set of coverings J_U of U consist of sieves S on U with $\bigcup S = U$.

We want to show that $(\mathscr{S}_X, J)$ is a Grothendieck topology.

(G1) For an open subset $U \subseteq X$, we have $\max(U) = \{V \subseteq U : V \text{ open}\}$. So $\bigcup \max(U) = U$ and therefore $\max(U) \in J_U$.

(G2) Suppose given open subsets $V \subseteq U \subseteq X$ and a covering S of U.
We have $(\mathrm{i}_V^U)^*(S) = \{W \subseteq V : W \in S\} = \{W' \cap V : W' \in S\}$.
So $\bigcup (\mathrm{i}_V^U)^*(S) = \bigcup \{W' \cap V : W' \in S\} = \big(\bigcup \{W' : W' \in S\}\big) \cap V = U \cap V = V$ and therefore $(\mathrm{i}_V^U)^*(S) \in J_V$.

(G3) Suppose given an open subset $U \subseteq X$ and a covering S of U. Suppose given a sieve T on U such that $(i_V^U)^*(T)$ covers V for each $V \in S$. We have to show that T covers U.
For each $V \in S$, we have $V = \bigcup (\mathrm{i}_V^U)^*(T) = \bigcup \{W \cap V : W \in T\} \subseteq \bigcup \{W : W \in T\} = \bigcup T$.
Thus $U = \bigcup S = \bigcup \{V \subseteq U : V \in S\} \subseteq \bigcup T \subseteq U$.
We conclude that $\bigcup T = U$ and thus $T \in J_U$.

Suppose given a site $\mathscr{S} = (\mathscr{S}, J)$ for the remainder of this section 2.2.

Definition 2.2.2.

(a) A functor $P \colon \mathscr{S}^{\mathrm{op}} \to \mathtt{Set}$ is called a *presheaf on* $\mathscr{S}$.
The functor category $\hat{\mathscr{S}} := \mathtt{Set}^{(\mathscr{S}^{\mathrm{op}})}$ is called the *category of presheaves on* $\mathscr{S}$.

(b) A functor $P \colon \mathscr{S}^{\mathrm{op}} \to \text{Mod-}R$ is called an *R-presheaf on* $\mathscr{S}$.
The functor category ${}_R\hat{\mathscr{S}} := (\text{Mod-}R)^{(\mathscr{S}^{\mathrm{op}})}$ is called the *category of R-presheaves on* $\mathscr{S}$.
Cf. definition 1.1.11.

Definition 2.2.3. Suppose given an object $A \in \mathrm{Ob}\,\mathscr{S}$ and a covering $S \in J_A$.

(a) Suppose given a presheaf P on $\mathscr{S}$. A transformation $\xi\colon \mathrm{F}_S \Rightarrow P$ is called a *matching family for P on S in $\mathscr{S}$.*

So $\xi_B\, P(f^{\mathrm{op}}) = \mathrm{F}_S(f^{\mathrm{op}})\,\xi_C$ for $C \xrightarrow{f} B$ in $\mathscr{S}$.

$$\begin{tikzcd} \mathrm{F}_S(B) \arrow[r, "\xi_B"] \arrow[d, "\mathrm{F}_S(f^{\mathrm{op}})"'] & P(B) \arrow[d, "P(f^{\mathrm{op}})"] \\ \mathrm{F}_S(C) \arrow[r, "\xi_C"] & P(C) \end{tikzcd}$$

(b) Recall that ${}_R\Upsilon\colon$ Mod-$R \to$ `Set` denotes the forgetful functor, cf. convention 37. Suppose given an R-presheaf P on $\mathscr{S}$. A matching family for ${}_R\Upsilon \circ P$ on S in $\mathscr{S}$ is called a *matching family for P on S in $\mathscr{S}$.*

Definition 2.2.4. Suppose given an object $A \in \mathrm{Ob}\,\mathscr{S}$ and a covering $S \in J_A$. Suppose given a presheaf P on $\mathscr{S}$. Suppose given a matching family $\xi\colon \mathrm{F}_S \Rightarrow P$ for P on S. A transformation $\zeta\colon \mathrm{y}_{\mathscr{S},A} \Rightarrow P$ with $\iota_S\,\zeta = \xi$ is called an *amalgamation of* ξ.

$$\begin{tikzcd} \mathrm{F}_S \arrow[r, "\xi"] \arrow[d, "\iota_S"'] & P \\ \mathrm{y}_{\mathscr{S},A} \arrow[ur, "\zeta"'] & \end{tikzcd}$$

So $\zeta_B\, P(f^{\mathrm{op}}) = \mathrm{y}_{\mathscr{S},A}(f^{\mathrm{op}})\,\zeta_C$ for $C \xrightarrow{f} B$ in $\mathscr{S}$.

$$\begin{tikzcd} \mathrm{y}_{\mathscr{S},A}(B) \arrow[r, "\zeta_B"] \arrow[d, "\mathrm{y}_{\mathscr{S},A}(f^{\mathrm{op}})"'] & P(B) \arrow[d, "P(f^{\mathrm{op}})"] \\ \mathrm{y}_{\mathscr{S},A}(C) \arrow[r, "\zeta_C"] & P(C) \end{tikzcd}$$

Cf. also remark 2.2.6.

Remark 2.2.5. Suppose given an object $A \in \mathrm{Ob}\,\mathscr{S}$ and a covering $S \in J_A$. Suppose given an R-presheaf P on $\mathscr{S}$.

Note that a matching family ξ for P on S in $\mathscr{S}$ is a matching family for ${}_R\Upsilon \circ P$ on S in $\mathscr{S}$ by definition, so we can speak of an amalgamation of ξ, being a transformation $\zeta\colon \mathrm{y}_{\mathscr{S},A} \Rightarrow {}_R\Upsilon \circ P$ such that $\iota_S\,\zeta = \xi$.

Remark 2.2.6. Suppose given an object $A \in \mathrm{Ob}\,\mathscr{S}$ and a covering $S \in J_A$. Suppose given a presheaf P on $\mathscr{S}$. Suppose given a matching family ξ for P on S.

By the Yoneda lemma 1.2.2, we have the bijection $\gamma_{P,A}\colon \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{y}_{\mathscr{S},A}, P) \to P(A)\colon \zeta \mapsto (1_A)\zeta_A$
.
Under this bijection, the amalgamations of ξ correspond to the elements $x \in P(A)$ with $(x)\,P(f^{\mathrm{op}}) = (f)\xi_B$ for $\big(B \xrightarrow{f} A\big) \in S$.

We will use this correspondence from now on usually without comment.
So we will identify an amalgamation ζ of ξ with $(1_A)\zeta_A \in P(A)$.

Proof. Suppose given $\zeta \in \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{y}_{\mathscr{S},A}, P)$.

For $\big(B \xrightarrow{f} A\big) \in S$, we have

$$(\gamma_{P,A}(\zeta))\,P(f^{\mathrm{op}}) = ((1_A)\zeta_A)\,P(f^{\mathrm{op}}) = (1_A)(\zeta_A\,P(f^{\mathrm{op}})) = (1_A)(\mathrm{y}_{\mathscr{S},A}(f^{\mathrm{op}})\,\zeta_B) = (f)\zeta_B\,.$$

So if $\iota_S\,\zeta = \xi$, then $(\gamma_{P,A}(\zeta))\,P(f^{\mathrm{op}}) = (f)\zeta_B = (f)(\iota_S\,\zeta)_B = (f)\xi_B$ for $\big(B \xrightarrow{f} A\big) \in S$.

Conversely, if $(\gamma_{P,A}(\zeta))\,P(f^{\mathrm{op}}) = (f)\xi_B$ for $\big(B \xrightarrow{f} A\big) \in S$, then

$$(f)(\iota_S\,\zeta)_B = (f)\zeta_B = (\gamma_{P,A}(\zeta))\,P(f^{\mathrm{op}}) = (f)\xi_B$$

for $\big(B \xrightarrow{f} A\big) \in S$ and, consequently, $\iota_S\,\zeta = \xi$.

□

Definition 2.2.7.

(a) A presheaf P on $\mathscr{S}$ is called *separated* if for each matching family ξ for P on each covering of each object in $\mathscr{S}$, there exists at most one amalgamation of ξ.

(b) An R-presheaf P on $\mathscr{S}$ is called *separated* if for each matching family ξ for P on each covering of each object in $\mathscr{S}$, there exists at most one amalgamation of ξ.

Definition 2.2.8.

(a) A presheaf P on $\mathscr{S}$ is called a *sheaf* on $\mathscr{S}$ if for each matching family ξ for P on each covering of each object in $\mathscr{S}$, there exists a unique amalgamation of ξ.

The full subcategory $\tilde{\mathscr{S}}$ of $\hat{\mathscr{S}}$ whose objects are the sheaves on $\mathscr{S}$ is called the *category of sheaves on $\mathscr{S}$*.

Let $\mathrm{E}_{\mathscr{S}}\colon \tilde{\mathscr{S}} \to \hat{\mathscr{S}}$ denote the inclusion functor. We abbreviate $\mathrm{E} = \mathrm{E}_{\mathscr{S}}$ if unambiguous.

(b) An R-presheaf P on $\mathscr{S}$ is called an *R-sheaf* on $\mathscr{S}$ if for each matching family ξ for P on each covering of each object in $\mathscr{S}$, there exists a unique amalgamation of ξ.

The full subcategory ${}_R\tilde{\mathscr{S}}$ of ${}_R\hat{\mathscr{S}}$ whose objects are the R-sheaves on $\mathscr{S}$ is called the *category of R-sheaves on $\mathscr{S}$*.

Let ${}_R\mathrm{E}_{\mathscr{S}}\colon {}_R\tilde{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ denote the inclusion functor. We abbreviate $\mathrm{E} = {}_R\mathrm{E}_{\mathscr{S}}$ if unambiguous.

Definition 2.2.9. Recall that ${}_R\Upsilon\colon \text{Mod-}R \to \texttt{Set}$ denotes the forgetful functor, cf. convention 37.

Let ${}_R\Upsilon_{\mathscr{S}} = {}_R\Upsilon^{(\mathscr{S}^{\mathrm{op}})}\colon {}_R\hat{\mathscr{S}} \to \hat{\mathscr{S}}$, cf. definition 1.1.12.

So we have ${}_R\Upsilon_{\mathscr{S}}\big(P \xrightarrow{\alpha} Q\big) = \big({}_R\Upsilon \circ P \xrightarrow{{}_R\Upsilon\star\alpha} {}_R\Upsilon \circ Q\big)$ for $P \xrightarrow{\alpha} Q$ in ${}_R\hat{\mathscr{S}}$.

Remark 2.2.10. Suppose given an R-presheaf P on $\mathscr{S}$.

The R-presheaf P is separated if and only if the presheaf ${}_R\Upsilon_{\mathscr{S}}(P) = {}_R\Upsilon \circ P$ is separated.

The R-presheaf P is an R-sheaf if and only if the presheaf ${}_R\Upsilon_{\mathscr{S}}(P) = {}_R\Upsilon \circ P$ is a sheaf.

Example (cont.)

We illustrate the previous definitions with the example of the presheaf of continuous real-valued functions on X.

For an open subset $U \subseteq X$, let $\mathcal{O}(U) = \{\varphi\colon U \to \mathbf{R}\colon \varphi \text{ is continuous}\}$.

For open subsets $V \subseteq U \subseteq X$, let $\mathcal{O}\big((\mathrm{i}_V^U)^{\mathrm{op}}\big)\colon \mathcal{O}(U) \to \mathcal{O}(V)\colon \varphi \mapsto \varphi|_V$.

This yields a presheaf $\mathcal{O} \in \mathrm{Ob}(\hat{\mathscr{S}}_X)$. Our aim is to prove that $\mathcal{O}$ is a sheaf.

Suppose given a covering S on an open subset $U \subseteq X$.
Suppose given a matching family $\xi\colon \mathrm{F}_S \Rightarrow \mathcal{O}$.

The matching family ξ consists of maps $\xi_V\colon S \cap \mathrm{Hom}_{\mathscr{S}_X}(V,U) \to \mathcal{O}(V)$ for each open subset $V \subseteq X$. Note that $S \cap \mathrm{Hom}_{\mathscr{S}_X}(V,U) = \emptyset$ if V is not an open subset of U contained on S, and that $S \cap \mathrm{Hom}_{\mathscr{S}_X}(V,U) = \{\mathrm{i}_V^U\}$ if V is an open subset of U contained in S. So for $V \subseteq U$ with $V \in S$, the image of ξ_V contains precisely one element $\psi_V \in \mathcal{O}(V)$. The tuple $(\psi_V)_{V \subseteq U, V \in S}$ carries the information of ξ and the naturality of ξ translates into $(\psi_V)|_W = \psi_W$ for $W \subseteq V$, where $W, V \in S$.

An amalgamation of ξ is a function $\varphi \in \mathcal{O}(U)$ with $\varphi|_V = \psi_V$ for $V \in S$, cf. remark 2.2.6.

Suppose given amalgamations φ and φ' of ξ. We want to show that $\varphi = \varphi'$.

Suppose given $u \in U$. Since $\bigcup S = U$, we may choose $V \in S$ such that $u \in V$.

We have

$$(u)\,\varphi = (u)\,\varphi|_V = (u)\,\psi_V = (u)\,\varphi'|_V = (u)\,\varphi'.$$

So $\varphi = \varphi'$. Thus the presheaf $\mathcal{O}$ is separated.

We want to construct an amalgamation $\varphi\colon U \to \mathbf{R}$ of ξ. For $u \in U$, we may choose $V \in S$ such that $u \in V$ since $\bigcup S = U$. Let $(u)\,\varphi := (u)\,\psi_V$. This definition is independent of the choice of V since in case $u \in V \cap V'$ for $V, V' \in S$, we have

$$(u)\,\psi_V = (u)\,\psi_V|_{V\cap V'} = (u)\,\psi_{V\cap V'} = (u)\,\psi_{V'}|_{V\cap V'} = (u)\,\psi_{V'}\,.$$

So $\varphi|_V = \psi_V$ for $V \in S$.

It remains to show that φ is continuous. Suppose given an open subset $I \subseteq \mathbf{R}$.

For $V \in S$, we have $V \cap \varphi^{-1}(I) = \psi_V^{-1}(I)$, which is an open subset of V since ψ_V is continuous.

We obtain

$$\begin{aligned}\varphi^{-1}(I) &= \varphi^{-1}(I) \cap U = \varphi^{-1}(I) \cap \bigcup\{V \subseteq U\colon V \in S\} = \bigcup\{V \cap \varphi^{-1}(I)\colon V \in S\} \\ &= \bigcup\{\psi_V^{-1}(I)\colon V \in S\}.\end{aligned}$$

So $\varphi^{-1}(I)$ is a union of open subsets of U and therefore $\varphi^{-1}(I)$ is open as well.

We conclude that ξ has a unique amalgamation. Thus $\mathcal{O}$ is a sheaf.

2.2.2 Elementary facts on the category of sheaves

Lemma 2.2.11.

(a) The full subcategory of sheaves $\tilde{\mathscr{S}} \subseteq \hat{\mathscr{S}}$ is closed under isomorphisms: Suppose given an isomorphism $P \xrightarrow{\alpha} P'$ in $\hat{\mathscr{S}}$. If $P \in \operatorname{Ob} \tilde{\mathscr{S}}$, then $P' \in \operatorname{Ob} \tilde{\mathscr{S}}$.

(b) The full subcategory of R-sheaves ${}_R\tilde{\mathscr{S}} \subseteq {}_R\hat{\mathscr{S}}$ is closed under isomorphisms: Suppose given an isomorphism $P \xrightarrow{\alpha} P'$ in ${}_R\hat{\mathscr{S}}$. If $P \in \operatorname{Ob}({}_R\tilde{\mathscr{S}})$, then $P' \in \operatorname{Ob}({}_R\tilde{\mathscr{S}})$.

Proof. Ad (a). Suppose given $A \in \operatorname{Ob} \mathscr{S}$ and $S \in J_A$. Suppose given a matching family $\xi' \colon \mathrm{F}_S \Rightarrow P'$. We want to show that there exists a unique amalgamation of ξ'.

Let $\xi := \xi' \cdot \alpha^{-1} \colon \mathrm{F}_S \Rightarrow P$. So ξ is a matching family for P.

Let $A_\xi \subseteq \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{y}_{\mathscr{S},A}, P)$ denote the set of amalgamations of ξ. Let $A_{\xi'} \subseteq \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{y}_{\mathscr{S},A}, P')$ denote the set of amalgamations of ξ'.

We *claim* that the map

$$I \colon A_\xi \to A_{\xi'} \colon \zeta \mapsto \zeta\alpha$$

is a bijection with inverse

$$J \colon A_{\xi'} \to A_\xi \colon \zeta' \mapsto \zeta'\alpha^{-1}.$$

The map I is well-defined since for $\zeta \in A_\xi$, we have $\iota_S\, \zeta\alpha = \xi\alpha = \xi'$.

The map J is well-defined since for $\zeta' \in A_{\xi'}$, we have $\iota_S\, \zeta'\alpha^{-1} = \xi'\alpha^{-1} = \xi$.

The maps I and J are mutually inverse by construction. This proves the *claim*.

If P is a sheaf, then the set A_ξ contains precisely one element. So the same is true for $A_{\xi'}$, thus P' is a sheaf.

Ad (b). By remark 2.2.10, P' is an R-sheaf if and only if ${}_R\Upsilon_{\mathscr{S}}(P')$ is a sheaf. The result now follows from (a) since ${}_R\Upsilon_{\mathscr{S}}$ is a functor and therefore preserves isomorphisms. □

Remark 2.2.12. Recall that $0_{\text{Mod-}R}$ denotes a zero object in Mod-R and that $0_{{}_R\hat{\mathscr{S}}}$ denotes a zero object in ${}_R\hat{\mathscr{S}}$ with $0_{{}_R\hat{\mathscr{S}}}(A) = 0_{\text{Mod-}R}$ for $A \in \operatorname{Ob} \mathscr{S}$, cf. convention 15.

The R-presheaf $0_{{}_R\hat{\mathscr{S}}}$ is an R-sheaf.

Proof. The presheaf $0_{{}_R\hat{\mathscr{S}}}$ is separated since the set $0_{{}_R\hat{\mathscr{S}}}(A) = 0_{\text{Mod-}R}$ contains precisely one element for $A \in \operatorname{Ob} \mathscr{S}$.

Suppose given $A \in \operatorname{Ob} \mathscr{S}$ and $S \in J_A$. Suppose given a matching family $\xi\colon \mathrm{F}_S \Rightarrow {}_R\Upsilon \circ 0_{{}_R\hat{\mathscr{S}}}$.

Note that ξ_B has target $0_{{}_R\hat{\mathscr{S}}}(B) = 0_{\text{Mod-}R}$ for $B \in \operatorname{Ob} \mathscr{S}$.

The zero element $0 \in 0_{\text{Mod-}R} = 0_{{}_R\hat{\mathscr{S}}}(A)$ is an amalgamation of ξ since we have

$(0)\, 0_{{}_R\hat{\mathscr{S}}}(f^{\mathrm{op}}) = 0 = (f)\xi_B$ for $\big(B \xrightarrow{f} A\big) \in S$.

We conclude that $0_{{}_R\hat{\mathscr{S}}}$ is an R-sheaf. □

Proposition 2.2.13. The category of R-sheaves ${}_R\tilde{\mathscr{S}}$ is a full additive subcategory of ${}_R\hat{\mathscr{S}}$, which is closed under isomorphisms, cf. convention 17.

Proof. By lemma 2.2.11.(b), the category of R-sheaves ${}_R\tilde{\mathscr{S}}$ is a full subcategory of ${}_R\hat{\mathscr{S}}$ which is closed under isomorphisms. By remark 2.2.12, ${}_R\tilde{\mathscr{S}}$ contains a zero object. It remains to show that a direct sum of two objects in ${}_R\tilde{\mathscr{S}}$ lies in ${}_R\tilde{\mathscr{S}}$.

Suppose given R-sheaves $P, Q \in \operatorname{Ob}({}_R\tilde{\mathscr{S}})$.
A direct sum of P and Q in ${}_R\hat{\mathscr{S}}$ is $P \oplus Q\colon \mathscr{S}^{\mathrm{op}} \to \text{Mod-}R$ with

$$(P \oplus Q)\big(A \xrightarrow{f^{\mathrm{op}}} B\big) = \left(P(A) \oplus Q(A) \xrightarrow{\left(\begin{smallmatrix} P(f^{\mathrm{op}}) & 0 \\ 0 & Q(f^{\mathrm{op}}) \end{smallmatrix}\right)} P(B) \oplus Q(B) \right)$$

for $B \xrightarrow{f} A$ in $\mathscr{S}$. Cf. [14, rem. 5.(f)]. We want to show that $P \oplus Q$ is an R-sheaf.

Suppose given $A \in \operatorname{Ob} \mathscr{S}$ and $S \in J_A$. Suppose given a matching family $\xi\colon \mathrm{F}_S \Rightarrow {}_R\Upsilon \circ (P \oplus Q)$.

Write $\left(\begin{smallmatrix} \xi_B^P & \xi_B^Q \end{smallmatrix}\right) := \xi_B\colon \mathrm{F}_S(B) \to P(B) \oplus Q(B)$ for $B \in \operatorname{Ob} \mathscr{S}$.

We have

$$\begin{aligned} \left(\begin{smallmatrix} \mathrm{F}_S(g^{\mathrm{op}})\xi_C^P & \mathrm{F}_S(g^{\mathrm{op}})\xi_C^Q \end{smallmatrix}\right) &= \mathrm{F}_S(g^{\mathrm{op}}) \cdot \left(\begin{smallmatrix} \xi_C^P & \xi_C^Q \end{smallmatrix}\right) = \mathrm{F}_S(g^{\mathrm{op}}) \cdot \xi_C = \xi_B \cdot (P \oplus Q)(g^{\mathrm{op}}) \\ &= \left(\begin{smallmatrix} \xi_B^P & \xi_B^Q \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} P(g^{\mathrm{op}}) & 0 \\ 0 & Q(g^{\mathrm{op}}) \end{smallmatrix}\right) = \left(\begin{smallmatrix} \xi_B^P\, P(g^{\mathrm{op}}) & \xi_B^Q\, Q(g^{\mathrm{op}}) \end{smallmatrix}\right) \end{aligned}$$

for $C \xrightarrow{g} B$ in $\mathscr{S}$ since ξ is natural.

$$\begin{tikzcd} \mathrm{F}_S(B) \arrow[r, "\left(\begin{smallmatrix} \xi_B^P & \xi_B^Q \end{smallmatrix}\right)"] \arrow[d, "\mathrm{F}_S(g^{\mathrm{op}})"'] & P(B) \oplus Q(B) \arrow[d, "\left(\begin{smallmatrix} P(g^{\mathrm{op}}) & 0 \\ 0 & Q(g^{\mathrm{op}}) \end{smallmatrix}\right)"] \\ \mathrm{F}_S(C) \arrow[r, "\left(\begin{smallmatrix} \xi_C^P & \xi_C^Q \end{smallmatrix}\right)"] & P(C) \oplus Q(C) \end{tikzcd}$$

The equations for each component show that $\xi^P := (\xi_B^P)_{B \in \operatorname{Ob} \mathscr{S}}$ and $\xi^Q := (\xi_B^Q)_{B \in \operatorname{Ob} \mathscr{S}}$ are natural as well. So ξ^P is a matching family for P and ξ^Q is a matching family for Q. Suppose given $(p, q) \in P(A) \oplus Q(A) = (P \oplus Q)(A)$.

For $\big(B \xrightarrow{f} A\big) \in S$, we have

$$(p, q)\,(P \oplus Q)(f^{\mathrm{op}}) = (p, q) \left(\begin{smallmatrix} P(f^{\mathrm{op}}) & 0 \\ 0 & Q(f^{\mathrm{op}}) \end{smallmatrix}\right) = \big((p)\,P(f^{\mathrm{op}}),\, (q)\,Q(f^{\mathrm{op}})\big)$$

and

$$(f)\xi_B = (f)\,(\,\xi_B^P\ \xi_B^Q\,) = \big((f)\xi_B^P\,,\,(f)\xi_B^Q\big).$$

So (p, q) is an amalgamation of ξ if and only if p is an amalgamation of ξ^P and q is an amalgamation of ξ^Q. We conclude that $P \oplus Q$ is an R-sheaf. $\square$

2.2.3 Elementary facts on coverings

Lemma 2.2.14. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$ and coverings S, S' of A, i.e. $S, S' \in J_A$. Then $S \cap S'$ is a covering of A, i.e. $S \cap S' \in J_A$.

Proof. Suppose given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$ with $f \in S \cap S'$. Then $gf \in S$ since S is a sieve and $gf \in S'$ since S' is a sieve. So $gf \in S \cap S'$. We conclude that $S \cap S'$ is a sieve, cf. definition 2.1.1.

Suppose given $\big(B \xrightarrow{f} A\big) \in S$.

We *claim* that $f^*(S \cap S') = f^*(S')$. We have $f^*(S \cap S') \subseteq f^*(S')$, cf. remark 2.1.4.

For $\big(C \xrightarrow{g} B\big) \in f^*(S')$, we have $gf \in S'$ and $gf \in S$ since $f \in S$, so $g \in f^*(S \cap S')$.

This proves the *claim*. By (G2), we have $f^*(S') \in J_B$, thus $f^*(S \cap S') = f^*(S') \in J_B$.

By (G3), we conclude that $S \cap S' \in J_B$. $\square$

Lemma 2.2.15. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$ and a covering $S \in J_A$. Suppose given a sieve T on A with $S \subseteq T$. Then T covers A, i.e. $T \in J_A$.

Proof. Suppose given $\big(B \xrightarrow{f} A\big) \in S$. Then $f^*(S) = \max(B)$ since $gf \in S$ for $g \in \max(B)$. By remark 2.1.4, we have $\max(B) = f^*(S) \subseteq f^*(T)$. We conclude that $f^*(T) = \max(B) \in J_B$ by (G1). Now (G3) yields $T \in J_A$. $\square$

Lemma 2.2.16. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$ and a covering $S \in J_A$. Suppose given coverings $T_f \in J_B$ for $\big(B \xrightarrow{f} A\big) \in S$. Then $\{gf : f \in S, g \in T_f\} \in J_A$.

Proof. Write $U := \{gf : f \in S, g \in T_f\}$.

The set U is a sieve on A since for $D \xrightarrow{h} C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$ with $f \in S$ and $g \in T_f$, we obtain $hg \in T_f$ and therefore $hgf \in U$.

Suppose given $\big(B \xrightarrow{f} A\big) \in S$. We have $f^*(U) \supseteq T_f$ since $gf \in U$ for $g \in T_f$. Therefore $f^*(U) \in J_B$ by lemma 2.2.15.

So $U \in J_A$ by (G3). $\square$

2.2.4 Reduced topologies

For possible practical purposes, we present a way how to obtain Grothendieck topologies. However, we will not use the results of section 2.2.4 in the sequel.

Lemma/Definition 2.2.17. For $A \in \operatorname{Ob}\mathscr{S}$, let $\operatorname{Cod}(A) := \{f \in \operatorname{Mor}\mathscr{S} \colon \operatorname{cod}(f) = A\}$. The power set of $\operatorname{Cod}(A)$ is denoted by $\mathscr{P}(\operatorname{Cod}(A))$.

A tuple $K = (K_A)_{A \in \operatorname{Ob}\mathscr{S}}$ with $K_A \subseteq \mathscr{P}(\operatorname{Cod}(A))$ is called a *reduced topology on $\mathscr{S}$* if the following three conditions hold.

(RT1) $\{1_A\} \in K_A$ for $A \in \operatorname{Ob}\mathscr{S}$.

(RT2) Suppose given $A \in \operatorname{Ob}\mathscr{S}$, $U \in K_A$ and $\big(B \xrightarrow{f} A\big) \in \operatorname{Mor}\mathscr{S}$. Then there exists $V \in K_B$ such that for all $\big(D \xrightarrow{d} B\big) \in V$, there exist $\big(C \xrightarrow{c} A\big) \in U$ and $\big(D \xrightarrow{g} C\big) \in \operatorname{Mor}\mathscr{S}$ such that $df = gc$.

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "c"] & A \end{tikzcd}$$

(RT3) Suppose given $A \in \operatorname{Ob}\mathscr{S}$, $U \in K_A$ and $V_c \in K_C$ for $\big(C \xrightarrow{c} A\big) \in U$. Then there exists $W \in K_A$ such that for all $\big(E \xrightarrow{e} A\big) \in W$, there exist $\big(C \xrightarrow{c} A\big) \in U$, $\big(D_c \xrightarrow{d} C\big) \in V_c$ and $g \in \operatorname{Mor}\mathscr{S}$ such that $e = gdc$.

$$\begin{tikzcd} & & E \arrow[dll, "g"'] \arrow[d, "e"] \\ D_c \arrow[r, "d"'] & C \arrow[r, "c"'] & A \end{tikzcd}$$

A reduced topology K gives rise to a Grothendieck topology $J^K = (J^K_A)_{A \in \operatorname{Ob}\mathscr{S}}$ on $\mathscr{S}$ by setting $J^K_A := \{S \in \operatorname{Sieves}(A) \colon \text{there exists } U \in K_A \text{ with } U \subseteq S\}$ for $A \in \operatorname{Ob}\mathscr{S}$.

Proof. Ad (G1). For $A \in \operatorname{Ob}\mathscr{S}$, we have $\{1_A\} \subseteq \max(A)$ by (RT1). Thus $\max(A) \in J^K_A$. So (G1) holds.

Ad (G2). Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$. Suppose given $S \in J^K_A$. We may choose $U \in K_A$ with $U \subseteq S$. By (RT2), we may choose $V \in K_B$ such that for all $\big(D \xrightarrow{d} B\big) \in V$, there exist $\big(C \xrightarrow{c} A\big) \in U$ and $\big(D \xrightarrow{g} C\big) \in \operatorname{Mor}\mathscr{S}$ such that $df = gc$. Since $c \in U \subseteq S$ and S is a sieve, $df = gc$ is in S and therefore $d \in f^*(S)$. So $V \subseteq f^*(S)$ and thus $f^*(S) \in J^K_B$. So (G2) holds.

Ad (G3). Suppose given $A \in \operatorname{Ob}\mathscr{S}$ and $S \in J^K_A$. We may choose $U \in K_A$ with $U \subseteq S$. Suppose given a sieve T on A such that $f^*(T) \in J^K_B$ for $\big(B \xrightarrow{f} A\big) \in S$. For $\big(C \xrightarrow{c} A\big) \in U$, we may choose $V_c \in K_C$ with $V_c \subseteq c^*(T)$. By (RT3), we may choose $W \in K_A$ such that for all $\left(E \xrightarrow{e} A\right) \in W$, there exist $\left(C \xrightarrow{c} A\right) \in U$, $\left(D_c \xrightarrow{d} C\right) \in V_c$ and $g \in \operatorname{Mor} \mathscr{S}$ such that $e = gdc$. Since $gdc \in T$ for $c \in U$, $d \in V_c \subseteq c^*(T)$ and $g \in \operatorname{Mor} \mathscr{S}$, we have $W \subseteq T$. Thus $T \in J_A^K$. So (G3) holds. □

Example 2.2.18. A prétopologie in the sense of [5, Exposé II, Def. 1.3] and a basis in the sense of [9, III. 2. Def. 2] are examples of reduced topologies.

Example (cont.)

We want to show that the Grothendieck topology $(\mathscr{S}_X, J)$ from above can be obtained from the following reduced topology $K = (K_U)_{U \subseteq X \text{ open}}$ on $\mathscr{S}_X$. For an open subset $U \subseteq X$, let K_U consist of the sets of open subsets of U whose union is U. So

$$K_U := \{C \subseteq \mathscr{P}(U) \colon \bigcup C = U, \, C \text{ consists of open subsets of } U\}.$$

- Ad (RT1). For an open subset $U \subseteq X$, we have $\bigcup\{U\} = U$.
- Ad (RT2). Suppose given open subsets $V \subseteq U \subseteq X$. Suppose given a family of open subsets $C = \{U_i \colon i \in I\} \in K_U$, where I is some index set. So $U_i \subseteq U$ for $i \in I$ and $\bigcup_{i \in I} U_i = U$. Then $\{V \cap U_i \colon i \in I\} \in K_V$ since $\bigcup_{i \in I} V \cap U_i = V \cap \bigcup_{i \in I} U_i = V \cap U = V$. Moreover, we have the following inclusions for $i \in I$.

$$\begin{tikzcd} V \cap U_i \arrow[r] \arrow[d] & V \arrow[d] \\ U_i \arrow[r] & U. \end{tikzcd}$$

- Ad (RT3). Suppose given an open subset $U \subseteq X$. Suppose given a set of open subsets $C = \{U_i \colon i \in I\} \in K_U$, where I is some set. So $U_i \subseteq U$ for $i \in I$ and $\bigcup_{i \in I} U_i = U$. For $i \in I$, suppose given a family of open subsets $D_i = \{V_{i,j} \colon j \in J_i\}$, where J_i is some set. So $V_{i,j} \subseteq U_i$ for $j \in J_i$ and $\bigcup_{j \in J_i} V_{i,j} = U_i$. Then $\{V_{i,j} \colon i \in I, j \in J_i\} \in K_U$ since $\bigcup_{i \in I} \bigcup_{j \in J_i} V_{i,j} = \bigcup_{i \in I} U_i = U$. Moreover, we have the following inclusions for $i \in I$ and $j \in J_i$.

$$\begin{tikzcd} & & V_{i,j} \arrow[lld] \arrow[d] \\ V_{i,j} \arrow[r] & U_i \arrow[r] & U \end{tikzcd}$$

By construction, $J^K = J$.

2.3 Sheafification

Suppose given a ring R.

Suppose given a site $(\mathscr{S}, J)$.

In this section, we construct the sheafification functor, which is left-adjoint to the inclusion functor from the category of (R-)sheaves into the category of (R-)presheaves on $\mathscr{S}$. We will use this adjunction to prove that the category of R-sheaves is abelian.

2.3.1 Construction of the sheafification functor

We will proceed in two steps. At first, we construct a functor that maps a presheaf P to a separated presheaf P^+. Then we show that P^{++} is a sheaf.

Remark 2.3.1. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$.

We consider the set of coverings J_A, ordered by inclusion, as a poset category. So J_A^{op} is a poset category as well. Note that for $S, S' \in J_A$, there exists a morphism $\big(S \rightarrow S'\big)^{\mathrm{op}}$ in J_A^{op} if and only if $S \subseteq S'$.

The category J_A^{op} is filtering, as we show using lemma 1.4.3:

- The category J_A^{op} is non-empty since $\max(A) \in J_A$ by (G1).
- For $S, S' \in J_A$, we have $S \cap S' \in J_A$ by lemma 2.2.14. So (F1) holds since $S \cap S' \subseteq S$ and $S \cap S' \subseteq S'$.

Lemma/Definition 2.3.2. Suppose given an object $A \in \operatorname{Ob} \mathscr{S}$.

(a) Suppose given a presheaf P on $\mathscr{S}$, i.e. $P \in \operatorname{Ob} \hat{\mathscr{S}}$.

The functor $P_A \colon J_A^{\mathrm{op}} \to \mathtt{Set}$ shall be defined as follows.

- For $S \in J_A$, let $P_A(S) := \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, P)$.
- For $S \rightarrow S'$ in J_A, let $P_A\big(\big(S \rightarrow S'\big)^{\mathrm{op}}\big) \colon \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_{S'}, P) \to \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, P) \colon \xi \mapsto \iota_S^{S'} \xi$.

So P_A is a system over J_A^{op} in $\mathtt{Set}$. Let $P^+(A) := \varinjlim P_A \in \operatorname{Ob}(\mathtt{Set})$, cf. definition 1.4.7.(a).

(b) Suppose given an R-presheaf P on $\mathscr{S}$, i.e. $P \in \operatorname{Ob}({}_R\hat{\mathscr{S}})$.

The functor $P_A \colon J_A^{\mathrm{op}} \to \text{Mod-}R$ shall be defined as follows.

- For $S \in J_A$, let $P_A(S) := \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, {}_R\Upsilon \circ P)$, equipped with the following R-module structure.
  Suppose given $\xi, \xi' \in \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, {}_R\Upsilon \circ P)$, $r \in R$ and $B \in \operatorname{Ob} \mathscr{S}$. To define $\xi + \xi'$ and ξr, we use the R-module structure on $\operatorname{Hom}_{\mathtt{Set}}(\mathrm{F}_S(B), ({}_R\Upsilon \circ P)(B))$, cf. remark 1.1.13.
  Let $(\xi + \xi')_B := \xi_B + \xi'_B$ and $(\xi r)_B := \xi_B\, r = \xi_B \cdot r$.
- For $S \rightarrow S'$ in J_A, let

$$P_A\big(\big(S \rightarrow S'\big)^{\mathrm{op}}\big) \colon \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_{S'}, {}_R\Upsilon \circ P) \to \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, {}_R\Upsilon \circ P) \colon \xi \mapsto \iota_S^{S'} \xi.$$

So P_A is a system over J_A^{op} in Mod-R and we have ${}_R\Upsilon \circ P_A = ({}_R\Upsilon \circ P)_A$ by construction.

Let $P^+(A) := \varinjlim P_A \in \operatorname{Ob}(\text{Mod-}R)$, cf. definition 1.4.7.(b).

We have ${}_R\Upsilon(P^+(A)) = {}_R\Upsilon(\varinjlim P_A) = \varinjlim({}_R\Upsilon \circ P_A) = \varinjlim({}_R\Upsilon \circ P)_A = ({}_R\Upsilon \circ P)^+(A)$ by loc. cit.

Proof. Ad (a).

We show that P_A is a well-defined functor. Suppose given $S \to S' \to S''$ in J_A .

For $\xi \in \mathrm{Hom}_{\mathscr{S}}(\mathrm{F}_S, P)$, we have $(\xi)\, P_A\big((S \to S)^{\mathrm{op}}\big) = \iota_S^S\, \xi = \xi$ and thus $P_A\big((S \to S)^{\mathrm{op}}\big) = 1_{\mathrm{Hom}_{\mathscr{S}}(\mathrm{F}_S, P)}$.

For $\xi \in \mathrm{Hom}_{\mathscr{S}}(\mathrm{F}_{S''}, P)$, we have

$$\begin{aligned}
(\xi)\, P_A\big((S' \to S'')^{\mathrm{op}}(S \to S')^{\mathrm{op}}\big) &= (\xi)\, P_A\big((S \to S'')^{\mathrm{op}}\big) \\
&= \iota_S^{S''}\, \xi \\
&= \iota_S^{S'}\, \iota_{S'}^{S''}\, \xi \\
&= (\iota_{S'}^{S''}\, \xi)\, P_A\big((S \to S')^{\mathrm{op}}\big) \\
&= (\xi)\, P_A\big((S' \to S'')^{\mathrm{op}}\big)\, P_A\big((S \to S')^{\mathrm{op}}\big),
\end{aligned}$$

so $P_A\big((S' \to S'')^{\mathrm{op}}\, (S \to S')^{\mathrm{op}}\big) = P_A\big((S' \to S'')^{\mathrm{op}}\big)\, P_A\big((S \to S')^{\mathrm{op}}\big)$.

Ad (b).

Suppose given $S \in J_A$.

Suppose given $\xi, \xi' \in P_A(S) = \mathrm{Hom}_{\mathscr{S}}(\mathrm{F}_S, {}_R\Upsilon \circ P)$ and $r \in R$.

We show that $\xi + \xi'$ and ξr are elements of $P_A(S)$, i.e. that they are natural.

For $C \xrightarrow{f} B$ in $\mathscr{S}$, we have

$$\begin{aligned}
(\xi + \xi')_B \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}}) &= (\xi_B + \xi'_B) \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}}) \\
&= \xi_B \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}}) + \xi'_B \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}}) \\
&= \mathrm{F}_S(f^{\mathrm{op}}) \cdot \xi_C + \mathrm{F}_S(f^{\mathrm{op}}) \cdot \xi'_C \\
&= \mathrm{F}_S(f^{\mathrm{op}}) \cdot (\xi_C + \xi'_C) \\
&= \mathrm{F}_S(f^{\mathrm{op}}) \cdot (\xi + \xi')_C
\end{aligned}$$

and

$$\begin{aligned}
(\xi r)_B \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}}) &= (\xi_B\, r) \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}}) \\
&= (\xi_B \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}})) \cdot r \\
&= (\mathrm{F}_S(f^{\mathrm{op}}) \cdot \xi_C) \cdot r \\
&= \mathrm{F}_S(f^{\mathrm{op}}) \cdot (\xi_C \cdot r) \\
&= \mathrm{F}_S(f^{\mathrm{op}}) \cdot (\xi r)_C\,,
\end{aligned}$$

cf. remark 1.1.13.

We verify the axioms for R-modules for $P_A(S)$.

The zero element in $P_A(S)$ is

$$0 = (0_B)_{B \in \operatorname{Ob} \mathscr{S}} \colon \ \mathrm{F}_S \Rightarrow {}_R\Upsilon \circ P$$

with $0_B = 0_{\operatorname{Hom}_{\mathbf{Set}}(\mathrm{F}_S(B), ({}_R\Upsilon \circ P)(B))}$ for $B \in \operatorname{Ob} \mathscr{S}$. It is natural since for $C \xrightarrow{f} B$ in $\mathscr{S}$, we have

$$0_B \cdot ({}_R\Upsilon \circ P)(f^{\mathrm{op}}) = 0 = \mathrm{F}_S(f^{\mathrm{op}}) \cdot 0_C \,.$$

Suppose given $\xi, \xi', \xi'' \in P_A(S)$, $B \in \operatorname{Ob} \mathscr{S}$ and $r, r' \in R$.

We have

$$(\xi + \xi')_B = \xi_B + \xi'_B = \xi'_B + \xi_B = (\xi' + \xi)_B \,,$$

$$(\xi + 0)_B = \xi_B + 0_B = \xi_B \,,$$

$$(\xi + \xi \cdot (-1))_B = \xi_B + \xi_B \cdot (-1) = 0_B$$

and

$$((\xi + \xi') + \xi'')_B = (\xi + \xi')_B + \xi''_B = \xi_B + \xi'_B + \xi''_B = \xi_B + (\xi' + \xi'')_B = (\xi + (\xi' + \xi''))_B \,.$$

So $\xi + \xi' = \xi' + \xi$, $\quad \xi + 0 = \xi$, $\quad \xi + \xi \cdot (-1) = 0$ and $(\xi + \xi') + \xi'' = \xi + (\xi' + \xi'')$.

We have

$$(\xi \cdot 1)_B = \xi_B \cdot 1 = \xi_B \,,$$

$$(\xi(rr'))_B = \xi_B \cdot (rr') = (\xi_B \cdot r)r' = (\xi r)_B \cdot r' = ((\xi r)r')_B \,,$$

$$((\xi + \xi')r)_B = (\xi + \xi')_B \cdot r = (\xi_B + \xi'_B)r = \xi_B \cdot r + \xi'_B \cdot r = (\xi r)_B + (\xi' r)_B = (\xi r + \xi' r)_B$$

and

$$(\xi(r + r'))_B = \xi_B \cdot (r + r') = \xi_B \cdot r + \xi_B \cdot r' = (\xi r + \xi r')_B \,.$$

So $\xi \cdot 1 = \xi$, $\quad \xi(rr') = (\xi r)r'$, $\quad (\xi + \xi')r = \xi r + \xi' r$ and $\xi(r + r') = \xi r + \xi r'$.

Suppose given $S \!\to\! S'$ in J_A . We show that $P_A((S \!\to\! S')^{\mathrm{op}})$ is R-linear.

Suppose given $\xi, \xi' \in P_A(S')$ and $r, r' \in R$.

We have

$$\begin{aligned}(\xi r + \xi' r')\, P_A((S \!\to\! S')^{\mathrm{op}}) &= \iota_S^{S'}(\xi r + \xi' r') = \iota_S^{S'}\xi \cdot r + \iota_S^{S'}\xi' \cdot r' \\ &= (\xi)\, P_A((S \!\to\! S')^{\mathrm{op}}) \cdot r + (\xi')\, P_A((S \!\to\! S')^{\mathrm{op}}) \cdot r',\end{aligned}$$

cf. remark 1.1.13.

The same calculation as in (a) now shows that P_A is a functor. $\square$

Remark 2.3.3.

(a) Suppose given an object $A \in \operatorname{Ob}\mathscr{S}$ and a presheaf P on $\mathscr{S}$. The set $P^+(A) = \varinjlim P_A$ consists of elements of the form $[\xi, S]$, where $S \in J_A$ and $\xi \in \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, P)$.

For $[\xi, S], [\xi', S'] \in P^+(A)$, we have $[\xi, S] = [\xi', S']$ if and only if there exists $T \in J_A$ with $T \subseteq S \cap S'$ such that $\iota_T^S \xi = \iota_T^{S'} \xi'$.

Cf. definitions 2.3.2.(a) and 1.4.7.(a).

(b) Suppose given an object $A \in \operatorname{Ob}\mathscr{S}$ and an R-presheaf P on $\mathscr{S}$. The set $P^+(A) = \varinjlim P_A$ consists of elements of the form $[\xi, S]$, where $S \in J_A$ and $\xi \in \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, {}_R\Upsilon \circ P)$.

For $[\xi, S], [\xi', S'] \in P^+(A)$, we have $[\xi, S] = [\xi', S']$ if and only if there exists $T \in J_A$ with $T \subseteq S \cap S'$ such that $\iota_T^S \xi = \iota_T^{S'} \xi'$. Moreover, for $T \in J_A$ with $T \subseteq S \cap S'$, we have $[\xi, S] + [\xi', S'] = [\iota_T^S \xi + \iota_T^{S'} \xi', T]$. For $r \in R$, we have $[\xi, S] \cdot r = [\xi r, S]$.

Cf. definitions 2.3.2.(b) and 1.4.7.(b).

Lemma/Definition 2.3.4. Suppose given a presheaf P on $\mathscr{S}$. Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$, a covering $S \in J_A$ and a matching family $\xi \in \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, P)$ for P on S.

Let $\xi_C^f \colon \mathrm{F}_{f^*(S)}(C) \to P(C) \colon h \mapsto (h)\xi_C^f := (hf)\xi_C$ for $C \in \operatorname{Ob}\mathscr{S}$.

We obtain a matching family $\xi^f := (\xi_C^f)_{C \in \operatorname{Ob}\mathscr{S}} \colon \mathrm{F}_{f^*(S)} \Rightarrow P$ for P on $f^*(S)$.

Proof. We need to show that ξ^f is natural.

Suppose given $D \xrightarrow{g} C$ in $\mathscr{S}$ and $h \in \mathrm{F}_{f^*(S)}(C) \subseteq \operatorname{Hom}_{\mathscr{S}}(C, B)$.

We have

$$\begin{aligned}(h)\xi_C^f\, P(g^{\mathrm{op}}) &= (hf)\xi_C\, P(g^{\mathrm{op}}) = (hf)\ \mathrm{F}_S(g^{\mathrm{op}})\,\xi_D = (ghf)\xi_D = (gh)\xi_D^f \\ &= (h)\ \mathrm{F}_{f^*(S)}(g^{\mathrm{op}})\,\xi_D^f\,,\end{aligned}$$

so $\xi_C^f\, P(g^{\mathrm{op}}) = \mathrm{F}_{f^*(S)}(g^{\mathrm{op}})\,\xi_D^f$.

$$\begin{tikzcd} \mathrm{F}_{f^*(S)}(C) \arrow[r, "\xi_C^f"] \arrow[d, "\mathrm{F}_{f^*(S)}(g^{\mathrm{op}})"'] & P(C) \arrow[d, "P(g^{\mathrm{op}})"] \\ \mathrm{F}_{f^*(S)}(D) \arrow[r, "\xi_D^f"] & P(D) \end{tikzcd}$$

□

Remark 2.3.5. Suppose given a presheaf P on $\mathscr{S}$. Suppose given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$, a covering $S \in J_A$ and a matching family $\xi \in \operatorname{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, P)$ for P on S.

We have $\xi^{(gf)} = \big(\xi^f\big)^g$ as matching families for P on $(gf)^*(S) = g^*(f^*(S))$, cf. remark 2.1.5.

Proof. For $D \in \operatorname{Ob}\mathscr{S}$ and $h \in \mathrm{F}_{(gf)^*(S)}(D)$, we have

$$(h)\xi_D^{(gf)} = (hgf)\xi_D = (hg)\xi_D^f = (h)(\xi^f)_D^g\,.$$

□

Remark 2.3.6. Suppose given $P \xrightarrow{\alpha} Q$ in $\hat{\mathscr{S}}$. So we have presheaves P and Q on $\mathscr{S}$ and a transformation $\alpha\colon P \Rightarrow Q$.

Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$, a covering $S \in J_A$ and a matching family $\xi \in \mathrm{Hom}_{\hat{\mathscr{S}}}(\mathrm{F}_S, P)$ for P on S.

We have $\xi^f \alpha = (\xi\alpha)^f$ as matching families for Q on $f^*(S)$.

Proof. For $C \in \mathrm{Ob}\,\mathscr{S}$ and $h \in \mathrm{F}_{f^*(S)}(C)$, we have

$$(h)(\xi^f \alpha)_C = (h)\xi^f_C\, \alpha_C = (hf)\xi_C\, \alpha_C = (hf)(\xi\alpha)_C = (h)(\xi\alpha)^f_C\,. \qquad \square$$

Lemma/Definition 2.3.7. Suppose given a morphism $\big(B \xrightarrow{f} A\big) \in \mathrm{Mor}\,\mathscr{S}$.

(a) Suppose given a presheaf P on $\mathscr{S}$. Let

$$P^+(f^{\mathrm{op}})\colon P^+(A) \to P^+(B)\colon [\xi, S] \mapsto [\xi^f, f^*(S)].$$

This is a well-defined map.

(b) Suppose given an R-presheaf P on $\mathscr{S}$. Let

$$P^+(f^{\mathrm{op}})\colon P^+(A) \to P^+(B)\colon [\xi, S] \mapsto [\xi^f, f^*(S)].$$

This is a well-defined R-linear map.

We have ${}_R\Upsilon(P^+(f^{\mathrm{op}})) = ({}_R\Upsilon \circ P)^+(f^{\mathrm{op}})$ by construction, cf. definition 2.3.2.

Proof. Ad (a).

Suppose given $[\xi, S], [\xi', S'] \in P^+(A)$ with $[\xi, S] = [\xi', S']$.
So there exists $T \in J_A$ with $T \subseteq S \cap S'$ such that $\iota^S_T\, \xi = \iota^{S'}_T \xi'$.

Thus for $C \in \mathrm{Ob}\,\mathscr{S}$ and $g \in \mathrm{F}_T(C)$, we have $(g)\xi_C = (g)\xi'_C$ as elements of $P(C)$.

We have to show that $[\xi^f, f^*(S)] = [\xi'^f, f^*(S')]$.

Note that $f^*(T) \in J_B$ with $f^*(T) \subseteq f^*(S) \cap f^*(S')$ by (G2) and remark 2.1.4.

It suffices to show that $\iota^{f^*(S)}_{f^*(T)}\, \xi^f = \iota^{f^*(S')}_{f^*(T)}\, \xi'^f\colon \mathrm{F}_{f^*(T)} \Rightarrow P$.

Suppose given $C \in \mathrm{Ob}\,\mathscr{S}$ and $h \in \mathrm{F}_{f^*(T)}(C)$. So $\big(C \xrightarrow{h} B \xrightarrow{f} A\big) \in T$.

We have

$$(h)\big(\iota^{f^*(S)}_{f^*(T)}\big)_C\, \xi^f_C = (hf)\xi_C = (hf)(\iota^S_T\, \xi)_C = (hf)(\iota^{S'}_T \xi')_C = (hf)\xi'_C = (h)\big(\iota^{f^*(S')}_{f^*(T)}\big)_C\, \xi'^f_C\,.$$

Thus $\big(\iota^{f^*(S)}_{f^*(T)}\, \xi^f\big)_C = \big(\iota^{f^*(S')}_{f^*(T)}\, \xi'^f\big)_C\,.$

Ad (b).

The same calculation as in (a) shows that $P^+(f^{\mathrm{op}})$ is well-defined. We show that it is R-linear.

Suppose given $r, r' \in R$ and $[\xi, S], [\xi', S'] \in P^+(A)$. Write $T := S \cap S' \in J_A$, cf. lemma 2.2.14. Note that $f^*(T) \in J_B$ with $f^*(T) \subseteq f^*(S) \cap f^*(S')$ by (G2) and remark 2.1.4.

We have

$$\begin{aligned}\big([\xi, S] \cdot r + [\xi', S'] \cdot r'\big)\, P^+(f^{\mathrm{op}}) &= \big([\xi r, S] + [\xi' r', S']\big)\, P^+(f^{\mathrm{op}}) \\ &= \big([\iota_T^S(\xi r) + \iota_T^{S'}(\xi' r'), T]\big)\, P^+(f^{\mathrm{op}}) \\ &= [(\iota_T^S(\xi r) + \iota_T^{S'}(\xi' r'))^f, f^*(T)]\end{aligned}$$

and

$$\begin{aligned}([\xi, S])\, P^+(f^{\mathrm{op}}) \cdot r + ([\xi', S'])\, P^+(f^{\mathrm{op}}) \cdot r' &= [\xi^f, f^*(S)] \cdot r + [\xi'^f, f^*(S')] \cdot r' \\ &= [\xi^f r, f^*(S)] + [\xi'^f r', f^*(S')] \\ &= [\iota_T^S(\xi^f r) + \iota_T^{S'}(\xi'^f r'), f^*(T)].\end{aligned}$$

So it suffices to show that $(\iota_T^S(\xi r) + \iota_T^{S'}(\xi' r'))^f = \iota_T^S(\xi^f r) + \iota_T^{S'}(\xi'^f r')$.

Suppose given $C \in \mathrm{Ob}\,\mathscr{S}$ and $h \in \mathrm{F}_{f^*(T)}(C)$.

We have

$$\begin{aligned}(h)(\iota_T^S(\xi r) + \iota_T^{S'}(\xi' r'))^f_C &= (hf)(\iota_T^S(\xi r) + \iota_T^{S'}(\xi' r'))_C \\ &= (hf)\xi_C \cdot r + (hf)\xi'_C \cdot r' \\ &= (h)\xi^f_C \cdot r + (h)\xi'^f_C \cdot r' \\ &= (h)(\xi^f_C\, r) + (h)(\xi'^f_C\, r') \\ &= (h)(\iota_T^S(\xi^f r) + \iota_T^{S'}(\xi'^f r'))_C\, .\end{aligned}$$

□

Lemma/Definition 2.3.8.

(a) Suppose given a presheaf P on $\mathscr{S}$. The constructions given in the definitions 2.3.2.(a) and 2.3.7.(a) yield a presheaf P^+ on $\mathscr{S}$.

(b) Suppose given an R-presheaf P on $\mathscr{S}$. The constructions given in the definitions 2.3.2.(b) and 2.3.7.(b) yield an R-presheaf on $\mathscr{S}$.

We have ${}_R\Upsilon_{\mathscr{S}}(P^+) = {}_R\Upsilon \circ P^+ = ({}_R\Upsilon \circ P)^+ = ({}_R\Upsilon_{\mathscr{S}}(P))^+$ by construction, cf. definition 2.2.9.

Proof. Ad (a). Suppose given $P \in \mathrm{Ob}\,\hat{\mathscr{S}}$. Suppose given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$. Suppose given $[\xi, S] \in P^+(A)$.

We have

$$[\xi, S]\, P^+(1_A^{\mathrm{op}}) = [\xi^{1_A}, (1_A)^*(S)] = [\xi, S] = [\xi, S]\, 1_{P^+(A)}$$

and

$$\begin{aligned}[\xi, S]\, P^+(f^{\mathrm{op}} \cdot g^{\mathrm{op}}) &= [\xi, S]\, P^+((gf)^{\mathrm{op}}) = [\xi^{(gf)}, (gf)^*(S)] \overset{\mathrm{R2.3.5}}{=} [(\xi^f)^g, g^*(f^*(S))] \\ &= [\xi^f, f^*(S)]\, P^+(g^{\mathrm{op}}) \\ &= [\xi, S]\, P^+(f^{\mathrm{op}}) \cdot P^+(g^{\mathrm{op}}).\end{aligned}$$

So $P^+(1_A) = 1_{P^+(A)}$ and $P^+(f^{\mathrm{op}} \cdot g^{\mathrm{op}}) = P^+(f^{\mathrm{op}}) \cdot P^+(g^{\mathrm{op}})$.

Ad (b). The same calculation as in (a) shows that P^+ is a functor. $\square$

Lemma/Definition 2.3.9.

(a) Suppose given $P \xrightarrow{\alpha} Q$ in $\hat{\mathscr{S}}$. So we have presheaves P and Q on $\mathscr{S}$ and a transformation $\alpha\colon P \Rightarrow Q$.

For $A \in \operatorname{Ob}\mathscr{S}$, let $\alpha_A^+\colon P^+(A) \to Q^+(A)\colon [\xi, S] \mapsto [\xi, S]\, \alpha_A^+ := [\xi\alpha, S]$.

This is a well-defined map.

We obtain a transformation $\alpha^+ := \left(\alpha_A^+\right)_{A \in \operatorname{Ob}\mathscr{S}}\colon P^+ \Rightarrow Q^+$.

(b) Suppose given $P \xrightarrow{\alpha} Q$ in ${}_R\hat{\mathscr{S}}$. So we have R-presheaves P and Q on $\mathscr{S}$ and a transformation $\alpha\colon P \Rightarrow Q$.

For $A \in \operatorname{Ob}\mathscr{S}$, let $\alpha_A^+\colon P^+(A) \to Q^+(A)\colon [\xi, S] \mapsto [\xi, S]\, \alpha_A^+ := \left[\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha), S\right]$,

where ${}_R\Upsilon_{\mathscr{S}}(\alpha) = {}_R\Upsilon \star \alpha = \left({}_R\Upsilon(\alpha_A)\right)_{A \in \operatorname{Ob}\mathscr{S}} \in \operatorname{Hom}_{\hat{\mathscr{S}}}({}_R\Upsilon \circ P, {}_R\Upsilon \circ Q)$, cf. definition 2.2.9.

This is a well-defined R-linear map.

We obtain a transformation $\alpha^+ := \left(\alpha_A^+\right)_{A \in \operatorname{Ob}\mathscr{S}}\colon P^+ \Rightarrow Q^+$.

We have ${}_R\Upsilon_{\mathscr{S}}(\alpha^+) = ({}_R\Upsilon_{\mathscr{S}}(\alpha))^+$ by construction, cf. definition 2.3.8 and lemma 2.3.7.(b).

Proof. Ad (a). We want to show that α_A^+ is well-defined.

Suppose given $[\xi, S], [\xi', S'] \in P^+(A)$ with $[\xi, S] = [\xi', S']$.

So there exists $T \in J_A$ with $T \subseteq S \cap S'$ such that $\iota_T^S\, \xi = \iota_T^{S'}\, \xi'$.

We have $\iota_T^S\, (\xi\alpha) = \iota_T^S\, \xi\alpha = \iota_T^{S'} \xi'\alpha = \iota_T^{S'}\, (\xi'\alpha)$, so $[\xi\alpha, S] = [\xi'\alpha, S']$.

We want to show that α^+ is natural.

Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$. For $[\xi, S] \in P^+(A)$, we have

$$\begin{aligned}[\xi, S]\, \alpha_A^+ \cdot Q^+(f^{\mathrm{op}}) &= [\xi\alpha, S]\, Q^+(f^{\mathrm{op}}) = [(\xi\alpha)^f, f^*(S)] \overset{\mathrm{R2.3.6}}{=} [\xi^f\alpha, f^*(S)] = [\xi^f, f^*(S)]\, \alpha_B^+ \\ &= [\xi, S]\, P^+(f^{\mathrm{op}}) \cdot \alpha_B^+ .\end{aligned}$$

So $\alpha_A^+ \cdot Q^+(f^{\mathrm{op}}) = P^+(f^{\mathrm{op}}) \cdot \alpha_B^+$.

$$\begin{tikzcd} P^+(A) \arrow[r, "\alpha_A^+"] \arrow[d, "P^+(f^{\mathrm{op}})"'] & Q^+(A) \arrow[d, "Q^+(f^{\mathrm{op}})"] \\ P^+(B) \arrow[r, "\alpha_B^+"] & Q^+(B) \end{tikzcd}$$

Ad (b).

The same calculation as in (a) shows that α_A^+ is a well-defined map. We show that it is R-linear.

Suppose given $[\xi, S], [\xi', S'] \in P^+(A)$ and $r, r' \in R$. Write $T := S \cap S' \in J_A$, cf. lemma 2.2.14.

Write $\hat{\alpha} := {}_R\Upsilon_{\mathscr{S}}(\alpha)$. We have

$$\begin{aligned}
\big([\xi, S] \cdot r + [\xi', S'] \cdot r'\big)\alpha_A^+ &= [\iota_T^S\, \xi r + \iota_T^{S'} \xi' r', T]\, \alpha_A^+ \\
&= [(\iota_T^S\, \xi r + \iota_T^{S'} \xi' r')\hat{\alpha}, T] \\
&= [\iota_T^S\, \xi \hat{\alpha} r + \iota_T^{S'} \xi' \hat{\alpha} r', T] \\
&= [\xi\hat{\alpha}, S] \cdot r + [\xi'\hat{\alpha}, S'] \cdot r' \\
&= [\xi, S]\, \alpha_A^+ \cdot r + [\xi, S']\, \alpha_A^+ \cdot r'.
\end{aligned}$$

$$\mathrm{F}_T \xrightarrow{\iota_T^S} \mathrm{F}_S \xrightarrow{\xi} {}_R\Upsilon \circ P \xrightarrow{\hat{\alpha}} {}_R\Upsilon \circ Q$$

Now the same calculation as in (a) shows that α^+ is natural. □

Lemma/Definition 2.3.10.

(a) The functor $(-)^+ \colon \hat{\mathscr{S}} \to \hat{\mathscr{S}}$ shall be defined as follows.

- For $P \in \operatorname{Ob} \hat{\mathscr{S}}$, let $(-)^+(P) := P^+$, cf. definition 2.3.8.(a).
- For $\alpha \in \operatorname{Mor} \hat{\mathscr{S}}$, let $(-)^+(\alpha) := \alpha^+$, cf. definition 2.3.9.(a).

(b) The functor $(-)^+ \colon {}_R\hat{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ shall be defined as follows.

- For $P \in \operatorname{Ob}({}_R\hat{\mathscr{S}})$, let $(-)^+(P) := P^+$, cf. definition 2.3.8.(b).
- For $\alpha \in \operatorname{Mor}({}_R\hat{\mathscr{S}})$, let $(-)^+(\alpha) := \alpha^+$, cf. definition 2.3.9.(b).

We do not distinguish between $(-)^+ \colon \hat{\mathscr{S}} \to \hat{\mathscr{S}}$ and $(-)^+ \colon {}_R\hat{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ in notation.

We have $(-)^+ \circ {}_R\Upsilon_{\mathscr{S}} = {}_R\Upsilon_{\mathscr{S}} \circ (-)^+$, cf. definitions 2.3.8 and 2.3.9.

Moreover, the functor $(-)^+ \colon {}_R\hat{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ is additive.

$$\begin{tikzcd}
{}_R\hat{\mathscr{S}} \arrow[r, "(-)^+"] \arrow[d, "{}_R\Upsilon_{\mathscr{S}}"'] & {}_R\hat{\mathscr{S}} \arrow[d, "{}_R\Upsilon_{\mathscr{S}}"] \\
\hat{\mathscr{S}} \arrow[r, "(-)^+"] & \hat{\mathscr{S}}
\end{tikzcd}$$

Proof. Ad (a). Suppose given $P \xrightarrow{\alpha} Q \xrightarrow{\beta} N$ in $\hat{\mathscr{S}}$.

For $A \in \operatorname{Ob} \mathscr{S}$ and $[\xi, S] \in P^+(A)$, we have

$$[\xi, S]\,\big(1_P^+\big)_A = [\xi \cdot 1_P, S] = [\xi, S] = [\xi, S]\,\big(1_{P^+}\big)_A$$

and

$$[\xi, S]\,(\alpha^+\beta^+)_A = [\xi, S]\,\alpha_A^+\,\beta_A^+ = [\xi\alpha, S]\,\beta_A^+ = [\xi\alpha\beta, S] = [\xi, S]\,(\alpha\beta)_A^+\,.$$

So $1_P^+ = 1_{P^+}$ and $(\alpha\beta)^+ = \alpha^+\beta^+$.

Ad (b). The same calculation as in (a) shows that $(-)^+$ is a functor.

We show that it is additive. Suppose given $P \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Q$ in ${}_R\hat{\mathscr{S}}$. It suffices to show that $(\alpha+\beta)^+ = \alpha^+ + \beta^+$.

Suppose given $A \in \operatorname{Ob}\mathscr{S}$ and $[\xi, S] \in P^+(A)$.

We have

$$[\xi, S](\alpha+\beta)_A^+ = [\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha+\beta), S]$$

and

$$[\xi, S](\alpha^+ + \beta^+)_A = [\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha), S] + [\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\beta), S] = [\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha) + \xi \cdot {}_R\Upsilon_{\mathscr{S}}(\beta), S].$$

So it remains to show that $\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha+\beta) = \xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha) + \xi \cdot {}_R\Upsilon_{\mathscr{S}}(\beta)$.

Suppose given $\big(B \xrightarrow{f} A\big) \in S$. We have

$$\begin{aligned}
(f)\big(\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha+\beta)\big)_B &= (f)\xi_B \cdot \big({}_R\Upsilon_{\mathscr{S}}(\alpha+\beta)\big)_B \\
&= (f)\xi_B \cdot (\alpha+\beta)_B \\
&= (f)\xi_B \cdot (\alpha_B + \beta_B) \\
&= (f)\xi_B \cdot \alpha_B + (f)\xi_B \cdot \beta_B \\
&= (f)\big(\xi_B \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha)_B + \xi_B \cdot {}_R\Upsilon_{\mathscr{S}}(\beta)_B\big) \\
&= (f)\big(\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha) + \xi \cdot {}_R\Upsilon_{\mathscr{S}}(\beta)\big)_B\,.
\end{aligned}$$

□

Proposition 2.3.11.

(a) Suppose given a presheaf P on $\mathscr{S}$. The presheaf P^+ is separated, cf. definition 2.2.7.(a).

(b) Suppose given an R-presheaf P on $\mathscr{S}$. The R-presheaf P^+ is separated, cf. definition 2.2.7.(b).

Proof. Ad (a).

Suppose given an object $A \in \operatorname{Ob}\mathscr{S}$, a covering $S \in J_A$ and a matching family $\xi\colon \mathrm{F}_S \Rightarrow P^+$.

Suppose given amalgamations $[\zeta, U], [\eta, V] \in P^+(A)$ of ξ, cf. definition 2.2.4, remark 2.2.6 and remark 2.3.3.(a). So $U, V \in J_A$, $\zeta\colon \mathrm{F}_U \Rightarrow P$ and $\eta\colon \mathrm{F}_V \Rightarrow P$. We have to show that $[\zeta, U] = [\eta, V]$.

For $\left(B \xrightarrow{f} A\right) \in S$, we have

$$[\zeta^f, f^*(U)] = [\zeta, U]\, P^+(f^{\mathrm{op}}) = (f)\xi_B = [\eta, V]\, P^+(f^{\mathrm{op}}) = [\eta^f, f^*(V)]$$

since $[\zeta, U]$ and $[\eta, V]$ are amalgamations of ξ.

So we may choose $W^f \in J_B$ with $W^f \subseteq f^*(U) \cap f^*(V)$ such that $\iota_{W^f}^{f^*(U)}\, \zeta^f = \iota_{W^f}^{f^*(V)}\, \eta^f$.

We have $T := \{gf \colon f \in S,\ g \in W^f\} \in J_A$, cf. lemma 2.2.16. Note that $T \subseteq U \cap V$.

Now it suffices to show that $\iota_T^U\, \zeta = \iota_T^V\, \eta$.

Suppose given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$ with $f \in S$ and $g \in W^f$, i.e. $gf \in T$.

We obtain

$$(gf)\zeta_C = (g)\zeta_C^f = (g)\big(\iota_{W^f}^{f^*(U)}\big)_C\, \zeta_C^f = (g)\big(\iota_{W^f}^{f^*(V)}\big)_C\, \eta_C^f = (g)\eta_C^f = (gf)\eta_C\,.$$

Thus $\iota_T^U\, \zeta = \iota_T^V\, \eta$.

Ad (b).

It suffices to show that ${}_R\Upsilon \circ P^+ = ({}_R\Upsilon \circ P)^+$ is separated, cf. definition 2.3.8 and remark 2.2.10. The result now follows from (a). $\square$

Proposition 2.3.12.

(a) Suppose given a separated presheaf P on $\mathscr{S}$. The presheaf P^+ is a sheaf, cf. definition 2.2.8.(a).

(b) Suppose given a separated R-presheaf P on $\mathscr{S}$. The R-presheaf P^+ is an R-sheaf, cf. definition 2.2.8.(b).

Proof. Ad (a).

Suppose given an object $A \in \operatorname{Ob}\mathscr{S}$, a covering $S \in J_A$ and a matching family $\xi \colon \mathrm{F}_S \Rightarrow P^+$.

Write $(f)\xi_B =: [\xi_f, T_f]$ for $\left(B \xrightarrow{f} A\right) \in S$. So $T_f \in J_B$ and $\xi_f \colon \mathrm{F}_{T_f} \Rightarrow P$.

Suppose given $C \xrightarrow{g} B$ in $\mathscr{S}$.

We observe that

$$[(\xi_f)^g, g^*(T_f)] = [\xi_f, T_f]\, P^+(g^{\mathrm{op}}) = (f)\xi_B\, P^+(g^{\mathrm{op}}) = (f)\, \mathrm{F}_S(g^{\mathrm{op}})\, \xi_C = (gf)\xi_C = [\xi_{gf}, T_{gf}]$$

in $P^+(C)$ since ξ is a transformation.

$$\begin{array}{ccc} \mathrm{F}_S(B) & \xrightarrow{\ \xi_B\ } & P^+(B) \\ {\scriptstyle \mathrm{F}_S(g^{\mathrm{op}})}\downarrow & & \downarrow{\scriptstyle P^+(g^{\mathrm{op}})} \\ \mathrm{F}_S(C) & \xrightarrow{\ \xi_C\ } & P^+(C) \end{array}$$

So we may choose $Q_{g,f} \in J_C$ with $Q_{g,f} \subseteq g^*(T_f) \cap T_{gf}$ such that $\iota_{Q_{g,f}}^{g^*(T_f)}(\xi_f)^g = \iota_{Q_{g,f}}^{T_{gf}}\,\xi_{gf}$, cf. remark 2.3.3.(a).

The presheaf P^+ is separated by proposition 2.3.11.(a). Thus it suffices to show that there exists an amalgamation $[\zeta, U] \in P^+(A)$ of ξ, where $U \in J_A$ and $\zeta\colon \mathrm{F}_U \Rightarrow P$.

Let $U := \{gf\colon f \in S,\ g \in T_f\}$. We have $U \in J_A$ by lemma 2.2.16.

For $C \in \operatorname{Ob}\mathscr{S}$, we define $\zeta_C\colon \mathrm{F}_U(C) \to P(C)$ as follows.

For $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{S}$ with $f \in S$ and $g \in T_f$, let $(gf)\zeta_C := (g)(\xi_f)_C \in P(C)$.

We want to show that ζ_C is a well-defined map. Suppose given $C \xrightarrow{g'} B' \xrightarrow{f'} A$ in $\mathscr{S}$ with $f' \in S$, $g' \in T_{f'}$ and $gf = g'f' =: u$.

We have $Q := Q_{g,f} \cap Q_{g',f'} \in J_C$ by lemma 2.2.14.

So we have the following diagram of inclusions in J_A .

$$\begin{tikzcd}
g^*(T_f) \cap T_u & & g'^*(T_{f'}) \cap T_u \\
Q_{g,f} \arrow[u] & & Q_{g',f'} \arrow[u] \\
& Q \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We *claim* that $(g)(\xi_f)_C \in P(C)$ and $(g')(\xi_{f'})_C \in P(C)$ are amalgamations of the matching family $\iota_Q^{T_u}\,\xi_u\colon \mathrm{F}_Q \Rightarrow P$.

Suppose given $\bigl(D \xrightarrow{h} C\bigr) \in Q$.

$$\begin{tikzcd}
& & B \arrow[dr, "f"] & \\
D \arrow[r, "h"] & C \arrow[ur, "g"] \arrow[rr, "u"] \arrow[dr, "g'"'] & & A \\
& & B' \arrow[ur, "f'"'] &
\end{tikzcd}$$

The naturality of ξ_f yields

$$\begin{aligned}
(g)(\xi_f)_C\,P(h^{\mathrm{op}}) &= (g)\,\mathrm{F}_{T_f}(h^{\mathrm{op}})\,(\xi_f)_D = (hg)(\xi_f)_D = (h)((\xi_f)^g)_D = (h)\bigl(\iota_{Q_{g,f}}^{g^*(T_f)}\bigr)_D\,\bigl((\xi_f)^g\bigr)_D \\
&= (h)\bigl(\iota_{Q_{g,f}}^{T_{gf}}\bigr)_D\,(\xi_{gf})_D = (h)\bigl(\iota_Q^{T_u}\bigr)_D\,(\xi_u)_D = (h)\bigl(\iota_Q^{T_u}\xi_u\bigr)_D\,.
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{F}_{T_f}(C) \arrow[r, "(\xi_f)_C"] \arrow[d, "\mathrm{F}_{T_f}(h^{\mathrm{op}})"'] & P(C) \arrow[d, "P(h^{\mathrm{op}})"] \\
\mathrm{F}_{T_f}(D) \arrow[r, "(\xi_f)_D"] & P(D)
\end{tikzcd}$$

Similarly, the naturality of $\xi_{f'}$ yields

$$\begin{aligned}
(g')(\xi_{f'})_C\,P(h^{\mathrm{op}}) &= (g')\,\mathrm{F}_{T_{f'}}(h^{\mathrm{op}})\,(\xi_{f'})_D = (hg')(\xi_{f'})_D = (h)((\xi_{f'})^{g'})_D \\
&= (h)\big(\iota_{Q_{g',f'}}^{g'^*(T_{f'})}\big)_D\,\big((\xi_{f'})^{g'}\big)_D = (h)\big(\iota_{Q_{g',f'}}^{T_{g'f'}}\big)_D\,(\xi_{g'f'})_D = (h)\big(\iota_Q^{T_u}\big)_D\,(\xi_u)_D \\
&= (h)\big(\iota_Q^{T_u}\xi_u\big)_D\,.
\end{aligned}$$

This proves the *claim*.

So $(g)(\xi_f)_C = (g')(\xi_{f'})_C$ since P is separated.

We need to show that $\zeta = (\zeta_C)_{C\in\mathrm{Ob}\,\mathscr{S}}$ is natural.

Suppose given $D\xrightarrow{h}C$ in $\mathscr{S}$.

Suppose given $C\xrightarrow{g}B\xrightarrow{f}A$ in $\mathscr{S}$ with $f\in S$ and $g\in T_f$, i.e. $gf\in U$.

We have

$$\begin{aligned}
(gf)\zeta_C\,P(h^{\mathrm{op}}) = (g)(\xi_f)_C\,P(h^{\mathrm{op}}) &= (g)\,\mathrm{F}_{T_f}(h^{\mathrm{op}})\,(\xi_f)_D = (hg)(\xi_f)_D = (hgf)\zeta_D \\
&= (gf)\,\mathrm{F}_U(h^{\mathrm{op}})\,\zeta_D\,.
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{F}_U(C) \arrow[r, "\zeta_C"] \arrow[d, "\mathrm{F}_U(h^{\mathrm{op}})"'] & P(C) \arrow[d, "P(h^{\mathrm{op}})"] \\
\mathrm{F}_U(D) \arrow[r, "\zeta_D"] & P(D)
\end{tikzcd}$$

Suppose given $\big(B\xrightarrow{f}A\big)\in S$.

We verify that $[\zeta,U]\,P^+(f^{\mathrm{op}}) = (f)\xi_B$, cf. remark 2.2.6.

We have $T_f\subseteq f^*(U)$ since $g\in T_f$ implies $gf\in U$, i.e. $g\in f^*(U)$. Moreover, $\iota_{T_f}^{f^*(U)}\,\zeta^f = \xi_f$ since for $\big(C\xrightarrow{g}B\big)\in T_f$, we have

$$(g)(\zeta^f)_C = (gf)\zeta_C = (g)(\xi_f)_C\,.$$

We conclude that

$$[\zeta,U]\,P^+(f^{\mathrm{op}}) = [\zeta^f, f^*(U)] = [\xi_f, T_f] = (f)\xi_B\,.$$

Ad (b).

It suffices to show that ${}_R\Upsilon\circ P^+ = ({}_R\Upsilon\circ P)^+$ is an R-sheaf, cf. definition 2.3.8 and remark 2.2.10. The result now follows from (a) since ${}_R\Upsilon\circ P$ is separated by loc. cit. $\square$

Definition 2.3.13.

(a) Recall that $\hat{\mathscr{S}}$ is the category of presheaves on $\mathscr{S}$, $\tilde{\mathscr{S}}$ is the category of sheaves on $\mathscr{S}$ and $\mathrm{E} = \mathrm{E}_{\mathscr{S}} \colon \tilde{\mathscr{S}} \to \hat{\mathscr{S}}$ is the inclusion functor. Cf. definitions 2.2.2.(a) and 2.2.8.(a).

Propositions 2.3.11.(a) and 2.3.12.(a) show that we may define the *sheafification functor* $(-)^{\sim}$ as follows.

Let

$$(-)^{\sim} := \big((-)^{+} \circ (-)^{+}\big)\big|^{\tilde{\mathscr{S}}} : \hat{\mathscr{S}} \to \tilde{\mathscr{S}}.$$

So $\mathrm{E} \circ (-)^{\sim} = (-)^{++}$.

Suppose given $P \xrightarrow{\alpha} Q$ in $\hat{\mathscr{S}}$.

We write $\big(\tilde{P} \xrightarrow{\tilde{\alpha}} \tilde{Q}\big) := (-)^{\sim}\big(P \xrightarrow{\alpha} Q\big) = \big(P^{++} \xrightarrow{\alpha^{++}} Q^{++}\big)$.

(b) Recall that ${}_R\hat{\mathscr{S}}$ is the category of R-presheaves on $\mathscr{S}$, ${}_R\tilde{\mathscr{S}}$ is the category of R-sheaves on $\mathscr{S}$ and $\mathrm{E} = {}_R\mathrm{E}_{\mathscr{S}} \colon {}_R\tilde{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ is the inclusion functor. Cf. definitions 2.2.2.(b) and 2.2.8.(b).

Propositions 2.3.11.(b) and 2.3.12.(b) show that we may define the *sheafification functor* $(-)^{\sim}$ as follows.

Let

$$(-)^{\sim} := \big((-)^{+} \circ (-)^{+}\big)\big|^{{}_R\tilde{\mathscr{S}}} : {}_R\hat{\mathscr{S}} \to {}_R\tilde{\mathscr{S}}.$$

So $\mathrm{E} \circ (-)^{\sim} = (-)^{++}$.

We write $\big(\tilde{P} \xrightarrow{\tilde{\alpha}} \tilde{Q}\big) := (-)^{\sim}\big(P \xrightarrow{\alpha} Q\big) = \big(P^{++} \xrightarrow{\alpha^{++}} Q^{++}\big)$.

Note that ${}_R\Upsilon_{\mathscr{S}}$ maps R-sheaves to sheaves by remark 2.2.10. Thus we may restrict ${}_R\Upsilon_{\mathscr{S}}|^{\tilde{\mathscr{S}}}_{{}_R\tilde{\mathscr{S}}} \colon {}_R\tilde{\mathscr{S}} \to \tilde{\mathscr{S}}$.

We have ${}_R\Upsilon_{\mathscr{S}}|^{\tilde{\mathscr{S}}}_{{}_R\tilde{\mathscr{S}}} \circ (-)^{\sim} = (-)^{\sim} \circ {}_R\Upsilon_{\mathscr{S}}$, cf. definition 2.3.10.(b).

$$\begin{tikzcd}
{}_R\tilde{\mathscr{S}} \arrow[r, "\mathrm{E}"] \arrow[d, "{}_R\Upsilon_{\mathscr{S}}|^{\tilde{\mathscr{S}}}_{{}_R\tilde{\mathscr{S}}}"'] & {}_R\hat{\mathscr{S}} \arrow[r, "(-)^{\sim}"] \arrow[d, "{}_R\Upsilon_{\mathscr{S}}"'] & {}_R\tilde{\mathscr{S}} \arrow[d, "{}_R\Upsilon_{\mathscr{S}}|^{\tilde{\mathscr{S}}}_{{}_R\tilde{\mathscr{S}}}"] \\
\tilde{\mathscr{S}} \arrow[r, "\mathrm{E}"] & \hat{\mathscr{S}} \arrow[r, "(-)^{\sim}"] & \tilde{\mathscr{S}}
\end{tikzcd}$$

2.3.2 Adjunction

2.3.2.1 Unit

Definition 2.3.14. Suppose given a presheaf $P \in \mathrm{Ob}\,\hat{\mathscr{S}}$, an object $A \in \mathrm{Ob}\,\mathscr{S}$ and an element $x \in P(A)$.

Let $\beta^x := (x)\gamma_{P,A}^{-1} \colon \mathrm{F}_{\max(A)} \Rightarrow P$, cf. lemma 1.2.2.

For an object $B \in \operatorname{Ob} \mathscr{S}$, we have $\beta^x_B \colon \operatorname{Hom}_{\mathscr{S}}(B, A) \to P(B) \colon f \mapsto (f)\beta^x_B = (x)\, P(f^{\mathrm{op}})$.

Note that x is an amalgamation of β^x, cf. remark 2.2.6. We will use this fact in lemma 2.3.27.

Remark 2.3.15. Suppose given an R-presheaf $P \in \operatorname{Ob}({}_R\hat{\mathscr{S}})$, an object $A \in \operatorname{Ob} \mathscr{S}$ and an element $x \in P(A)$. Since $x \in ({}_R\Upsilon \circ P)(A)$, we have $\beta^x \colon \mathrm{F}_{\max(A)} \Rightarrow {}_R\Upsilon \circ P$.

For $x, x' \in P(A)$ and $r, r' \in R$, we have $\beta^x \cdot r + \beta^{x'} \cdot r' = \beta^{xr + x'r'}$.

Proof. For $B \xrightarrow{f} A$ in $\mathscr{S}$, we have

$$\begin{aligned}(f)\,(\beta^x \cdot r + \beta^{x'} \cdot r')_B &= (f)\,\beta^x_B \cdot r + (f)\,\beta^{x'} \cdot r' = (x)\,P(f^{\mathrm{op}}) \cdot r + (x')\,P(f^{\mathrm{op}}) \cdot r' \\ &= (xr + x'r')\,P(f^{\mathrm{op}}) = (f)\,\beta^{xr+x'r'}_B .\end{aligned}$$ □

Lemma/Definition 2.3.16.

(a) Suppose given a presheaf $P \in \operatorname{Ob} \hat{\mathscr{S}}$.

For an object $A \in \operatorname{Ob} \mathscr{S}$, let $(\mathrm{e}_P)_A \colon P(A) \to P^+(A) \colon x \mapsto (x)\,(\mathrm{e}_P)_A := [\beta^x, \max(A)]$, cf. remark 2.3.3.(a).

This yields a morphism of presheaves $\mathrm{e}_P := \big((\mathrm{e}_P)_A\big)_{A \in \operatorname{Ob} \mathscr{S}} \colon P \Rightarrow P^+$.

(b) Suppose given a R-presheaf $P \in \operatorname{Ob}({}_R\hat{\mathscr{S}})$.

For an object $A \in \operatorname{Ob} \mathscr{S}$, let $(\mathrm{e}_P)_A \colon P(A) \to P^+(A) \colon x \mapsto (x)\,(\mathrm{e}_P)_A := [\beta^x, \max(A)]$, cf. remark 2.3.3.(b).

This yields a morphism of R-presheaves $\mathrm{e}_P := \big((\mathrm{e}_P)_A\big)_{A \in \operatorname{Ob} \mathscr{S}} \colon P \Rightarrow P^+$.

We do not distinguish between the morphism e_P of presheaves and the morphism e_P of R-presheaves in notation.

Proof. Ad (a). Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$.

We want to show that $(\mathrm{e}_P)_A\, P^+(f^{\mathrm{op}}) = P(f^{\mathrm{op}})\,(\mathrm{e}_P)_B$.

Note that $f^*(\max(A)) = \max(B)$.

For $x \in P(A)$, we have

$$(x)\,(\mathrm{e}_P)_A\, P^+(f^{\mathrm{op}}) = [\beta^x, \max(A)]\, P^+(f^{\mathrm{op}}) = \big[(\beta^x)^f, f^*(\max(A))\big] = \big[(\beta^x)^f, \max(B)\big]$$

and

$$(x)\, P(f^{\mathrm{op}})\,(\mathrm{e}_P)_B = \big[\beta^{(x)P(f^{\mathrm{op}})}, \max(B)\big].$$

So it suffices to show that $(\beta^x)^f = \beta^{(x)P(f^{\mathrm{op}})} \colon \mathrm{F}_{\max(B)} \Rightarrow P$.

Suppose given $C \xrightarrow{g} B$ in $\mathscr{S}$.

We have

$$(g)\left((\beta^x)^f\right)_C = (gf)\beta^x_C = (x)\,P((gf)^{\mathrm{op}}) = (x)\,P(f^{\mathrm{op}})\,P(g^{\mathrm{op}}) = (g)\left(\beta^{(x)P(f^{\mathrm{op}})}\right)_C.$$

We conclude that e_P is natural.

$$\begin{tikzcd} P(A) \arrow[r, "(\mathrm{e}_P)_A"] \arrow[d, "P(f^{\mathrm{op}})"'] & P^+(A) \arrow[d, "P^+(f^{\mathrm{op}})"] \\ P(B) \arrow[r, "(\mathrm{e}_P)_B"] & P^+(B) \end{tikzcd}$$

Ad (b). We show that $(\mathrm{e}_P)_A$ is R-linear.

Suppose given $x, x' \in P(A)$ and $r, r' \in R$.

We have

$$(x)\,(\mathrm{e}_P)_A \cdot r + (x')\,(\mathrm{e}_P)_A \cdot r' = [\beta^x, \max(A)] \cdot r + [\beta^{x'}, \max(A)] \cdot r' = [\beta^x \cdot r + \beta^{x'} \cdot r', \max(A)]$$

and

$$(xr + x'r')\,(\mathrm{e}_P)_A = [\beta^{xr+x'r'}, \max(A)]$$

By remark 2.3.15, we have $\beta^x \cdot r + \beta^{x'} \cdot r' = \beta^{xr+x'r'}$.

Thus $[\beta^x \cdot r + \beta^{x'} \cdot r', \max(A)] = [\beta^{xr+x'r'}, \max(A)]$.

The same calculation as in (a) now shows that e_P is natural. □

Lemma/Definition 2.3.17.

(a) The tuple $\mathrm{e} := (\mathrm{e}_P)_{P \in \mathrm{Ob}\,\hat{\mathscr{S}}}$ is a transformation $\mathrm{e}\colon 1_{\hat{\mathscr{S}}} \Rightarrow (-)^+$.

(b) The tuple $\mathrm{e} := (\mathrm{e}_P)_{P \in \mathrm{Ob}(_R\hat{\mathscr{S}})}$ is a transformation $\mathrm{e}\colon 1_{_R\hat{\mathscr{S}}} \Rightarrow (-)^+$.

We do not distinguish between $\mathrm{e}\colon 1_{\hat{\mathscr{S}}} \Rightarrow (-)^+$ and $\mathrm{e}\colon 1_{_R\hat{\mathscr{S}}} \Rightarrow (-)^+$ in notation.

Proof. Ad (a). Suppose given $P \xrightarrow{\alpha} Q$ in $\hat{\mathscr{S}}$.

For $A \in \mathrm{Ob}\,\mathscr{S}$ and $x \in P(A)$, we have

$$(x)\,(\mathrm{e}_P\,\alpha^+)_A = (x)\,(\mathrm{e}_P)_A\,\alpha^+_A = [\beta^x, \max(A)]\,\alpha^+_A = [\beta^x\alpha, \max(A)]$$

and

$$(x)\,(\alpha\,\mathrm{e}_Q)_A = (x)\,\alpha_A\,(\mathrm{e}_Q)_A = \left[\beta^{(x)\alpha_A}, \max(A)\right].$$

So it suffices to show that $\beta^x \alpha = \beta^{(x)\alpha_A} \colon \mathrm{F}_{\max(A)} \Rightarrow Q$.

For $B \xrightarrow{f} A$ in $\mathscr{S}$, we have

$$(f)\,(\beta^x \alpha)_B = (f)\,\beta^x_B\,\alpha_B = (x)\,P(f^{\mathrm{op}})\,\alpha_B = (x)\,\alpha_A\,Q(f^{\mathrm{op}}) = (f)\,\beta^{(x)\alpha_A}_B .$$

$$\begin{tikzcd} P \arrow[r, "\mathrm{e}_P"] \arrow[d, "\alpha"'] & P^+ \arrow[d, "\alpha^+"] \\ Q \arrow[r, "\mathrm{e}_Q"] & Q^+ \end{tikzcd}$$

Ad (b). The same calculation as in (a) shows that e is natural. □

Remark 2.3.18.

(a) For a presheaf $P \in \mathrm{Ob}\,\hat{\mathscr{S}}$, we have $(\mathrm{e}_P)^+ = \mathrm{e}_{P^+} \colon P^+ \Rightarrow P^{++}$.

(b) For an R-presheaf $P \in \mathrm{Ob}({}_R\hat{\mathscr{S}})$, we have $(\mathrm{e}_P)^+ = \mathrm{e}_{P^+} \colon P^+ \Rightarrow P^{++}$.
Moreover, e_{P^+} is a monomorphism in ${}_R\hat{\mathscr{S}}$.

Proof. Ad (a). Suppose given $A \in \mathrm{Ob}\,\mathscr{S}$ and $[\xi, S] \in P^+(A)$.

We have $[\xi, S]\,(\mathrm{e}_P)^+_A = [\xi\,\mathrm{e}_P, S]$ and $[\xi, S]\,(\mathrm{e}_{P^+})_A = [\beta^{[\xi,S]}, \max(A)]$.

So it suffices to show that $\xi\,\mathrm{e}_P = \iota_S^{\max(A)} \beta^{[\xi,S]} \colon \mathrm{F}_S \Rightarrow P^+$.

For $\big(B \xrightarrow{f} A\big) \in S$, we have

$$(f)\,(\xi\,\mathrm{e}_P)_B = (f)\,\xi_B\,(\mathrm{e}_P)_B = [\beta^{(f)\xi_B}, \max(B)]$$

and

$$(f)\,\big(\iota_S^{\max(A)} \beta^{[\xi,S]}\big)_B = (f)\,\big(\beta^{[\xi,S]}\big)_B = [\xi, S]\,P^+(f^{\mathrm{op}}) = [\xi^f, f^*(S)].$$

So it suffices to show that $\iota_{f^*(S)}^{\max(B)} \beta^{(f)\xi_B} = \xi^f \colon \mathrm{F}_{f^*(S)} \Rightarrow P$.

For $\big(C \xrightarrow{g} B\big) \in f^*(S)$, we have

$$(g)\,(\iota_{f^*(S)}^{\max(B)} \beta^{(f)\xi_B})_C = (g)\,(\beta^{(f)\xi_B})_C = (f)\,\xi_B\,P(g^{\mathrm{op}}) = (f)\,\mathrm{F}_S(g^{\mathrm{op}})\,\xi_C = (gf)\,\xi_C = (g)\,(\xi^f)_C\,.$$

Ad (b). The same calculation as in (a) shows that $(\mathrm{e}_P)^+ = \mathrm{e}_{P^+}$.

To show that e_{P^+} is a monomorphism in ${}_R\hat{\mathscr{S}}$, it suffices to show that $(\mathrm{e}_{P^+})_A \colon P^+(A) \to P^{++}(A)$ is a monomorphism in Mod-R for $A \in \mathrm{Ob}\,\mathscr{S}$ by remark 1.1.9.(b).

Suppose given $A \in \mathrm{Ob}\,\mathscr{S}$ and $[\xi, S] \in P^+(A)$ with $[\xi, S]\,(\mathrm{e}_{P^+})_A = 0$. We have to show that $[\xi, S] = 0$.

Since $[\beta^{[\xi,S]}, \max(A)] = [\xi, S]\,(\mathrm{e}_{P^+})_A = 0$, we may choose $T \in J_A$ with $\iota_T\, \beta^{[\xi,S]} = 0$.

So for $\big(B \xrightarrow{f} A\big) \in T$, we have $[\xi^f, f^*(S)] = [\xi, S]\, P^+(f^{\mathrm{op}}) = (f)\beta_B^{[\xi,S]} = 0$. Therefore we may choose $U_f \in J_B$ with $U_f \subseteq f^*(S)$ such that $\iota_{U_f}^{f^*(S)}\, \xi^f = 0$.

By lemma 2.2.16, we have $V := \{gf \colon f \in T, g \in U_f\} \in J_A$. Moreover $V \subseteq S$ since $U_f \subseteq f^*(S)$ for $f \in T$.

For $\big(B \xrightarrow{f} A\big) \in T$ and $\big(C \xrightarrow{g} B\big) \in U_f$, we have $(gf)\xi_C = (g)\xi_B^f = 0$. We conclude that $\iota_V^S\, \xi = 0$ and therefore $[\xi, S] = 0$. So $(\mathrm{e}_{P^+})_A$ is a monomorphism in Mod-R. □

Corollary 2.3.19. Suppose given an R-presheaf $P \colon \mathscr{S}^{\mathrm{op}} \to \text{Mod-}R$.

We have $P^+ \cong 0$ in ${}_R\hat{\mathcal{A}}$ if and only if $\tilde{P} \cong 0$ in ${}_R\hat{\mathcal{A}}$.

Proof. Suppose that $P^+ \cong 0$ in ${}_R\hat{\mathcal{A}}$. Then $\tilde{P} = P^{++} \cong 0$ in ${}_R\hat{\mathcal{A}}$ since $(-)^+ \colon {}_R\hat{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ is additive, cf. definition 2.3.10.(b).

Conversely, suppose that $\tilde{P} \cong 0$ in ${}_R\hat{\mathcal{A}}$. Since $P^+ \xrightarrow{\mathrm{e}_{P^+}} \tilde{P}$ is a monomorphism in ${}_R\hat{\mathscr{S}}$ by remark 2.3.18.(b), we conclude that $P^+ \cong 0$ in ${}_R\hat{\mathcal{A}}$. □

Remark 2.3.20.

(a) By convention 24, we have

$$\mathrm{e} \star \mathrm{e} = (\mathrm{e} \star 1_{\hat{\mathscr{S}}}) \cdot ((-)^+ \star \mathrm{e}) = \mathrm{e} \cdot ((-)^+ \star \mathrm{e}) \colon 1_{\hat{\mathscr{S}}} \Rightarrow (-)^+ \circ (-)^+$$

and

$$\mathrm{e} \star \mathrm{e} = (1_{\hat{\mathscr{S}}} \star \mathrm{e}) \cdot (\mathrm{e} \star (-)^+) = \mathrm{e} \cdot (\mathrm{e} \star (-)^+) \colon 1_{\hat{\mathscr{S}}} \Rightarrow (-)^+ \circ (-)^+,$$

so $\mathrm{e} \cdot ((-)^+ \star \mathrm{e}) = \mathrm{e} \cdot (\mathrm{e} \star (-)^+)$.

Remark 2.3.18.(a) shows that even $(-)^+ \star \mathrm{e} = \mathrm{e} \star (-)^+$ since $((-)^+ \star \mathrm{e})_P = (\mathrm{e}_P)^+$ and $(\mathrm{e} \star (-)^+) = \mathrm{e}_{P^+}$ for $P \in \mathrm{Ob}\, \hat{\mathscr{S}}$.

(b) By convention 24, we have

$$\mathrm{e} \star \mathrm{e} = (\mathrm{e} \star 1_{{}_R\hat{\mathscr{S}}}) \cdot ((-)^+ \star \mathrm{e}) = \mathrm{e} \cdot ((-)^+ \star \mathrm{e}) \colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$$

and

$$\mathrm{e} \star \mathrm{e} = (1_{{}_R\hat{\mathscr{S}}} \star \mathrm{e}) \cdot (\mathrm{e} \star (-)^+) = \mathrm{e} \cdot (\mathrm{e} \star (-)^+) \colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim,$$

so $\mathrm{e} \cdot ((-)^+ \star \mathrm{e}) = \mathrm{e} \cdot (\mathrm{e} \star (-)^+)$.

Remark 2.3.18.(b) shows that even $(-)^+ \star \mathrm{e} = \mathrm{e} \star (-)^+$ since $((-)^+ \star \mathrm{e})_P = (\mathrm{e}_P)^+$ and $(\mathrm{e} \star (-)^+)_P = \mathrm{e}_{P^+}$ for $P \in \mathrm{Ob}({}_R\hat{\mathscr{S}})$.

Definition 2.3.21.

(a) Recall that $(-)^{++} = (-)^+ \circ (-)^+ = \mathrm{E} \circ (-)^\sim \colon \hat{\mathscr{S}} \to \hat{\mathscr{S}}$, cf. definition 2.3.13.(a).

Let $\varepsilon := \mathrm{e} \star \mathrm{e} \colon 1_{\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$.

For a presheaf $P \in \mathrm{Ob}\,\hat{\mathscr{S}}$, we have $\varepsilon_P = \mathrm{e}_P \cdot \mathrm{e}_{P^+} = \mathrm{e}_P \cdot (\mathrm{e}_P)^+ \colon P \Rightarrow P^{++}$.

Cf. convention 24 and remark 2.3.20.(a).

(b) Recall that $(-)^{++} = (-)^+ \circ (-)^+ = \mathrm{E} \circ (-)^\sim \colon {}_R\hat{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$, cf. definition 2.3.13.(b).

Let $\varepsilon := \mathrm{e} \star \mathrm{e} \colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$.

For an R-presheaf $P \in \mathrm{Ob}({}_R\hat{\mathscr{S}})$, we have $\varepsilon_P = \mathrm{e}_P \cdot \mathrm{e}_{P^+} = \mathrm{e}_P \cdot (\mathrm{e}_P)^+ \colon P \Rightarrow P^{++}$.

Cf. convention 24 and remark 2.3.20.(b).

We do not distinguish between $\varepsilon \colon 1_{\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$ and $\varepsilon \colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$ in notation.

2.3.2.2 Counit

Lemma/Definition 2.3.22.

(a) Suppose given a sheaf $P \in \mathrm{Ob}\,\tilde{\mathscr{S}}$ and an object $A \in \mathrm{Ob}\,\mathscr{S}$.

For a covering $S \in J_A$ and a matching family $\xi \colon \mathrm{F}_S \Rightarrow P$, let $\mathrm{x}_\xi \in P(A)$ denote the unique amalgamation of ξ, i.e. $(\mathrm{x}_\xi)\, P(f^{\mathrm{op}}) = (f)\xi_B$ for all $B \xrightarrow{f} A$ in S.

Let $(\mathrm{n}_P)_A \colon P^+(A) \to P(A) \colon [\xi, S] \mapsto [\xi, S]\, (\mathrm{n}_P)_A := \mathrm{x}_\xi$.

This yields a morphism of presheaves $\mathrm{n}_P := \big((\mathrm{n}_P)_A\big)_{A \in \mathrm{Ob}\,\mathscr{S}} \colon P^+ \Rightarrow P$.

(b) Suppose given an R-sheaf $P \in \mathrm{Ob}({}_R\tilde{\mathscr{S}})$ and an object $A \in \mathrm{Ob}\,\mathscr{S}$.

For a covering $S \in J_A$ and a matching family $\xi \colon \mathrm{F}_S \Rightarrow {}_R\Upsilon \circ P$, let $\mathrm{x}_\xi \in P(A)$ denote the unique amalgamation of ξ, i.e. $(\mathrm{x}_\xi)\, P(f^{\mathrm{op}}) = (f)\xi_B$ for all $B \xrightarrow{f} A$ in S.

Let $(\mathrm{n}_P)_A \colon P^+(A) \to P(A) \colon [\xi, S] \mapsto [\xi, S]\, (\mathrm{n}_P)_A := \mathrm{x}_\xi$.

This yields a morphism of R-presheaves $\mathrm{n}_P := \big((\mathrm{n}_P)_A\big)_{A \in \mathrm{Ob}\,\mathscr{S}} \colon P^+ \Rightarrow P$.

We do not distinguish between the morphism n_P of presheaves and the morphism n_P of R-presheaves in notation.

Proof. Ad (a). We show that $(\mathrm{n}_P)_A$ is well-defined.

Suppose given $[\xi, S] = [\xi', S'] \in P^+(A)$. We may choose $T \subseteq S \cap S'$ such that $\iota_T^S\, \xi = \iota_T^{S'} \xi'$.

The element x_ξ is an amalgamation of $\iota_T^S\, \xi$ since we have $(\mathrm{x}_\xi)\, P(f^{\mathrm{op}}) = (f)\xi_B = (f)(\iota_T^S\, \xi)_B$ for $\big(B \xrightarrow{f} A\big) \in T \subseteq S$.

Similarly, the element $\mathrm{x}_{\xi'}$ is an amalgamation of $\iota_T^{S'}\xi'$ since we have
$(\mathrm{x}_{\xi'})\,P(f^{\mathrm{op}}) = (f)\xi'_B = (f)(\iota_T^{S'}\xi')_B$ for $\big(B \xrightarrow{f} A\big) \in T \subseteq S'$.
So x_ξ and $\mathrm{x}_{\xi'}$ are amalgamations of $\iota_T^S\,\xi = \iota_T^{S'}\xi'$. We conclude that $\mathrm{x}_\xi = \mathrm{x}_{\xi'}$ since P is a sheaf.

We show that n_P is natural.

Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$.

For $[\xi, S] \in P^+(A)$, we have

$$[\xi, S]\,(\mathrm{n}_P)_A\,P(f^{\mathrm{op}}) = (\mathrm{x}_\xi)\,P(f^{\mathrm{op}})$$

and

$$[\xi, S]\,P^+(f^{\mathrm{op}})\,(\mathrm{n}_P)_B = [\xi^f, f^*(S)]\,(\mathrm{n}_P)_B = \mathrm{x}_{\xi^f}\ .$$

So it suffices to show that $(\mathrm{x}_\xi)\,P(f^{\mathrm{op}})$ is an amalgamation of ξ^f.

For $\big(C \xrightarrow{g} B\big) \in f^*(S)$, we have $gf \in S$ and thus

$$(\mathrm{x}_\xi)\,P(f^{\mathrm{op}})\,P(g^{\mathrm{op}}) = (\mathrm{x}_\xi)\,P((gf)^{\mathrm{op}}) = (gf)\xi_C = (g)\xi^f_C\,.$$

We conclude that $(\mathrm{n}_P)_A\,P(f^{\mathrm{op}}) = P^+(f^{\mathrm{op}})\,(\mathrm{n}_P)_B$.

$$\begin{tikzcd}
P^+(A) \arrow[r, "(\mathrm{n}_P)_A"] \arrow[d, "P^+(f^{\mathrm{op}})"'] & P(A) \arrow[d, "P(f^{\mathrm{op}})"] \\
P^+(B) \arrow[r, "(\mathrm{n}_P)_B"] & P(B)
\end{tikzcd}$$

Ad (b). The same calculation as in (a) shows that $(\mathrm{n}_P)_A$ is well-defined.

We show that $(\mathrm{n}_P)_A$ is R-linear.

Suppose given coverings $S, S' \in J_A$, matching families $\xi\colon\ \mathrm{F}_S \Rightarrow {}_R\Upsilon \circ P$, $\ \xi'\colon\ \mathrm{F}_{S'} \Rightarrow {}_R\Upsilon \circ P$ and $r, r' \in R$. Write $T := S \cap S'$.

We have

$$\big([\xi, S]\cdot r + [\xi', S']\cdot r'\big)\,(\mathrm{n}_P)_A = [\iota_T^S\,\xi\cdot r + \iota_T^{S'}\xi'\cdot r', T]\,(\mathrm{n}_P)_A = \mathrm{x}_{\iota_T^S\,\xi\cdot r + \iota_T^{S'}\,\xi'\cdot r'}$$

and

$$[\xi, S]\,(\mathrm{n}_P)_A\cdot r + [\xi', S']\,(\mathrm{n}_P)_A\cdot r' = \mathrm{x}_\xi \cdot r + \mathrm{x}_{\xi'} \cdot r'.$$

So it suffices to show that $\mathrm{x}_\xi \cdot r + \mathrm{x}_{\xi'} \cdot r' \in P(A)$ is an amalgamation of
$\iota_T^S\,\xi\cdot r + \iota_T^{S'}\xi'\cdot r'\colon\ \mathrm{F}_T \Rightarrow {}_R\Upsilon \circ P$.

For $\big(B \xrightarrow{f} A\big) \in T$, we have

$$\begin{aligned}(\mathrm{x}_\xi \cdot r + \mathrm{x}_{\xi'} \cdot r')\, P(f^{\mathrm{op}}) &= (\mathrm{x}_\xi)\, P(f^{\mathrm{op}}) \cdot r + (\mathrm{x}_{\xi'})\, P(f^{\mathrm{op}}) \cdot r' = (f)\xi_B \cdot r + (f)\xi'_B \cdot r' \\ &= (f)\, (\iota^S_T\, \xi)_B \cdot r + (f)\, (\iota^{S'}_T \xi')_B \cdot r' = (f)\, (\iota^S_T\, \xi \cdot r + \iota^{S'}_T \xi' \cdot r')_B\, .\end{aligned}$$

The same calculation as in (a) shows that n_P is natural. □

Lemma/Definition 2.3.23.

(a) The tuple $\mathrm{n} := (\mathrm{n}_P)_{P \in \mathrm{Ob}\, \tilde{\mathscr{S}}}$ is a transformation $\mathrm{n}\colon (-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \Rightarrow 1_{\tilde{\mathscr{S}}}$.

(b) The tuple $\mathrm{n} := (\mathrm{n}_P)_{P \in \mathrm{Ob}({}_R\tilde{\mathscr{S}})}$ is a transformation $\mathrm{n}\colon (-)^+|^{{}_R\tilde{\mathscr{S}}}_{{}_R\tilde{\mathscr{S}}} \Rightarrow 1_{{}_R\tilde{\mathscr{S}}}$.

We do not distinguish between $\mathrm{n}\colon (-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \Rightarrow 1_{\tilde{\mathscr{S}}}$ and $\mathrm{n}\colon (-)^+|^{{}_R\tilde{\mathscr{S}}}_{{}_R\tilde{\mathscr{S}}} \Rightarrow 1_{{}_R\tilde{\mathscr{S}}}$ in notation.

Proof. Ad (a). By proposition 2.3.12.(a), $(-)^+$ maps $\tilde{\mathscr{S}}$ to $\tilde{\mathscr{S}}$.

Suppose given $P \xrightarrow{\alpha} Q$ in $\tilde{\mathscr{S}}$.

For $A \in \mathrm{Ob}\, \mathscr{S}$ and $[\xi, S] \in P^+(A)$, we have

$$[\xi, S]\, (\mathrm{n}_P\, \alpha)_A = [\xi, S]\, (\mathrm{n}_P)_A\, \alpha_A = (\mathrm{x}_\xi)\, \alpha_A$$

and

$$[\xi, S]\, (\alpha^+\, \mathrm{n}_Q)_A = [\xi, S]\, \alpha^+_A\, (n_Q)_A = [\xi\alpha, S]\, (n_Q)_A = \mathrm{x}_{\xi\alpha}\, .$$

So it suffices to show that $(\mathrm{x}_\xi)\, \alpha_A$ is an amalgamation of $\xi\alpha\colon \mathrm{F}_S \Rightarrow Q$.

For $\big(B \xrightarrow{f} A\big) \in S$, we have

$$(\mathrm{x}_\xi)\, \alpha_A\, Q(f^{\mathrm{op}}) = (\mathrm{x}_\xi)\, P(f^{\mathrm{op}})\, \alpha_B = (f)\, \xi_B\, \alpha_B = (f)\, (\xi\alpha)_B\, .$$

$$\begin{tikzcd} P^+ \arrow[r, "\mathrm{n}_P"] \arrow[d, "\alpha^+"'] & P \arrow[d, "\alpha"] \\ Q^+ \arrow[r, "\mathrm{n}_Q"] & Q \end{tikzcd}$$

Ad (b). By proposition 2.3.12.(b), $(-)^+$ maps ${}_R\tilde{\mathscr{S}}$ to ${}_R\tilde{\mathscr{S}}$. The same calculation as in (a) shows that n is natural. □

Remark 2.3.24.

(a) For a sheaf $P \in \mathrm{Ob}\, \tilde{\mathscr{S}}$, we have $(\mathrm{n}_P)^+ = \mathrm{n}_{P^+}\colon P^{++} \Rightarrow P^+$.

(b) For an R-sheaf $P \in \mathrm{Ob}({}_R\tilde{\mathscr{S}})$, we have $(\mathrm{n}_P)^+ = \mathrm{n}_{P^+}\colon P^{++} \Rightarrow P^+$.

Proof. Ad (a). Suppose given $A \in \operatorname{Ob}\mathscr{S}$ and $[\xi, S] \in P^{++}(A)$, where $\xi\colon \mathrm{F}_S \Rightarrow P^+$.

We have $[\xi, S]\,(\mathrm{n}_P)^+_A = [\xi\,\mathrm{n}_P, S]$ and $[\xi, S]\,(\mathrm{n}_{P^+})_A = \mathrm{x}_\xi$. Write $[\zeta, T] := \mathrm{x}_\xi \in P^+(A)$, where $\zeta\colon \mathrm{F}_T \Rightarrow P$.

So it suffices to show that $[\xi\,\mathrm{n}_P, S] = [\zeta, T]$ in $P^+(A)$.

Suppose given $\big(B \xrightarrow{f} A\big) \in S \cap T$. We have

$$\begin{aligned}(f)\,(\xi\,\mathrm{n}_P)_B &= (f)\,\xi_B\,(\mathrm{n}_P)_B = (\mathrm{x}_\xi)\,P^+(f^{\mathrm{op}})\,(\mathrm{n}_P)_B = [\zeta, T]\,P^+(f^{\mathrm{op}})\,(\mathrm{n}_P)_B = [\zeta^f, f^*(T)]\,(\mathrm{n}_P)_B \\ &= \mathrm{x}_{\zeta^f} \in P(B).\end{aligned}$$

So we have to show that $\mathrm{x}_{\zeta^f} = (f)\,\zeta_B$, i.e. that $(f)\,\zeta_B \in P(B)$ is an amalgamation of $\zeta^f\colon \mathrm{F}_{f^*(T)} \Rightarrow P$.

In fact, for $\big(C \xrightarrow{g} B\big) \in f^*(T)$, we have

$$(f)\,\zeta_B\,P(g^{\mathrm{op}}) = (f)\;\mathrm{F}_T(g^{\mathrm{op}})\,\zeta_C = (gf)\,\zeta_C = (g)\,\zeta^f_C\,.$$

Ad (b). The same calculation as in (a) shows that $(\mathrm{n}_P)^+ = \mathrm{n}_{P^+}$. $\square$

Remark 2.3.25.

(a) By convention 24, we have

$$\mathrm{n}\star\mathrm{n} = \Big((-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \star \mathrm{n}\Big)\cdot(\mathrm{n}\star 1_{\tilde{\mathscr{S}}}) = \Big((-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \star \mathrm{n}\Big)\cdot\mathrm{n}\colon (-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \circ (-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \Rightarrow 1_{\tilde{\mathscr{S}}}$$

and

$$\mathrm{n}\star\mathrm{n} = \Big(\mathrm{n}\star(-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}}\Big)\cdot(1_{\tilde{\mathscr{S}}} \star \mathrm{n}) = \Big(\mathrm{n}\star(-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}}\Big)\cdot\mathrm{n}\colon (-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \circ (-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \Rightarrow 1_{\tilde{\mathscr{S}}}\,,$$

so $\Big((-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}} \star \mathrm{n}\Big)\cdot\mathrm{n} = \Big(\mathrm{n}\star(-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}}\Big)\cdot\mathrm{n}$.

Remark 2.3.24.(a) shows that even $(-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}}\star\mathrm{n} = \mathrm{n}\star(-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}}$ since $\Big((-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}}\star\mathrm{n}\Big)_P = (\mathrm{n}_P)^+$ and $\Big(\mathrm{n}\star(-)^+|^{\tilde{\mathscr{S}}}_{\tilde{\mathscr{S}}}\Big)_P = \mathrm{n}_{P^+}$ for $P \in \operatorname{Ob}\tilde{\mathscr{S}}$.

(b) By convention 24, we have

$$\mathrm{n}\star\mathrm{n} = \Big((-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}} \star \mathrm{n}\Big)\cdot(\mathrm{n}\star 1_{_R\tilde{\mathscr{S}}}) = \Big((-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}} \star \mathrm{n}\Big)\cdot\mathrm{n}\colon (-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}} \circ (-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}} \Rightarrow 1_{_R\tilde{\mathscr{S}}}$$

and

$$\mathrm{n}\star\mathrm{n} = \Big(\mathrm{n}\star(-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}}\Big)\cdot(1_{_R\tilde{\mathscr{S}}} \star \mathrm{n}) = \Big(\mathrm{n}\star(-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}}\Big)\cdot\mathrm{n}\colon (-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}} \circ (-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}} \Rightarrow 1_{_R\tilde{\mathscr{S}}}\,,$$

so $\Big((-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}} \star \mathrm{n}\Big)\cdot\mathrm{n} = \Big(\mathrm{n}\star(-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}}\Big)\cdot\mathrm{n}$.

Remark 2.3.24.(b) shows that even $(-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}}\star\mathrm{n} = \mathrm{n}\star(-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}}$ since $\Big((-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}}\star\mathrm{n}\Big)_P = (\mathrm{n}_P)^+$ and $\Big(\mathrm{n}\star(-)^+|^{_R\tilde{\mathscr{S}}}_{_R\tilde{\mathscr{S}}}\Big)_P = \mathrm{n}_{P^+}$ for $P \in \operatorname{Ob}({}_R\tilde{\mathscr{S}})$.

Definition 2.3.26.

(a) Recall that $(-)^+|_{\mathscr{S}}^{\tilde{\mathscr{S}}} \circ (-)^+|_{\tilde{\mathscr{S}}}^{\mathscr{S}} = (-)^\sim \circ \mathrm{E} \colon \tilde{\mathscr{S}} \to \tilde{\mathscr{S}}$, cf. definition 2.3.13.(a).
Let $\eta := \mathrm{n} \star \mathrm{n} \colon (-)^\sim \circ \mathrm{E} \Rightarrow 1_{\tilde{\mathscr{S}}}$.
For a sheaf $P \in \operatorname{Ob} \tilde{\mathscr{S}}$, we have $\eta_P = \mathrm{n}_{P^+} \cdot \mathrm{n}_P = (\mathrm{n}_P)^+ \cdot \mathrm{n}_P \colon P^{++} \Rightarrow P$.
Cf. convention 24 and remark 2.3.25.(a).

(b) Recall that $(-)^+|_{R\tilde{\mathscr{S}}}^{R\tilde{\mathscr{S}}} \circ (-)^+|_{R\tilde{\mathscr{S}}}^{R\tilde{\mathscr{S}}} = (-)^\sim \circ \mathrm{E} \colon {}_R\tilde{\mathscr{S}} \to {}_R\tilde{\mathscr{S}}$, cf. definition 2.3.13.(b).
Let $\eta := \mathrm{n} \star \mathrm{n} \colon (-)^\sim \circ \mathrm{E} \Rightarrow 1_{{}_R\tilde{\mathscr{S}}}$.
For an R-sheaf $P \in \operatorname{Ob}({}_R\tilde{\mathscr{S}})$, we have $\eta_P = \mathrm{n}_{P^+} \cdot \mathrm{n}_P = (\mathrm{n}_P)^+ \cdot \mathrm{n}_P \colon P^{++} \Rightarrow P$.
Cf. convention 24 and remark 2.3.25.(b).

We do not destinguish between $\eta \colon (-)^\sim \circ \mathrm{E} \Rightarrow 1_{\tilde{\mathscr{S}}}$ and $\eta \colon (-)^\sim \circ \mathrm{E} \Rightarrow 1_{{}_R\tilde{\mathscr{S}}}$ in notation.

2.3.2.3 The sheafification theorem

Lemma 2.3.27.

(a) Suppose given a sheaf $P \in \operatorname{Ob} \tilde{\mathscr{S}}$. We have $\mathrm{e}_P \cdot \mathrm{n}_P = 1_P \colon P \Rightarrow P$.

(b) Suppose given an R-sheaf $P \in \operatorname{Ob}({}_R\tilde{\mathscr{S}})$. We have $\mathrm{e}_P \cdot \mathrm{n}_P = 1_P \colon P \Rightarrow P$.

Proof. Ad (a). Suppose given $A \in \operatorname{Ob} \mathscr{S}$ and $x \in P(A)$.

We have

$$(x)(\mathrm{e}_P\ \mathrm{n}_P)_A = (x)(\mathrm{e}_P)_A\,(\mathrm{n}_P)_A = [\beta^x, \max(A)](\mathrm{n}_P)_A = x = (x)(1_P)_A$$

since $(x)\,P(f^{\mathrm{op}}) = (f)\beta^x_B$ for $B \xrightarrow{f} A$ in $\mathscr{S}$ shows that x is an amalgamation of β^x.

Ad (b). The same calculation as in (a) shows that $\mathrm{e}_P \cdot \mathrm{n}_P = 1_P$. □

Lemma 2.3.28.

(a) Suppose given a sheaf $P \in \operatorname{Ob} \tilde{\mathscr{S}}$. We have $\mathrm{n}_P \cdot \mathrm{e}_P = 1_{P^+} \colon P^+ \Rightarrow P^+$.

(b) Suppose given an R-sheaf $P \in \operatorname{Ob}({}_R\tilde{\mathscr{S}})$. We have $\mathrm{n}_P \cdot \mathrm{e}_P = 1_{P^+} \colon P^+ \Rightarrow P^+$.

Proof. Ad (a). Suppose given $A \in \operatorname{Ob} \mathscr{S}$ and $[\xi, S] \in P^+(A)$.

Note that $(f)(\beta^{\mathrm{x}_\xi})_B = (\mathrm{x}_\xi)\,P(f^{\mathrm{op}}) = (f)\xi_B$ for $\big(B \xrightarrow{f} A\big) \in S$. So $\iota_S^{\max(A)} \beta^{\mathrm{x}_\xi} = \xi$ and therefore $[\beta^{\mathrm{x}_\xi}, \max(A)] = [\xi, S]$.

We obtain

$$[\xi, S]\,(\mathrm{n}_P)_A\,(\mathrm{e}_P)_A = (\mathrm{x}_\xi)(\mathrm{e}_P)_A = [\beta^{\mathrm{x}_\xi}, \max(A)] = [\xi, S]\ = [\xi, S]\,(1_{P^+})_A\,.$$

Ad (b). The same calculation as in (a) shows that $\mathrm{n}_P \cdot \mathrm{e}_P = 1_{P^+}$. □

Corollary 2.3.29.

(a) Suppose given a presheaf $P \in \operatorname{Ob} \hat{\mathscr{S}}$. The presheaf P is a sheaf if and only if e_P is an isomorphism in $\hat{\mathscr{S}}$.

(b) Suppose given an R-presheaf $P \in \operatorname{Ob}({}_R\hat{\mathscr{S}})$. The R-presheaf P is an R-sheaf if and only if e_P is an isomorphism in ${}_R\hat{\mathscr{S}}$.

Proof. Ad (a). If P is a sheaf, then e_P is an isomorphism with inverse n_P by lemmata 2.3.27.(a) and 2.3.28.(a). If e_P is an isomorphism, then $(\mathrm{e}_P)^+$ is also an isomorphism since $(-)^+ \colon \hat{\mathscr{S}} \to \hat{\mathscr{S}}$ is a functor. Consequently, P is a sheaf since it is isomorphic to the sheaf P^{++} via $\mathrm{e}_P \cdot (\mathrm{e}_P)^+$. Cf. lemma 2.2.11.(a).

Ad (b). The same argument as in (a) shows that P is a R-sheaf if and only if e_P is an isomorphism. $\square$

Theorem 2.3.30.

(a) Recall that $\mathscr{S}$ is a site, $\hat{\mathscr{S}}$ is the category of presheaves on $\mathscr{S}$ and $\tilde{\mathscr{S}}$ is the category of sheaves on $\mathscr{S}$. Recall that $\mathrm{E} \colon \tilde{\mathscr{S}} \to \hat{\mathscr{S}}$ is the full and faithful inclusion functor and $(-)^\sim \colon \hat{\mathscr{S}} \to \tilde{\mathscr{S}}$ is the sheafification functor. Cf. definitions 2.2.1, 2.2.2.(a), 2.2.8.(a) and 2.3.13.(a).

The diagram

$$\begin{tikzcd} \hat{\mathscr{S}} \arrow[r, bend left, "(-)^\sim"] \arrow[r, phantom, "\perp"] & \tilde{\mathscr{S}} \arrow[l, bend left, "\mathrm{E}"] \end{tikzcd}$$

with unit $\varepsilon \colon 1_{\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$ and counit $\eta \colon (-)^\sim \circ \mathrm{E} \Rightarrow 1_{\tilde{\mathscr{S}}}$ is an adjunction.

Cf. convention 43 and definitions 2.3.21.(a) and 2.3.26.(a).

So $(-)^\sim \colon \hat{\mathscr{S}} \to \tilde{\mathscr{S}}$ is left-adjoint to $\mathrm{E} \colon \tilde{\mathscr{S}} \to \hat{\mathscr{S}}$.

(b) Recall that R is a ring, $\mathscr{S}$ is a site, ${}_R\hat{\mathscr{S}}$ is the category of R-presheaves on $\mathscr{S}$ and ${}_R\tilde{\mathscr{S}}$ is the category of R-sheaves on $\mathscr{S}$. Recall that $\mathrm{E} \colon {}_R\tilde{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ is the full and faithful inclusion functor and $(-)^\sim \colon {}_R\hat{\mathscr{S}} \to {}_R\tilde{\mathscr{S}}$ is the sheafification functor. Cf. definitions 2.2.1, 2.2.2.(b), 2.2.8.(b) and 2.3.13.(b).

The diagram

$$\begin{tikzcd} {}_R\hat{\mathscr{S}} \arrow[r, bend left, "(-)^\sim"] \arrow[r, phantom, "\perp"] & {}_R\tilde{\mathscr{S}} \arrow[l, bend left, "\mathrm{E}"] \end{tikzcd}$$

with unit $\varepsilon \colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$ and counit $\eta \colon (-)^\sim \circ \mathrm{E} \Rightarrow 1_{{}_R\tilde{\mathscr{S}}}$ is an adjunction.

Cf. convention 43 and definitions 2.3.21.(b) and 2.3.26.(b).

So $(-)^\sim \colon {}_R\hat{\mathscr{S}} \to {}_R\tilde{\mathscr{S}}$ is left-adjoint to $\mathrm{E} \colon {}_R\tilde{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$.

Proof. Ad (a). For $P \in \operatorname{Ob} \hat{\mathscr{S}}$, we have

$$\begin{aligned}((-)^{\sim} \star \varepsilon)_P \cdot (\eta \star (-)^{\sim})_P = (\varepsilon_P)^{++} \cdot \eta_{P^{++}} &= (\mathrm{e}_P \cdot \mathrm{e}_{P^+})^{++} \cdot \mathrm{n}_{P^{+++}} \cdot \mathrm{n}_{P^{++}} \\ &= (\mathrm{e}_P)^{++} \cdot (\mathrm{e}_{P^+})^{++} \cdot \mathrm{n}_{P^{+++}} \cdot \mathrm{n}_{P^{++}} \overset{\text{R2.3.18.}(a)}{=} \mathrm{e}_{P^{++}} \cdot \mathrm{e}_{P^{+++}} \cdot \mathrm{n}_{P^{+++}} \cdot \mathrm{n}_{P^{++}} \\ &\overset{\text{L2.3.27.}(a)}{=} \mathrm{e}_{P^{++}} \cdot \mathrm{n}_{P^{++}} \overset{\text{L2.3.27.}(a)}{=} 1_{P^{++}} ,\end{aligned}$$

so $((-)^{\sim} \star \varepsilon) \cdot (\eta \star (-)^{\sim}) = 1_{(-)^{\sim}}$.

$$\begin{tikzcd}
(-)^{\sim} \arrow[r, Rightarrow, "(-)^{\sim}\star\varepsilon"] \arrow[dr, Rightarrow, "1_{(-)^{\sim}}"'] & (-)^{\sim}\circ \mathrm{E} \circ (-)^{\sim} \arrow[d, Rightarrow, "\eta\star(-)^{\sim}"] \\
& (-)^{\sim}
\end{tikzcd}$$

For $P \in \operatorname{Ob} \tilde{\mathscr{S}}$, we have

$$(\varepsilon \star E)_P \cdot (E \star \eta)_P = \varepsilon_P \cdot \eta_P = \mathrm{e}_P \cdot \mathrm{e}_{P^+} \cdot \mathrm{n}_{P^+} \cdot \mathrm{n}_P \overset{\text{L2.3.27.}(a)}{=} \mathrm{e}_P \cdot \mathrm{n}_P \overset{\text{L2.3.27.}(a)}{=} 1_P ,$$

so $(\varepsilon \star E) \cdot (E \star \eta) = 1_{\mathrm{E}}$.

$$\begin{tikzcd}
\mathrm{E} \arrow[r, Rightarrow, "\varepsilon\star\mathrm{E}"] \arrow[dr, Rightarrow, "1_{\mathrm{E}}"'] & \mathrm{E} \circ (-)^{\sim} \circ \mathrm{E} \arrow[d, Rightarrow, "\mathrm{E}\star\eta"] \\
& \mathrm{E}
\end{tikzcd}$$

So $(-)^{\sim} \dashv E$.

Ad (b). The same calculation as in (a) shows that $(-)^{\sim} \dashv \mathrm{E}$. $\square$

2.3.3 The category of R-sheaves is abelian

Lemma 2.3.31. Suppose given a monomorphism $P \xrightarrow{\alpha} Q$ in ${}_R\hat{\mathscr{S}}$. Then ${}_R\Upsilon_{\mathscr{S}}(\alpha) \colon {}_R\Upsilon \circ P \to {}_R\Upsilon \circ Q$ is monomorphic in $\hat{\mathscr{S}}$ as well.

Proof. For $A \in \operatorname{Ob} \mathcal{A}$, the morphism α_A is a monomorphism in Mod-R by remark 1.1.9.(b) and therefore injective. So ${}_R\Upsilon_{\mathscr{S}}(\alpha)_A = {}_R\Upsilon(\alpha_A)$ is injective as well. Thus ${}_R\Upsilon_{\mathscr{S}}(\alpha)$ is a monomorphism in $\hat{\mathscr{S}}$. $\square$

Lemma 2.3.32. Suppose given a monomorphism $P \xrightarrow{\alpha} Q$ in ${}_R\hat{\mathscr{S}}$. Then $P^+ \xrightarrow{\alpha^+} Q^+$ is monomorphic in ${}_R\hat{\mathscr{S}}$ as well.

Proof. Suppose given $N \xrightarrow{\beta} P^+$ in ${}_R\hat{\mathscr{S}}$ with $\beta\alpha^+ = 0$.

We want to show that $\beta = 0$.

Suppose given $A \in \operatorname{Ob} \mathscr{S}$ and $x \in N(A)$. We have to prove that $(x)\beta_A = 0$ in $P^+(A)$.

Write $[\xi, S] := (x)\beta_A \in P^+(A)$.

We have $[\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha), S] = [\xi, S]\,\alpha_A^+ = (x)\beta_A\,\alpha_A^+ = 0$. So there exists $T \in J_A$ with $T \subseteq S$ such that $\iota_T^S\,\xi \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha) = 0$. Since ${}_R\Upsilon_{\mathscr{S}}(\alpha)$ is a monomorphism by lemma 2.3.31, we conclude that $\iota_T^S\,\xi = 0$ and thus $[\xi, S] = 0$. □

Lemma 2.3.33. Suppose given $K \xrightarrow{k} P \xrightarrow{\alpha} Q$ in ${}_R\hat{\mathscr{S}}$ such that k is a kernel of α. Then ${}_R\Upsilon_{\mathscr{S}}(k)$ has the following factorisation property in $\hat{\mathscr{S}}$.

Suppose given $T \xrightarrow{t} {}_R\Upsilon \circ P$ in $\hat{\mathscr{S}}$ with $(x)t_A \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha)_A = 0$ for $A \in \operatorname{Ob} \mathcal{A}$ and $x \in T(A)$. Then there exists a unique morphism $T \xrightarrow{u} {}_R\Upsilon \circ K$ in $\hat{\mathscr{S}}$ such that $u \cdot {}_R\Upsilon_{\mathscr{S}}(k) = t$.

$$\begin{tikzcd}
{}_R\Upsilon \circ K \arrow[r, "{}_R\Upsilon_{\mathscr{S}}(k)"] & {}_R\Upsilon \circ P \arrow[r, "{}_R\Upsilon_{\mathscr{S}}(\alpha)"] & {}_R\Upsilon \circ Q \\
T \arrow[u, "u"] \arrow[ur, "t"'] & &
\end{tikzcd}$$

Proof. For $A \in \operatorname{Ob} \mathcal{A}$, let $L(A) := \{x \in P(A) \colon (x)\alpha_A = 0\}$ be the usual kernel of α_A and let $L(A) \xrightarrow{l_A} P(A)$ be the inclusion morphism. For $B \xrightarrow{f} A$ in $\mathscr{S}$, let $L(f)$ be the induced morphism between the kernels $L(A)$ and $L(B)$. This yields the functor $L \colon \mathscr{S}^{\mathrm{op}} \to \text{Mod-}R$ and the transformation $l = (l_A)_{A \in \operatorname{Ob} \mathscr{S}} \colon L \to P$ as in remark 1.1.9.(b). Moreover, l is a kernel of α.

It suffices to show the factorisation property for ${}_R\Upsilon_{\mathscr{S}}(l)$. Note that ${}_R\Upsilon_{\mathscr{S}}(l)$ is monomorphic by lemma 2.3.31.

For $A \in \operatorname{Ob} \mathcal{A}$ and $x \in T(A)$, we have $(x)t_A\,\alpha_A = (x)t_A \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha)_A = 0$ and therefore $(x)t_A \in L(A)$. Thus we may restrict $u_A := t_A|^{L(A)}$. So $u_A\,l_A = t_A$ by construction. It remains to show that $u := (u_A)_{A \in \operatorname{Ob} \mathscr{S}}$ is natural. For $B \xrightarrow{f} A$ in $\mathscr{S}$, we have

$$u_A\,L(f^{\mathrm{op}})\,l_B = u_A\,l_A\,P(f^{\mathrm{op}}) = t_A\,P(f^{\mathrm{op}}) = T(f^{\mathrm{op}})\,t_B = T(f^{\mathrm{op}})\,u_B\,l_B\,.$$

So $u_A\,L(f^{\mathrm{op}}) = T(f^{\mathrm{op}})\,u_B$ since l_B is monomorphic.

$$\begin{tikzcd}
T(A) \arrow[r, "u_A"] \arrow[d, "T(f^{\mathrm{op}})"'] & L(A) \arrow[r, "l_A"] \arrow[d, "L(f^{\mathrm{op}})"] & P(A) \arrow[d, "P(f^{\mathrm{op}})"] \\
T(B) \arrow[r, "u_B"] & L(B) \arrow[r, "l_B"] & P(B)
\end{tikzcd}$$

□

Lemma 2.3.34. Suppose given $K \xrightarrow{k} P \xrightarrow{\alpha} Q$ in ${}_R\hat{\mathscr{S}}$ such that k is a kernel of α. Then k^+ is a kernel of α^+ in ${}_R\hat{\mathscr{S}}$.

Proof. The morphism k^+ is a monomorphism by lemma 2.3.32.

Suppose given $T \xrightarrow{t} P^+$ in ${}_R\hat{\mathscr{S}}$ such that $t\alpha^+ = 0$.

$$\begin{tikzcd} K^+ \arrow[r, "k^+"] & P^+ \arrow[r, "\alpha^+"] & Q^+ \\ T \arrow[ur, "t"'] \end{tikzcd}$$

We have to find $T \xrightarrow{u} K^+$ in ${}_R\hat{\mathscr{S}}$ such that $uk^+ = t$.

Suppose given $A \in \operatorname{Ob}\mathscr{S}$. We want to define $u_A \colon T(A) \to K^+(A)$ as follows.

Suppose given $x \in T(A)$. Write $[\xi_x, S_x] := (x)t_A \in P^+(A)$, where $S_x \in J_A$ and $\xi_x \colon \mathrm{F}_{S_x} \Rightarrow {}_R\Upsilon \circ P$.

We have $[\xi_x \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha), S_x] = [\xi_x, S_x]\,\alpha_A^+ = (x)t_A\,\alpha_A^+ = 0$. So we may choose $U_x \subseteq S_x$ such that $U_x \in J_A$ and $\iota_{U_x}^{S_x}\,\xi_x \cdot {}_R\Upsilon_{\mathscr{S}}(\alpha) = 0$.

By lemma 2.3.33, there exists $\mathrm{F}_{U_x} \xrightarrow{\zeta_x} {}_R\Upsilon \circ K$ in ${}_R\hat{\mathscr{S}}$ such that $\zeta_x\,k = \iota_{U_x}^{S_x}\,\xi_x$.

$$\begin{tikzcd} {}_R\Upsilon \circ K \arrow[r, "{}_R\Upsilon_{\mathscr{S}}(k)"] & {}_R\Upsilon \circ P \arrow[r, "{}_R\Upsilon_{\mathscr{S}}(\alpha)"] & {}_R\Upsilon \circ Q \\ \mathrm{F}_{U_x} \arrow[u, "\zeta_x"] \arrow[r, "\iota_{U_x}^{S_x}"'] & \mathrm{F}_{S_x} \arrow[u, "\xi_x"'] \end{tikzcd}$$

Let $(x)u_A := [\zeta_x, U_x] \in K^+(A)$.

We have to show that $u := (u_A)_{A \in \operatorname{Ob}\mathscr{S}}$ is natural.

Suppose given $B \xrightarrow{f} A$ in $\mathscr{S}$ and $x \in T(A)$.

Since t is natural, we have, in $P^+(B)$,

$$[\xi_x^f, f^*(S_x)] = [\xi_x, S_x]\,P^+(f^{\mathrm{op}}) = (x)t_A\,P^+(f^{\mathrm{op}}) = (x)\,T(f^{\mathrm{op}})\,t_B = [\xi_{(x)T(f^{\mathrm{op}})}, S_{(x)T(f^{\mathrm{op}})}].$$

So we may choose $W_x^f \subseteq f^*(S_x) \cap S_{(x)T(f^{\mathrm{op}})}$ such that $W_x^f \in J_B$ and $\iota_{W_x^f}^{f^*(S_x)}\,\xi_x^f = \iota_{W_x^f}^{S_{(x)T(f^{\mathrm{op}})}}\,\xi_{(x)T(f^{\mathrm{op}})}$.

$$\begin{tikzcd} T(A) \arrow[r, "t_A"] \arrow[d, "T(f^{\mathrm{op}})"'] & P^+(A) \arrow[d, "P^+(f^{\mathrm{op}})"] \\ T(B) \arrow[r, "t_B"] & P^+(B) \end{tikzcd}$$

We have

$$(x)u_A\,K^+(f^{\mathrm{op}}) = [\zeta_x, U_x]\,K^+(f^{\mathrm{op}}) = [\zeta_x^f, f^*(U_x)]$$

and

$$(x)\,T(f^{\mathrm{op}})\,u_B = [\zeta_{(x)T(f^{\mathrm{op}})}, U_{(x)T(f^{\mathrm{op}})}].$$

Let $V := W_x^f \cap U_{(x)T(f^{\mathrm{op}})} \cap f^*(U_x)$. So $V \in J_B$, cf. lemma 2.2.14.

It suffices to show that $\iota_V^{f^*(U_x)}\, \zeta_x^f \cdot {}_R\Upsilon_{\mathscr{S}}(k) = \iota_V^{U_{(x)T(f^{\mathrm{op}})}}\, \zeta_{(x)T(f^{\mathrm{op}})} \cdot {}_R\Upsilon_{\mathscr{S}}(k)$ since ${}_R\Upsilon_{\mathscr{S}}(k)$ is a monomorphism by lemma 2.3.31.

For $\big(C \xrightarrow{g} B\big) \in V$, we have

$$\begin{aligned}
(g)\,(\iota_V^{f^*(U_x)}\, \zeta_x^f \cdot {}_R\Upsilon_{\mathscr{S}}(k))_C &= (g)\,(\zeta_x^f \cdot {}_R\Upsilon_{\mathscr{S}}(k))_C = (gf)\,(\zeta_x \cdot {}_R\Upsilon_{\mathscr{S}}(k))_C = (gf)\,(\iota_{U_x}^{S_x}\, \xi_x)_C \\
&= (gf)\,(\xi_x)_C = (g)\,(\xi_x^f)_C = (g)\,(\iota_V^{W_x^f}\, \iota_{W_x^f}^{f^*(S_x)}\, \xi_x^f)_C \\
&= (g)\,(\iota_V^{W_x^f}\, \iota_{W_x^f}^{S_{(x)T(f^{\mathrm{op}})}}\, \xi_{(x)T(f^{\mathrm{op}})})_C \\
&= (g)\,(\iota_V^{U_{(x)T(f^{\mathrm{op}})}}\, \iota_{U_{(x)T(f^{\mathrm{op}})}}^{S_{(x)T(f^{\mathrm{op}})}}\, \xi_{(x)T(f^{\mathrm{op}})})_C \\
&= (g)\,(\iota_V^{U_{(x)T(f^{\mathrm{op}})}}\, \zeta_{(x)T(f^{\mathrm{op}})} \cdot {}_R\Upsilon_{\mathscr{S}}(k))_C\,.
\end{aligned}$$

$$\begin{tikzcd}
T(A) \arrow[r, "u_A"] \arrow[d, "T(f^{\mathrm{op}})"'] & K^+(A) \arrow[d, "K^+(f^{\mathrm{op}})"] \\
T(B) \arrow[r, "u_B"] & K^+(B)
\end{tikzcd}$$

We show that $uk^+ = t$.

For $A \in \mathrm{Ob}\,\mathscr{S}$ and $x \in T(A)$, we have

$$(x)u_A\, k_A^+ = [\zeta_x, U_x]\, k_A^+ = [\zeta_x \cdot {}_R\Upsilon_{\mathscr{S}}(k), U_x] = [\iota_{U_x}^{S_x}\, \xi_x, U_x] = [\xi_x, S_x] = (x)t_A\,. \qquad \square$$

Theorem 2.3.35. Recall that R is a ring, $\mathscr{S}$ is a site and ${}_R\tilde{\mathscr{S}}$ is the category of R-sheaves on $\mathscr{S}$. Cf. definitions 2.2.1 and 2.2.8.(b).

The category of R-sheaves ${}_R\tilde{\mathscr{S}}$ is abelian.

Recall that ${}_R\hat{\mathscr{S}}$ is the category of R-presheaves on $\mathscr{S}$, $\mathrm{E}\colon {}_R\tilde{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ is the inclusion functor and $(-)^{\sim}\colon {}_R\hat{\mathscr{S}} \to {}_R\tilde{\mathscr{S}}$ is the sheafification functor. Cf. definitions 2.2.2.(b) and 2.3.13.(b).

The inclusion functor E is left-exact and the sheafification functor $(-)^{\sim}$ is exact.

We give formulas for kernels and cokernels in ${}_R\tilde{\mathscr{S}}$ in remark 2.3.36.

Proof. We use lemma 1.3.2.(c) to prove the assertions. The diagram

$$\begin{tikzcd}
{}_R\hat{\mathscr{S}} \arrow[r, bend left, "(-)^{\sim}"] \arrow[r, phantom, "\perp"] & {}_R\tilde{\mathscr{S}} \arrow[l, bend left, "\mathrm{E}"]
\end{tikzcd}$$

with unit $\varepsilon\colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^{\sim}$ and counit $\eta\colon (-)^{\sim} \circ \mathrm{E} \Rightarrow 1_{{}_R\tilde{\mathscr{S}}}$ is an adjunction, cf. theorem 2.3.30.(b).

The category ${}_R\hat{\mathscr{S}} = (\text{Mod-}R)^{(\mathscr{S}^{\text{op}})}$ is abelian since it is a functor category with values in an abelian category, cf. remark 1.1.9.(c).

The category ${}_R\tilde{\mathscr{S}}$ is additive and the inclusion functor $\mathrm{E}\colon {}_R\tilde{\mathscr{S}} \to {}_R\hat{\mathscr{S}}$ is full and faithful, cf. proposition 2.2.13 and definition 2.3.13.(b).

It remains to show that $(-)^\sim$ preserves kernels.

Suppose given $f \in \operatorname{Mor}({}_R\hat{\mathscr{S}})$ and a kernel $k \in \operatorname{Mor}({}_R\hat{\mathscr{S}})$ of f in ${}_R\hat{\mathscr{S}}$. By applying lemma 2.3.34 twice, we conclude that k^{++} is a kernel of f^{++} in ${}_R\hat{\mathscr{S}}$. By lemma 1.1.8, $k^{++} = \tilde{k}$ is also a kernel of $f^{++} = \tilde{f}$ in ${}_R\tilde{\mathscr{S}}$. $\square$

Remark 2.3.36. Recall that the diagram

$$\begin{tikzcd} {}_R\hat{\mathscr{S}} \arrow[r, bend left, "(-)^\sim"] \arrow[r, phantom, "\perp"] & {}_R\tilde{\mathscr{S}} \arrow[l, bend left, "\mathrm{E}"] \end{tikzcd}$$

with unit $\varepsilon\colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow \mathrm{E} \circ (-)^\sim$ and counit $\eta\colon (-)^\sim \circ \mathrm{E} \Rightarrow 1_{{}_R\tilde{\mathscr{S}}}$ is an adjunction, cf. theorem 2.3.30.(b).

Suppose given $P \xrightarrow{f} Q$ in ${}_R\tilde{\mathscr{S}}$. Suppose given a kernel $K \xrightarrow{k} P$ of f in ${}_R\hat{\mathscr{S}}$ and a cokernel $Q \xrightarrow{c} C$ of f in ${}_R\hat{\mathscr{S}}$.

A kernel of f in ${}_R\tilde{\mathscr{S}}$ is given by k.

A cokernel of f in ${}_R\tilde{\mathscr{S}}$ is given by $\eta_Q^{-1} \cdot \tilde{c} = \varepsilon_Q \cdot \tilde{c} = c \cdot \varepsilon_C = c \cdot \mathrm{e}_C \cdot \mathrm{e}_{C^+}$.

Moreover, the diagram

$$\begin{tikzcd} K \arrow[r, "k"] & P \arrow[r, "f"] & Q \arrow[r, "c"] \arrow[d, "\mathrm{e}_Q"'] & C \arrow[d, "\mathrm{e}_C"] \\ & & Q^+ \arrow[r, "c^+"] \arrow[d, "\mathrm{e}_{Q^+}"'] & C^+ \arrow[d, "\mathrm{e}_{C^+}"] \\ & & \tilde{Q} \arrow[r, "\tilde{c}"] & \tilde{C} \end{tikzcd}$$

commutes.

Proof. We show that a kernel of f in ${}_R\tilde{\mathscr{S}}$ is given by k. Choose a kernel $L \xrightarrow{l} P$ of f in ${}_R\tilde{\mathscr{S}}$. Then l is a kernel of f in ${}_R\hat{\mathscr{S}}$ as well since E is left-exact, cf. theorem 2.3.35. So $L \cong K$ in ${}_R\hat{\mathscr{S}}$. Since ${}_R\tilde{\mathscr{S}}$ is closed under isomorphisms by proposition 2.2.13, we conclude that K is a sheaf. Thus k is a kernel of f in ${}_R\tilde{\mathscr{S}}$ as well.

A cokernel of f in ${}_R\tilde{\mathscr{S}}$ is given by $\eta_Q^{-1} \cdot \tilde{c} = \varepsilon_Q \cdot \tilde{c}$ since we have $\eta_Q^{-1} = \mathrm{E}(\eta_Q^{-1}) = \varepsilon_{\mathrm{E}(Q)} = \varepsilon_Q$, cf. lemma 1.3.2.(c). The formulas now follow from loc. cit. and from the fact that $\mathrm{e}\colon 1_{{}_R\hat{\mathscr{S}}} \Rightarrow (-)^+$ is natural, cf. definition 2.3.17.(b). $\square$

2.4 Exact categories as sites

Suppose given a ring R.

Suppose given an exact category $\mathcal{A} = (\mathcal{A}, \mathcal{E})$. Cf. definition 1.5.1.

We will endow $\mathcal{A}$ with a Grothendieck topology $\mathrm{J}^{\mathcal{E}}$ in definition 2.4.1 below. We will often write $\mathcal{A} = (\mathcal{A}, \mathrm{J}^{\mathcal{E}})$ for the resulting site.

So $\mathcal{A}$ can be viewed as an exact category or as a site, depending on the context.

2.4.1 Coverings

Lemma/Definition 2.4.1. For $A \in \operatorname{Ob}\mathcal{A}$, let $\mathrm{J}^{\mathcal{E}}_A := \{S \in \operatorname{Sieves}(A) \colon \text{there exists } p \in \mathcal{E}_{\mathrm{e}} \cap S\}$. The tuple $\mathrm{J}^{\mathcal{E}} := \big(\mathrm{J}^{\mathcal{E}}_A\big)_{A \in \operatorname{Ob}\mathcal{A}}$ is a Grothendieck topology on $\mathcal{A}$.

Proof. For $A \in \operatorname{Ob}\mathcal{A}$, we have $1_A \in \mathcal{E}_{\mathrm{e}} \cap \max(A)$ by (E3). So $\max(A) \in \mathrm{J}^{\mathcal{E}}_A$ and therefore (G1) holds.

Suppose given $B \xrightarrow{f} A$ in $\mathcal{A}$ and $S \in \mathrm{J}^{\mathcal{E}}_A$. Choose $\big(P \overset{p}{\twoheadrightarrow} A\big) \in \mathcal{E}_{\mathrm{e}} \cap S$. By (E7), there exists a pullback

$$\begin{tikzcd} P' \arrow[r, "p'", two heads] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "f"] \\ P \arrow[r, "p", two heads] & A \end{tikzcd}$$

in $\mathcal{A}$ with $p' \in \mathcal{E}_{\mathrm{e}}$. We have $p' \in f^*(S)$ since $p'f = f'p \in S$. So $f^*(S) \in \mathrm{J}^{\mathcal{E}}_B$ and therefore (G2) holds.

Suppose given $A \in \operatorname{Ob}\mathcal{A}$ and $S \in \mathrm{J}^{\mathcal{E}}_A$. Suppose given a sieve T on A such that $f^*(T) \in J^{\mathcal{E}}_B$ for $\big(B \xrightarrow{f} A\big) \in S$. Choose $\big(B \overset{p}{\twoheadrightarrow} A\big) \in \mathcal{E}_{\mathrm{e}} \cap S$. Since $p^*(T) \in \mathrm{J}^{\mathcal{E}}_B$, we may choose $q \in \mathcal{E}_{\mathrm{e}} \cap p^*(T)$. So $qp \in T$ and $qp \in \mathcal{E}_{\mathrm{e}}$ by (E5). We conclude that $T \in \mathrm{J}^{\mathcal{E}}_A$ and therefore (G3) holds. $\square$

For the remainder of this section 2.4, we will work with the site $\mathcal{A} = (\mathcal{A}, \mathrm{J}^{\mathcal{E}})$.

So for $A \in \operatorname{Ob}\mathcal{A}$, a covering of A is a sieve on A that contains a pure epimorphism.

Remark 2.4.2. Suppose given a pure epimorphism $P \overset{p}{\twoheadrightarrow} A$ in $\mathcal{A}$. The sieve generated by p covers A, i.e.

$$\mathrm{S}_p = \big\{g \in \operatorname{Mor}\mathcal{A} \colon \text{there exists } f \in \operatorname{Mor}\mathcal{A} \text{ such that } g = fp\big\} \in \mathrm{J}^{\mathcal{E}}_A \, .$$

Cf. definition 2.1.6.

Lemma 2.4.3. Suppose given an R-presheaf $F \in \mathrm{Ob}({}_R\hat{\mathcal{A}})$. Suppose given an object $A \in \mathrm{Ob}\,\mathcal{A}$ and $[\xi, S] \in F^+(A)$.

There exists a pure epimorphism $\big(P \xrightarrow{p} A\big) \in S$ with $(p)\xi_P = 0$ if and only if $[\xi, S] = 0$.

Proof. Suppose given a pure epimorphism $\big(P \xrightarrow{p} A\big) \in S$ with $(p)\xi_P = 0$.

Since $\mathrm{S}_p \subseteq S$, we have $[\xi, S] = [\iota^S_{\mathrm{S}_p}\xi, \mathrm{S}_p]$. For $B \xrightarrow{f} P$ in $\mathcal{A}$, we have

$$(fp)\xi_B = (p)\ \mathrm{F}_S(f^{\mathrm{op}})\,\xi_B = (p)\xi_P\,F(f^{\mathrm{op}}) = (0)\,F(f^{\mathrm{op}}) = 0.$$

So $\iota^S_{\mathrm{S}_p}\xi = 0$ and thus $[\iota^S_{\mathrm{S}_p}\xi, \mathrm{S}_p] = 0$.

Conversely, suppose that $[\xi, S] = 0 = [0, S]$. So there exists a covering $T \subseteq S$ of A with $\iota^S_T\,\xi = 0$. Choosing a pure epimorphism $\big(P \xrightarrow{p} A\big) \in T \subseteq S$, we obtain $(p)\xi_P = 0$. □

2.4.2 Sheaves are quasi-left-exact functors

Definition 2.4.4. An R-presheaf $F\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}R$ is called *quasi-left-exact* if and only if for each pullback of the form

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & P \arrow[d, "p"] \\ P \arrow[r, "p"] & A. \end{tikzcd}$$

in $\mathcal{A}$, where p is a pure epimorphism, the sequence

$$F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F(g'^{\,\mathrm{op}}) - F(g^{\mathrm{op}})} F(Q)$$

is left-exact in Mod-R.

We will see in lemma 2.4.7 that the quasi-left-exact R-presheaves are precisely the R-sheaves.

The term quasi-left-exact was chosen since in case F is additive it means that F is left-exact. Cf. lemma 2.4.8 and proposition 2.4.13 below.

Remark 2.4.5. Thomason and Trobaugh [15, p. 400] use the term "left exact" for what we call quasi-left-exact.

Remark 2.4.6. An R-presheaf $F\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}R$ is quasi-left-exact if and only if for each pure short exact sequence $B \xrightarrow{i} P \xrightarrow{p} A$ in $\mathcal{A}$, the sequence

$$F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F\left(\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right) - F\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right)} F(B \oplus P)$$

is left-exact in Mod-R.

Proof. Suppose given a pure short exact sequence $B \overset{i}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} A$ in $\mathcal{A}$.

We show that the diagram

$$\begin{tikzcd} B \oplus P \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)"'] & P \arrow[d, "p"] \\ P \arrow[r, "p"] & A \end{tikzcd}$$

is a pullback.

Suppose given $x, y \colon T \to P$ in $\mathcal{A}$ such that $xp = yp$. So $(y - x)p = 0$.

For $\left(\begin{smallmatrix} s & t \end{smallmatrix}\right) : T \to B \oplus P$ with $\left(\begin{smallmatrix} s & t \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = x$ and $\left(\begin{smallmatrix} s & t \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right) = y$, we necessarily have $t = x$ and $si = y - x$. Since i is monomorphic, s is uniquely determined by x and y.

It now suffices to show that there exists a morphism $\left(\begin{smallmatrix} s & t \end{smallmatrix}\right) : T \to B \oplus P$ with $\left(\begin{smallmatrix} s & t \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = x$ and $\left(\begin{smallmatrix} s & t \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right) = y$.

Since i is a kernel of p, there exists $w \colon T \to B$ with $wi = y - x$. The morphism $\left(\begin{smallmatrix} w & x \end{smallmatrix}\right) : T \to B \oplus P$ satisfies $\left(\begin{smallmatrix} w & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = x$ and $\left(\begin{smallmatrix} w & x \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right) = y - x + x = y$.

Now suppose given an arbitrary pullback

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & P \arrow[d, "p"] \\ P \arrow[r, "p"] & A. \end{tikzcd}$$

in $\mathcal{A}$.

Since the diagram

$$\begin{tikzcd} B \oplus P \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)"'] & P \arrow[d, "p"] \\ P \arrow[r, "p"] & A \end{tikzcd}$$

is a pullback as well, there is an isomorphism $\left(\begin{smallmatrix} u & v \end{smallmatrix}\right) : Q \to B \oplus P$ such that $\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = g$ and $\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right) = g'$.

The diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(p^{\mathrm{op}})"] & F(P) \arrow[rr, "F\left(\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right) - F\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right)"] \arrow[drr, "F(g'^{\mathrm{op}}) - F(g^{\mathrm{op}})"'] & & F(B \oplus P) \arrow[d, "F(\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)^{\mathrm{op}})"] \\ & & & F(Q) \end{tikzcd}$$

commutes since

$$\begin{aligned}\Big(F\big(\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\big) - F\big(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\big)\Big)\, F(\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)^{\mathrm{op}}) &= F\big(\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\big)\, F(\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)^{\mathrm{op}}) - F\big(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\big)\, F(\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)^{\mathrm{op}}) \\ &= F\Big(\big(\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)\big)^{\mathrm{op}}\Big) - F\Big(\big(\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\big)^{\mathrm{op}}\Big) \\ &= F({g'}^{\mathrm{op}}) - F(g^{\mathrm{op}}).\end{aligned}$$

Since $F(\left(\begin{smallmatrix} u & v \end{smallmatrix}\right)^{\mathrm{op}})$ is an isomorphism, the sequence

$$F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F\left(\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right) - F\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right)} F(B \oplus P)$$

is left-exact in Mod-R if and only if the sequence

$$F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F({g'}^{\mathrm{op}}) - F(g^{\mathrm{op}})} F(Q)$$

is left-exact in Mod-R. $\square$

Lemma 2.4.7. Suppose given an R-presheaf $F\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}R$.

The R-presheaf F is an R-sheaf if and only if F is quasi-left-exact.

Proof. Suppose that F is quasi-left-exact.

Suppose given $A \in \operatorname{Ob}\mathcal{A}$, $S \in J_A$ and $\xi\colon\ \mathrm{F}_S \Rightarrow {}_R\Upsilon \circ F$.

We have to show that ξ has a unique amalgamation $x \in F(A)$.

Choose a pure epimorphism $\big(P \overset{p}{\longrightarrow\!\!\!\!\!+\,} A\big) \in S$ and a pullback

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner"] & P \arrow[d, "p"] \\ P \arrow[r, "p"] & A. \end{tikzcd}$$

The sequence $F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F({g'}^{\mathrm{op}}) - F(g^{\mathrm{op}})} F(Q)$ is left-exact since F is quasi-left-exact.

We have

$$\begin{aligned}(p)\xi_P\,(F({g'}^{\mathrm{op}}) - F(g^{\mathrm{op}})) &= (p)(\mathrm{F}_S({g'}^{\mathrm{op}})\,\xi_Q - \mathrm{F}_S(g^{\mathrm{op}})\,\xi_Q) = (g'p)\xi_Q - (gp)\xi_Q \\ &= (gp)\xi_Q - (gp)\xi_Q = 0.\end{aligned}$$

So there exists precisely one $x \in F(A)$ with $(x)\,F(p^{\mathrm{op}}) = (p)\xi_P$. In particular, there exists at most one amalgamation of ξ.

It suffices to show that x is an amalgamation.

Suppose given $\big(B \xrightarrow{f} A\big) \in S$.

Choose a pullback

$$\begin{tikzcd} P' \arrow[r, "p'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "f"] \\ P \arrow[r, "p", "\shortmid" marking] & A. \end{tikzcd}$$

We have

$$\begin{aligned} (x)\, F(f^{\mathrm{op}})\, F(p'^{\,\mathrm{op}}) &= (x)\, F(p^{\mathrm{op}})\, F(f'^{\,\mathrm{op}}) = (p)\xi_P\, F(f'^{\,\mathrm{op}}) = (p)\ \mathrm{F}_S(f'^{\,\mathrm{op}})\, \xi_{P'} = (f'p)\xi_{P'} \\ &= (p'f)\xi_{P'} = (f)\ \mathrm{F}_S(p'^{\,\mathrm{op}})\, \xi_{P'} = (f)\xi_B\, F(p'^{\,\mathrm{op}}). \end{aligned}$$

The morphism $F(p'^{\,\mathrm{op}})$ is monomorphic since F is quasi-left-exact. Thus $(x)\, F(f^{\mathrm{op}}) = (f)\xi_B$.

Conversely, suppose that F is an R-sheaf.

Suppose given a pure epimorphism $P \xrightarrow{p} A$ and a pullback

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & P \arrow[d, "p", "\shortmid" marking] \\ P \arrow[r, "p", "\shortmid" marking] & A. \end{tikzcd}$$

We have to show that the sequence $F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F(g'^{\,\mathrm{op}}) - F(g^{\mathrm{op}})} F(Q)$ is left-exact.

Suppose given $y \in F(P)$ with $(y)(F(g'^{\,\mathrm{op}}) - F(g^{\mathrm{op}})) = 0$, i.e. $(y)\, F(g'^{\,\mathrm{op}}) = (y)\, F(g^{\mathrm{op}})$.

Recall that $\mathrm{S}_p \in \mathrm{J}_A^{\mathcal{E}}$, cf. remark 2.4.2.

We want to define $\xi\colon\ \mathrm{F}_{\mathrm{S}_p} \Rightarrow {}_R\Upsilon \circ F$.

For $B \in \operatorname{Ob} \mathcal{A}$, let $\xi_B\colon\ \mathrm{S}_p \cap \operatorname{Hom}_{\mathcal{A}}(B, A) \to F(B)\colon\ fp \mapsto (y)\, F(f^{\mathrm{op}})$.

This is well-defined since in case $fp = f'p$ for $f, f' \in \operatorname{Hom}_{\mathcal{A}}(B, P)$, there exists $u\colon B \to Q$ such that $f = ug$ and $f' = ug'$ by the pullback property and so

$$(y)\, F(f^{\mathrm{op}}) = (y)\, F(g^{\mathrm{op}})\, F(u^{\mathrm{op}}) = (y)\, F(g'^{\,\mathrm{op}})\, F(u^{\mathrm{op}}) = (y)\, F(f'^{\,\mathrm{op}}).$$

For $C \xrightarrow{g} B \xrightarrow{f} P$ in $\mathcal{A}$, we have

$$(fp)\xi_B\, F(g^{\mathrm{op}}) = (y)\, F(f^{\mathrm{op}})\, F(g^{\mathrm{op}}) = (y)\, F((gf)^{\mathrm{op}}) = (gfp)\xi_C = (fp)\ \mathrm{F}_{\mathrm{S}_p}(g^{\mathrm{op}})\, \xi_C\, .$$

Thus ξ is natural.

Since F is an R-sheaf, there exists a unique $x \in F(A)$ such that $(x)\, F(p^{\mathrm{op}}) = y$ since this equation implies $(x)\, F((fp)^{\mathrm{op}}) = (x)\, F(p^{\mathrm{op}})\, F(f^{\mathrm{op}}) = (y)\, F(f^{\mathrm{op}}) = (fp)\xi_B$ for $B \xrightarrow{f} P$ in $\mathcal{A}$.

We conclude that the sequence $F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F(g'^{\,\mathrm{op}}) - F(g^{\mathrm{op}})} F(Q)$ is left-exact. $\square$

Lemma 2.4.8. Suppose given an additive R-presheaf $F\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}R$, i.e. suppose F to be an additive functor. The following three statements are equivalent.

(a) The R-presheaf F is a sheaf.

(b) The R-presheaf F is quasi-left-exact.

(c) The R-presheaf F is left-exact.

Proof. The statements (a) and (b) are equivalent by lemma 2.4.7. We use remark 2.4.6 to show that (b) and (c) are equivalent.

Suppose given a pure short exact sequence $B \xrightarrow{i} P \xrightarrow{p} A$ in $\mathcal{A}$.

The diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(p^{\mathrm{op}})"] & F(P) \arrow[r, "F(i^{\mathrm{op}})"] \arrow[dr, "F\left(\left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)^{\mathrm{op}}\right)"'] & F(B) \arrow[d, "F\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^{\mathrm{op}}\right)"] \\
& & F(B \oplus P) \arrow[d, "F\left(\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)^{\mathrm{op}}\right)"] \\
& & F(B)
\end{tikzcd}$$

commutes.

Since F is additive, we have

$$F\left(\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right) - F\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right) = F\left(\left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)^{\mathrm{op}}\right).$$

The morphism $F\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^{\mathrm{op}}\right)$ is monomorphic since $F\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^{\mathrm{op}}\right) \cdot F\left(\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)^{\mathrm{op}}\right) = F\left(1^{\mathrm{op}}\right) = 1$.

Thus the sequence

$$F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F\left(\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right) - F\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^{\mathrm{op}}\right)} F(B \oplus P)$$

is left-exact in Mod-R if and only if the sequence

$$F(A) \xrightarrow{F(p^{\mathrm{op}})} F(P) \xrightarrow{F(i^{\mathrm{op}})} F(B)$$

is left-exact in Mod-R. □

Lemma 2.4.9. Suppose given an R-presheaf $F\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}R$. If F is an additive functor, then F^+ is additive as well.

Proof. At first, we want to show that $F^+(0_{\mathcal{A}}) \cong 0_{\text{Mod-}R}$ in Mod-R.
Suppose given $[\xi, S] \in F^+(0_{\mathcal{A}})$. Choose a pure epimorphism $\big(P \twoheadrightarrow 0_{\mathcal{A}}\big) \in S$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{F}_S(0_{\mathcal{A}}) \arrow[r, "\xi_{0_{\mathcal{A}}}"] \arrow[d, "\mathrm{F}_S(0^{\mathrm{op}})"'] & F(0_{\mathcal{A}}) \arrow[d, "F(0^{\mathrm{op}})"] \\
\mathrm{F}_S(P) \arrow[r, "\xi_P"] & F(P)
\end{tikzcd}$$

in `Set`. We have $(0)\xi_P = (0)\,\mathrm{F}_S(0^{\mathrm{op}})\,\xi_P = (0)\xi_{0_{\mathcal{A}}}\,F(0^{\mathrm{op}}) = 0$ since $F(0^{\mathrm{op}}) = 0$ by assumption.

We conclude that $[\xi, S] = 0$ and, consequently, $F^+(0_{\mathcal{A}}) \cong 0_{\text{Mod-}R}$ in Mod-R.

Now suppose given $A, B \in \operatorname{Ob} \mathcal{A}$. Write $i := \begin{pmatrix} 1 & 0 \end{pmatrix} : A \to A \oplus B$ and $j := \begin{pmatrix} 0 & 1 \end{pmatrix} : B \to A \oplus B$.

It suffices to show that $\begin{pmatrix} F^+(i^{\mathrm{op}}) & F^+(j^{\mathrm{op}}) \end{pmatrix} : F^+(A \oplus B) \to F^+(A) \oplus F^+(B)$ is monomorphic in Mod-R, cf. lemma 1.1.3.

Suppose given $[\xi, S] \in F^+(A \oplus B)$ with $[\xi, S]\begin{pmatrix} F^+(i^{\mathrm{op}}) & F^+(j^{\mathrm{op}}) \end{pmatrix} = 0$. So $\xi \colon \mathrm{F}_S \Rightarrow F$. We have to show that $[\xi, S] = 0$.

Since $[\xi, S]\begin{pmatrix} F^+(i^{\mathrm{op}}) & F^+(j^{\mathrm{op}}) \end{pmatrix} = \big([\xi^i, i^*(S)], [\xi^j, j^*(S)]\big)$, we have $[\xi^i, i^*(S)] = 0$ and $[\xi^j, j^*(S)] = 0$.

By lemma 2.4.3, we may choose pure epimorphisms $\big(P_A \xrightarrow{p_A} A\big) \in i^*(S)$ and $\big(P_B \xrightarrow{p_B} B\big) \in j^*(S)$ with $(p_A)\xi^i_{P_A} = 0$ and $(p_B)\xi^j_{P_B} = 0$.

Choose a pure epimorphism $\big(C \xrightarrow{e} A \oplus B\big) \in S$.

By (E7), there exist pullbacks

$$\begin{tikzcd}
C_A \arrow[r, "e_A"] \arrow[d, "c_A"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "i"] \\
C \arrow[r, "e"] & A \oplus B
\end{tikzcd}
\qquad
\begin{tikzcd}
C_B \arrow[r, "e_B"] \arrow[d, "c_B"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "j"] \\
C \arrow[r, "e"] & A \oplus B
\end{tikzcd}$$

in $\mathcal{A}$ with e_A and e_B purely epimorphic.

Again by (E7), there exist pullbacks

$$\begin{tikzcd}
Q_A \arrow[r, "q_A"] \arrow[d, "f_A"'] \arrow[dr, phantom, "\ulcorner"] & C_A \arrow[d, "e_A"] \\
P_A \arrow[r, "p_A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_B \arrow[r, "q_B"] \arrow[d, "f_B"'] \arrow[dr, phantom, "\ulcorner"] & C_B \arrow[d, "e_B"] \\
P_B \arrow[r, "p_B"] & B
\end{tikzcd}$$

in $\mathcal{A}$ with q_A and q_B purely epimorphic.

By (E5) and lemma 1.5.11, the morphism $\begin{pmatrix} q_A e_A & 0 \\ 0 & q_B e_B \end{pmatrix} : Q_A \oplus Q_B \to A \oplus B$ is purely epimorphic.

We have

$$\begin{pmatrix} q_A e_A & 0 \\ 0 & q_B e_B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} q_A\, e_A\, i + \begin{pmatrix} 0 \\ 1 \end{pmatrix} q_B\, e_B\, j = \Big(\begin{pmatrix} 1 \\ 0 \end{pmatrix} q_A\, c_A + \begin{pmatrix} 0 \\ 1 \end{pmatrix} q_B\, c_B\Big) e \in S.$$

It suffices to show that $\left(\begin{smallmatrix} q_A e_A & 0 \\ 0 & q_A e_B \end{smallmatrix}\right) \xi_{Q_A \oplus Q_B} = 0$ by lemma 2.4.3.

Note that the morphism $\left(\begin{smallmatrix} F((1\ 0)^{\mathrm{op}}) & F((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right) : F(Q_A \oplus Q_B) \to F(Q_A) \oplus F(Q_B)$ is an isomorphism since F is additive.

Since ξ is natural, we have

$$\begin{aligned}
\xi_{Q_A \oplus Q_B} \left(\begin{smallmatrix} F((1\ 0)^{\mathrm{op}}) & F((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right) &= \left(\begin{smallmatrix} \xi_{Q_A \oplus Q_B} F((1\ 0)^{\mathrm{op}}) & \xi_{Q_A \oplus Q_B} F((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} \mathrm{F}_S((1\ 0)^{\mathrm{op}}) \xi_{Q_A} & \mathrm{F}_S((0\ 1)^{\mathrm{op}}) \xi_{Q_B} \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} \mathrm{F}_S((1\ 0)^{\mathrm{op}}) & \mathrm{F}_S((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right) \begin{pmatrix} \xi_{Q_A} & 0 \\ 0 & \xi_{Q_B} \end{pmatrix}.
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{F}_S(Q_A \oplus Q_B) \arrow[r, "\xi_{Q_A \oplus Q_B}"] \arrow[d, "\left(\begin{smallmatrix} \mathrm{F}_S((1\ 0)^{\mathrm{op}}) & \mathrm{F}_S((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right)"'] & F(Q_A \oplus Q_B) \arrow[d, "\left(\begin{smallmatrix} F((1\ 0)^{\mathrm{op}}) & F((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right)"] \\
\mathrm{F}_S(Q_A) \oplus \mathrm{F}_S(Q_B) \arrow[r, "\left(\begin{smallmatrix} \xi_{Q_A} & 0 \\ 0 & \xi_{Q_B} \end{smallmatrix}\right)"] & F(Q_A) \oplus F(Q_B)
\end{tikzcd}$$

We obtain

$$\begin{aligned}
\left(\begin{smallmatrix} q_A e_A & 0 \\ 0 & q_A e_B \end{smallmatrix}\right) \xi_{Q_A \oplus Q_B} \left(\begin{smallmatrix} F((1\ 0)^{\mathrm{op}}) & F((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right) &= \left(\begin{smallmatrix} q_A e_A & 0 \\ 0 & q_A e_B \end{smallmatrix}\right) \left(\begin{smallmatrix} \mathrm{F}_S((1\ 0)^{\mathrm{op}}) & \mathrm{F}_S((0\ 1)^{\mathrm{op}}) \end{smallmatrix}\right) \begin{pmatrix} \xi_{Q_A} & 0 \\ 0 & \xi_{Q_B} \end{pmatrix} \\
&= \left(\left(\begin{smallmatrix} q_A e_A & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & q_B e_B \end{smallmatrix}\right) \right) \begin{pmatrix} \xi_{Q_A} & 0 \\ 0 & \xi_{Q_B} \end{pmatrix} \\
&= \left(q_A e_A i, q_B e_B j\right) \begin{pmatrix} \xi_{Q_A} & 0 \\ 0 & \xi_{Q_B} \end{pmatrix} \\
&= \left(f_A p_A i, f_B p_B j\right) \begin{pmatrix} \xi_{Q_A} & 0 \\ 0 & \xi_{Q_B} \end{pmatrix} \\
&= \left((f_A p_A i)\xi_{Q_A}, (f_B p_B j)\xi_{Q_B}\right) \\
&= \left((p_A i)\ \mathrm{F}_S(f_A^{\mathrm{op}})\, \xi_{Q_A}, (p_B j)\ \mathrm{F}_S(f_B^{\mathrm{op}})\, \xi_{Q_B}\right) \\
&= \left((p_A i)\, \xi_{P_A}\, F(f_A^{\mathrm{op}}), (p_B j)\, \xi_{P_B}\, F(f_B^{\mathrm{op}})\right) \\
&= \left((p_A)\, \xi^i_{P_A}\, F(f_A^{\mathrm{op}}), (p_B)\, \xi^j_{P_B}\, F(f_B^{\mathrm{op}})\right) \\
&= \left((0)\, F(f_A^{\mathrm{op}}), (0)\, F(f_B^{\mathrm{op}})\right) \\
&= 0.
\end{aligned}$$

Thus $\left(\begin{smallmatrix} q_A e_A & 0 \\ 0 & q_A e_B \end{smallmatrix}\right) \xi_{Q_A \oplus Q_B} = 0$. □

2.4.3 Effaceable presheaves

Definition 2.4.10. An R-presheaf $F \colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}R$ is called *effaceable* if and only if for each $A \in \operatorname{Ob} \mathcal{A}$ and each $x \in F(A)$ there exists a pure epimorphism $P \overset{p}{\twoheadrightarrow} A$ with $(x)\, F(p^{\mathrm{op}}) = 0$.

Lemma 2.4.11. Suppose given an R-presheaf $F \in \operatorname{Ob}({}_R\hat{\mathcal{A}})$.

The following five statements are equivalent.

(i) F is effaceable.

(ii) $F^+ \cong 0$ in ${}_R\hat{\mathcal{A}}$

(iii) $\tilde{F} \cong 0$ in ${}_R\hat{\mathcal{A}}$

(iv) $\mathrm{e}_F = 0\colon F \Rightarrow F^+$ in ${}_R\hat{\mathcal{A}}$, cf. definition 2.3.16.(b)

(v) $\varepsilon_F = 0\colon F \Rightarrow \tilde{F}$

Proof. The equivalence of (ii) and (iii) was shown in corollary 2.3.19.

Since $\varepsilon_F = \mathrm{e}_F \cdot \mathrm{e}_{F^+}$, the equivalence of (iv) and (v) follows from e_{F^+} monomorphic, which was shown in remark 2.3.18.(b).

Ad (i) $\Rightarrow$ (ii).

Suppose given $A \in \operatorname{Ob}\mathcal{A}$ and $[\xi, S] \in F^+(A)$. So $\xi\colon\ \mathrm{F}_S \Rightarrow {}_R\Upsilon \circ F$.

Choose a pure epimorphism $\big(P \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} A\big) \in S$.

Since F is effaceable, we may choose a pure epimorphism $Q \overset{q}{\longrightarrow\!\!\!\!\!\!\!+\;\;} P$ for $(p)\xi_P \in F(P)$ such that $(p)\xi_P\, F(q^{\mathrm{op}}) = 0$.

Recall that $\mathrm{S}_{qp} \in \mathrm{J}_A^{\mathcal{E}}$, cf. remark 2.4.2.

For $U \xrightarrow{f} Q$ in $\mathcal{A}$, we have

$$(fqp)\xi_U = (p)\ \mathrm{F}_S((fq)^{\mathrm{op}})\,\xi_U = (p)\xi_P\, F((fq)^{\mathrm{op}}) = (p)\xi_P\, F(q^{\mathrm{op}})\, F(f^{\mathrm{op}}) = (0)\, F(f^{\mathrm{op}}) = 0.$$

So $\iota^S_{\mathrm{S}_{qp}}\, \xi = 0$ and therefore $[\xi, S] = 0 \in F^+(A)$.

We conclude that $F^+(A) \cong 0$ in Mod-R for $A \in \operatorname{Ob}\mathcal{A}$.

Ad (ii) $\Rightarrow$ (iv).

The codomain of e_F is a zero object by assumption.

Ad (iv) $\Rightarrow$ (i).

Suppose given $A \in \operatorname{Ob}\mathcal{A}$ and $x \in F(A)$. We have $[\beta^x, \max(A)] = (x)\,(\mathrm{e}_F)_A = 0 \in F^+(A)$.

By lemma 2.4.3, there exists a pure epimorphism $P \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} A$ in $\mathcal{A}$ with $(x)\, F(p^{\mathrm{op}}) = (p)\beta^x_P = 0$. $\square$

2.4.4 An example: quasi-left-exact but not left-exact

Lemma 2.4.12. Suppose given an abelian category $\mathcal{B}$. Suppose given the following commutative diagram in $\mathcal{B}$.

$$\begin{tikzcd}
X' \arrow[r, "x'"] \arrow[d, "f'"'] & X \arrow[r, "x''"] \arrow[d, "f"] & X'' \arrow[d, "f''"] \\
Y' \arrow[r, "y'"] \arrow[d, "g'"'] & Y \arrow[r, "y''"] \arrow[d, "g"] & Y'' \arrow[d, "g''"] \\
Z' \arrow[r, "z'"] & Z \arrow[r, "z''"] & Z''
\end{tikzcd}$$

Suppose that the rows and columns are short exact sequences, i.e. that

$$X' \xrightarrow{x'} X \xrightarrow{x''} X'', \qquad Y' \xrightarrow{y'} Y \xrightarrow{y''} Y'', \qquad Z' \xrightarrow{z'} Z \xrightarrow{z''} Z'', \qquad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z',$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \text{ and } X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'$$

are short exact sequences.

The sequence $X' \xrightarrow{x'f} Y \xrightarrow{(\, g \; y'' \,)} Z \oplus Y''$ is left-exact.

Proof. Suppose given $T \xrightarrow{t} Y$ in $\mathcal{B}$ with $t \cdot (\, g \; y'' \,) = 0$. So $tg = 0$ and $ty'' = 0$. We conclude that there exist $T \xrightarrow{a} X$ and $T \xrightarrow{b} Y'$ in $\mathcal{B}$ with $af = t$ and $by' = t$.

Since the diagram

$$\begin{tikzcd}
X' \arrow[r, "x'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "y'"] & Y
\end{tikzcd}$$

is a pullback by the kernel-cokernel-criterion lemma 1.1.2.(a), there exists $T \xrightarrow{u} X'$ such that $ux' = a$ and $uf' = b$.

Thus $ux'f = af = t$.

The induced morphism is unique since $x'f$ is a monomorphism as composite of monomorphisms.

$$\begin{tikzcd}
T \arrow[dr, "u"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & & \\
& X' \arrow[r, "x'"'] \arrow[d, "f'"] & X \arrow[r, "x''"] \arrow[d, "f"] & X'' \arrow[d, "f''"] \\
& Y' \arrow[r, "y'"] \arrow[d, "g'"'] & Y \arrow[r, "y''"] \arrow[d, "g"] & Y'' \arrow[d, "g''"] \\
& Z' \arrow[r, "z'"] & Z \arrow[r, "z''"] & Z''
\end{tikzcd}$$

$\square$

Example 2.4.13. Suppose given a field K. For $\big(X \xrightarrow{f} Y\big), \big(X' \xrightarrow{f'} Y'\big) \in \mathrm{Mor}(\text{mod-}K)$, let $X \otimes X' \xrightarrow{f\otimes f'} Y \otimes Y'$ denote the tensor product over K.

We consider the site $(\text{mod-}K)^{\mathrm{op}}$, cf. definitions 2.2.1 and 2.4.1.

The K-presheaf $F\colon\ \text{mod-}K \to \text{Mod-}K$ defined by

$$F\big(X \xrightarrow{f} Y\big) = \big(X \otimes X \xrightarrow{f\otimes f} Y \otimes Y\big)$$

for $X \xrightarrow{f} Y$ in mod-K is quasi-left-exact but not left-exact.

I learned this from Fernando Muro [11].

Proof. We show that F is not left-exact. It suffices to show that F is not additive.

If F was additive, then

$$F(K \oplus K) \xrightarrow{\left(\, F\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \ \ F\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right)} F(K) \oplus F(K)$$

$$= (K \oplus K) \otimes (K \oplus K) \xrightarrow{\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \ \ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right)} (K \otimes K) \oplus (K \otimes K)$$

would be an isomorphism in Mod-K, cf. [14, prop. 4]. Abbreviate $f := \left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right)$.

Since f is K-linear, we may argue by counting dimensions. In fact, $\dim_K((K{\oplus}K){\otimes}(K{\oplus}K)) = 4$ and $\dim_K((K \otimes K) \oplus (K \otimes K)) = 2$. So f can not be an isomorphism and thus F can not be left-exact.

We show that F is quasi-left-exact.

Suppose given a short exact sequence $C \xrightarrow{i} B \xrightarrow{p} A$ in mod-K.

We have to show that the sequence

$$F(C) \xrightarrow{F(i)} F(B) \xrightarrow{F(p\ 1) - F(0\ 1)} F(A \oplus B)$$

$$= C \otimes C \xrightarrow{i\otimes i} B \otimes B \xrightarrow{(p\ 1)\otimes(p\ 1) - (0\ 1)\otimes(0\ 1)} (A \oplus B) \otimes (A \oplus B)$$

is left-exact in Mod-K, cf. remark 2.4.6.

Note that

$$(A{\oplus}B){\otimes}(A{\oplus}B) \xrightarrow{\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \ \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \ \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right)} (A{\otimes}A){\oplus}(A{\otimes}B){\oplus}(B{\otimes}A){\oplus}(B{\otimes}B)$$

and

$$(A{\otimes}A){\oplus}(A{\otimes}B){\oplus}(B{\otimes}A){\oplus}(B{\otimes}B) \xrightarrow{\left(\begin{smallmatrix}(1\ 0)\otimes(1\ 0)\\ (1\ 0)\otimes(0\ 1)\\ (0\ 1)\otimes(1\ 0)\\ (0\ 1)\otimes(0\ 1)\end{smallmatrix}\right)} (A{\oplus}B){\otimes}(A{\oplus}B)$$

are mutually inverse in Mod-K.

We have

$$\Big(\left(\begin{smallmatrix} p & 1 \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} p & 1 \end{smallmatrix} \right) - \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix} \right) \Big) \cdot \Big(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\otimes\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \; \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\otimes\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \; \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\otimes\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \; \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\otimes\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \Big) = \left(\begin{smallmatrix} p\otimes p & p\otimes 1 & 1\otimes p & 0 \end{smallmatrix} \right).$$

So it remains to show that $i \otimes i$ is a kernel of

$$B \otimes B \xrightarrow{\left(\begin{smallmatrix} p\otimes p & p\otimes 1 & 1\otimes p & 0 \end{smallmatrix} \right)} (A \otimes A) \oplus (A \otimes B) \oplus (B \otimes A) \oplus (B \otimes B)$$

For $X \xrightarrow{f} Y$ in mod-K, let $\ker(f) \subseteq X$ denote the usual kernel.

Note that $p \otimes p = (1 \otimes p)(p \otimes 1) = (p \otimes 1)(1 \otimes p)$, so $\ker(p \otimes p) \supseteq \ker(p \otimes 1) \cap \ker(1 \otimes p)$.

We have $\ker(\left(\begin{smallmatrix} p\otimes p & p\otimes 1 & 1\otimes p & 0 \end{smallmatrix} \right)) = \ker(p \otimes p) \cap \ker(p \otimes 1) \cap \ker(1 \otimes p) = \ker(p \otimes 1) \cap \ker(1 \otimes p)$.

So it suffices to show that $i \otimes i$ is a kernel of $\left(\begin{smallmatrix} p\otimes 1 & 1\otimes p \end{smallmatrix} \right)$.

This follows from lemma 2.4.12 applied to the diagram

$$\begin{tikzcd}
C \otimes C \arrow[r, "1\otimes i"] \arrow[d, "i\otimes 1"'] & C \otimes B \arrow[r, "1\otimes p"] \arrow[d, "i\otimes 1"] & C \otimes A \arrow[d, "i\otimes 1"] \\
B \otimes C \arrow[r, "1\otimes i"] \arrow[d, "p\otimes 1"'] & B \otimes B \arrow[r, "1\otimes p"] \arrow[d, "p\otimes 1"] & B \otimes A \arrow[d, "p\otimes 1"] \\
A \otimes C \arrow[r, "1\otimes i"] & A \otimes B \arrow[r, "1\otimes p"] & A \otimes A.
\end{tikzcd}$$

The rows and columns are short exact since for $X \in \mathrm{Ob}(\text{mod-}K)$, the functors $- \otimes X\colon \text{mod-}K \to \text{Mod-}K$ and $X \otimes -\colon \text{mod-}K \to \text{Mod-}K$ are exact. $\square$

Remark 2.4.14. If we compose the K-presheaf $F\colon \text{mod-}K \to \text{Mod-}K$ from the preceding example 2.4.13 with the forgetful functor $\Upsilon\colon \text{Mod-}K \to \text{Mod-}\mathbf{Z}$, we obtain the $\mathbf{Z}$-presheaf $\Upsilon \circ F\colon \text{mod-}K \to \text{Mod-}\mathbf{Z}$ which is also quasi-left-exact but not left-exact.

2.5 The Gabriel-Quillen-Laumon immersion theorem

Suppose given an exact category $\mathcal{A} = (\mathcal{A}, \mathcal{E})$. Cf. definition 1.5.1.

Recall that we endowed $\mathcal{A}$ with the Grothendieck topology $\mathrm{J}^{\mathcal{E}}$ in definition 2.4.1. We will often write $\mathcal{A} = (\mathcal{A}, \mathrm{J}^{\mathcal{E}})$ for the resulting site.

So $\mathcal{A}$ can be viewed as an exact category or as a site, depending on the context.

2.5.1 The Gabriel-Quillen-Laumon category

Definition 2.5.1. Let $\mathrm{GQL}(\mathcal{A}, \mathcal{E})$ be the full subcategory of ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ whose objects are the left-exact functors from $\mathcal{A}^{\mathrm{op}}$ to Mod-$\mathbf{Z}$. We call $\mathrm{GQL}(\mathcal{A}, \mathcal{E})$ the *Gabriel-Quillen-Laumon category of* $(\mathcal{A}, \mathcal{E})$.

We abbreviate $\mathrm{GQL}(\mathcal{A}) = \mathrm{GQL}(\mathcal{A}, \mathcal{E})$ if unambiguous.

For historical remarks, see e.g. [2, rem. A.4]. Some denote the Gabriel-Quillen-Laumon category by $\mathrm{Lex}(\mathcal{A}, \mathcal{E})$.

Theorem 2.5.2. Recall that $\mathcal{A} = (\mathcal{A}, \mathcal{E})$ is an exact category, yielding the site $\mathcal{A} = (\mathcal{A}, \mathrm{J}^{\mathcal{E}})$ as in definition 2.4.1. Recall that the category of $\mathbf{Z}$-sheaves ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ is abelian by theorem 2.3.35.

The category $\mathrm{GQL}(\mathcal{A})$ is a full additive subcategory of ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ and it is closed under isomorphisms. Moreover, kernels and cokernels of morphisms in $\mathrm{GQL}(\mathcal{A})$ formed in ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ lie again in $\mathrm{GQL}(\mathcal{A})$:

Suppose given $K \xrightarrow{k} F \xrightarrow{\alpha} G \xrightarrow{c} C$ in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ such that k is a kernel of α and c is a cokernel of α in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$. If $F, G \in \mathrm{Ob}(\mathrm{GQL}(\mathcal{A}))$, then $K, C \in \mathrm{Ob}(\mathrm{GQL}(\mathcal{A}))$.

In particular, $\mathrm{GQL}(\mathcal{A})$ is an abelian category.

We give formulas for kernels and cokernels in $\mathrm{GQL}(\mathcal{A})$ in remark 2.5.5.

Proof. By lemmata 2.4.7 and 2.4.8, the objects of $\mathrm{GQL}(\mathcal{A})$ are $\mathbf{Z}$-sheaves and therefore objects of ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$. So $\mathrm{GQL}(\mathcal{A})$ is a full subcategory of ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$. Moreover, an object $F \in \mathrm{Ob}({}_{\mathbf{Z}}\tilde{\mathcal{A}})$ is an object of $\mathrm{GQL}(\mathcal{A})$ if and only if F is an additive functor. By remark 1.1.9.(e), $\mathrm{GQL}(\mathcal{A})$ contains the zero objects of ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ and it contains $F \oplus G$ for $F, G \in \mathrm{Ob}(\mathrm{GQL}(\mathcal{A}))$. Thus $\mathrm{GQL}(\mathcal{A})$ is a full additive subcategory of ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$. It is closed under isomorphisms by remark 1.1.9.(f).

Suppose given $F \xrightarrow{\alpha} G$ in $\mathrm{GQL}(\mathcal{A})$. Suppose given a kernel $K \xrightarrow{k} F$ and a cokernel $G \xrightarrow{c} C$ of α in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$.

A kernel of f in ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ is given by $K \xrightarrow{k} F$ and a cokernel of f in ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ is given by $G \xrightarrow{\eta_G^{-1} \cdot \tilde{c}} \tilde{C}$, where $\eta \colon (-)^{\sim} \circ \mathrm{E} \Rightarrow 1_{{}_R\tilde{\mathscr{F}}}$ is the counit of the adjunction

$$\begin{tikzcd} {}_{\mathbf{Z}}\hat{\mathcal{A}} \arrow[r, bend left, "(-)^{\sim}"] \arrow[r, phantom, "\perp"] & {}_{\mathbf{Z}}\tilde{\mathcal{A}}. \arrow[l, bend left, "\mathrm{E}"] \end{tikzcd}$$

as in theorem 2.3.30.(b). Cf. remark 2.3.36.

The functors K and C are additive by remark 1.1.9.(b). By applying lemma 2.4.9 twice, we conclude that $\tilde{C}$ is additive as well. So kernels and cokernels of morphisms in $\mathrm{GQL}(\mathcal{A})$ formed in ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ lie again in $\mathrm{GQL}(\mathcal{A})$.

The induced morphism in a kernel-cokernel-factorisation of α is an isomorphism in the abelian category ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ and thus also in the full subcategory $\mathrm{GQL}(\mathcal{A})$ of ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$. $\square$

Remark 2.5.3. We have the following inclusion diagram of full subcategories of ${}_R\hat{\mathcal{A}}$.

$$\begin{tikzcd}
 & {}_{\mathbf{Z}}\hat{\mathcal{A}} & \\
{}_{\mathbf{Z}}\tilde{\mathcal{A}} \arrow[ur, hook, "\mathrm{E}"] & & \mathrm{Add}(\mathcal{A}^{\mathrm{op}}, \text{Mod-}\mathbf{Z}) \arrow[ul, hook] \\
 & \mathrm{GQL}(\mathcal{A}) \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Note that all these categories are abelian.

The inclusion functors from $\mathrm{GQL}(\mathcal{A})$ to ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ and from $\mathrm{Add}(\mathcal{A}^{\mathrm{op}}, \text{Mod-}\mathbf{Z})$ to ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ are exact.

The inclusion functors from $\mathrm{GQL}(\mathcal{A})$ to $\mathrm{Add}(\mathcal{A}^{\mathrm{op}}, \text{Mod-}\mathbf{Z})$ and from ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ to ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ are left-exact.

Cf. theorem 2.3.35 and remark 1.1.9.(h).

Definition 2.5.4. Suppose given $A \xrightarrow{f} B$ in $\mathcal{A}$.

For $D \in \mathrm{Ob}\,\mathcal{A}$, we have the usual kernel $\mathrm{K}_f(D) \xrightarrow{(\mathrm{k}_f)_D} \mathrm{Hom}_{\mathcal{A}}(D, A)$ of $({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f))_D$ in Mod-$\mathbf{Z}$, where $\mathrm{K}_f(D) = \{h \in \mathrm{Hom}_{\mathcal{A}}(D, A) \colon hf = 0\}$ and $(\mathrm{k}_f)_D$ is the inclusion morphism.

Remark 1.1.9.(b) yields the functor $\mathrm{K}_f \colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}\mathbf{Z}$ and the transformation

$\mathrm{k}_f \colon \mathrm{K}_f \Rightarrow {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A)$ such that k_f is a kernel of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f)$ in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$.

For $E \xrightarrow{h} D$ in $\mathcal{A}$ and $g \in \mathrm{K}_f(D)$, we have $(g)\,\mathrm{K}_f(h^{\mathrm{op}}) = hg$.

For $D \in \mathrm{Ob}\,\mathcal{A}$, we have the usual cokernel $\mathrm{Hom}_{\mathcal{A}}(D, B) \xrightarrow{(\mathrm{c}^\circ_f)_D} \mathrm{Hom}_{\mathcal{A}}(D, B)/\,\mathrm{Hom}_{\mathcal{A}}(D, A)f$ of $({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f))_D$ in Mod-$\mathbf{Z}$, where $(\mathrm{c}^\circ_f)_D$ is the residue class morphism.
We abbreviate $\overline{g} := g + \mathrm{Hom}_{\mathcal{A}}(D, A)f$ for $g \in \mathrm{Hom}_{\mathcal{A}}(D, B)$ if unambiguous.

Remark 1.1.9.(b) yields the functor $\mathrm{C}^\circ_f \colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}\mathbf{Z}$ and the transformation

$\mathrm{c}^\circ_f \colon {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \Rightarrow \mathrm{C}^\circ_f$ such that c°_f is a cokernel of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f)$ in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$. For $E \xrightarrow{h} D$ in $\mathcal{A}$ and $g \in \mathrm{Hom}_{\mathcal{A}}(D, B)$, we have $\mathrm{C}^\circ_f(D) = \mathrm{Hom}_{\mathcal{A}}(D, B)/\,\mathrm{Hom}_{\mathcal{A}}(D, A)f$ and $\overline{g}\,\mathrm{C}^\circ_f(h^{\mathrm{op}}) = \overline{hg} \in \mathrm{Hom}_{\mathcal{A}}(E, B)/\,\mathrm{Hom}_{\mathcal{A}}(E, A)f$.

Moreover, let $\big({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \xrightarrow{\mathrm{c}_f} \mathrm{C}_f\big) := \big({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \xrightarrow{\varepsilon_{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B)} \cdot (\mathrm{c}^\circ_f)^{++}} (\mathrm{C}^\circ_f)^{++}\big)$.

$$\begin{tikzcd}
{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \arrow[r, "\mathrm{c}^\circ_f"] \arrow[d, "\varepsilon_{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B)}"'] \arrow[dr, "\mathrm{c}_f"] & \mathrm{C}^\circ_f \arrow[d, "\varepsilon_{\mathrm{C}^\circ_f}"] \\
{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B)^{++} \arrow[r, "(\mathrm{c}^\circ_f)^{++}"'] & \mathrm{C}_f
\end{tikzcd}$$

Remark 2.5.5. Recall that the diagram

$$\begin{tikzcd} {}_{\mathbf{Z}}\hat{\mathcal{A}} \arrow[r, bend left, "(-)^{\sim}"] \arrow[r, phantom, "\perp"] & {}_{\mathbf{Z}}\tilde{\mathcal{A}} \arrow[l, bend left, "\mathrm{E}"] \end{tikzcd}$$

with unit $\varepsilon\colon 1_{{}_{\mathbf{Z}}\hat{\mathcal{A}}} \Rightarrow \mathrm{E}\circ(-)^{\sim}$ and counit $\eta\colon (-)^{\sim}\circ \mathrm{E} \Rightarrow 1_{{}_{\mathbf{Z}}\tilde{\mathcal{A}}}$ is an adjunction, cf. theorem 2.3.30.(b).

Suppose given $F \xrightarrow{\alpha} G$ in $\mathrm{GQL}(\mathcal{A}) \subseteq {}_{\mathbf{Z}}\tilde{\mathcal{A}}$. Suppose given a kernel $K \xrightarrow{k} F$ of α in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ and a cokernel $G \xrightarrow{c} C$ of α in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$.

A kernel of α in $\mathrm{GQL}(\mathcal{A})$ is given by k.

A cokernel of α in $\mathrm{GQL}(\mathcal{A})$ is given by $\eta_G^{-1}\cdot\tilde{c} = \varepsilon_G\cdot\tilde{c} = c\cdot\varepsilon_C = c\cdot \mathrm{e}_C\cdot \mathrm{e}_{C^+}$.

Moreover, the diagram

$$\begin{tikzcd} K \arrow[r, "k"] & F \arrow[r, "\alpha"] & G \arrow[r, "c"] \arrow[d, "\mathrm{e}_G"] \arrow[dd, bend right=60, "\varepsilon_G"'] & C \arrow[d, "\mathrm{e}_C"'] \arrow[dd, bend left=60, "\varepsilon_C"] \\ & & G^+ \arrow[r, "c^+"] \arrow[d, "\mathrm{e}_{G^+}"] & C^+ \arrow[d, "\mathrm{e}_{C^+}"'] \\ & & \tilde{G} \arrow[r, "\tilde{c}"] & \tilde{C} \end{tikzcd}$$

commutes.

Cf. remark 2.3.36 and theorem 2.5.2.

In particular, for $A \xrightarrow{f} B$ in $\mathcal{A}$ and $\alpha = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f)$, we may choose $k = \mathrm{k}_f$ and $c = \mathrm{c}_f^{\circ}$. Thus a cokernel of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f)$ in $\mathrm{GQL}(\mathcal{A})$ is given by $\mathrm{c}_f = \mathrm{c}_f^{\circ}\cdot\varepsilon_{\mathrm{C}_f^{\circ}} = \mathrm{c}_f^{\circ}\cdot \mathrm{e}_{\mathrm{C}_f^{\circ}}\cdot \mathrm{e}_{(\mathrm{C}_f^{\circ})^+}$.

Cf. definition 2.5.4.

2.5.2 The immersion theorem

Definition 2.5.6.
Recall that for $X \in \mathrm{Ob}\,\mathcal{A}$, the $\mathbf{Z}$-Yoneda functor ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(X) = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A},X}\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}\mathbf{Z}$ is left-exact, cf. definition 1.2.3.(b). Thus we may define

$$\mathrm{Y}_{(\mathcal{A},\mathcal{E})} := {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\big|^{\mathrm{GQL}(\mathcal{A},\mathcal{E})}\colon \mathcal{A} \to \mathrm{GQL}(\mathcal{A},\mathcal{E}).$$

We abbreviate $\mathrm{Y} = \mathrm{Y}_{\mathcal{A}} = \mathrm{Y}_{(\mathcal{A},\mathcal{E})}$ if unambiguous.

Remark 2.5.7. Suppose given $F \xrightarrow{\alpha} G$ in $\mathrm{GQL}(\mathcal{A})$. Suppose given a cokernel $G \xrightarrow{c} C$ of α in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$. The morphism α is epimorphic in $\mathrm{GQL}(\mathcal{A})$ if and only if C is effaceable, cf. lemma 2.4.11 and remark 2.5.5. Recall that C is effaceable if and only if for all $X \in \mathrm{Ob}\,\mathcal{A}$ and all $x \in C(X)$, there exists a pure epimorphism $P \overset{p}{\twoheadrightarrow} X$ in $\mathcal{A}$ such that $(x)\,C(p^{\mathrm{op}}) = 0$, cf. definition 2.4.10.

In particular, suppose given $A \xrightarrow{f} B$ in $\mathcal{A}$. The morphism ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f)$ is epimorphic in $\mathrm{GQL}(\mathcal{A})$ if and only if C_f° is effaceable. Now C_f° is effaceable if and only if for all $X \xrightarrow{h} B$ in $\mathcal{A}$, there exists $P \xrightarrow{u} A$ and a pure epimorphism $P \xrightarrow{p} X$ in $\mathcal{A}$ such that $ph = uf$, cf. definition 2.5.4.

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{f} & B \end{array}$$

Proof. Suppose given $A \xrightarrow{f} B$, $X \xrightarrow{h} B$ and a pure epimorphism $P \xrightarrow{p} X$ in $\mathcal{A}$.

Note that $\mathrm{C}_f^\circ(X) = \mathrm{Hom}_{\mathcal{A}}(X, B) / \mathrm{Hom}_{\mathcal{A}}(X, A)f$.

We have $(\overline{h})\, \mathrm{C}_f^\circ(p^{\mathrm{op}}) = \overline{ph} \in \mathrm{Hom}_{\mathcal{A}}(P, B) / \mathrm{Hom}_{\mathcal{A}}(P, A)f$.

So $(\overline{h})\, \mathrm{C}_f^\circ(p^{\mathrm{op}}) = 0$ if and only if there exists $P \xrightarrow{u} A$ in $\mathcal{A}$ such that $ph = uf$. □

Theorem 2.5.8. Recall that $\mathcal{A} = (\mathcal{A}, \mathcal{E})$ is an exact category and that the Gabriel-Quillen-Laumon category $\mathrm{GQL}(\mathcal{A})$ is the full subcategory of ${}_{\mathbf{Z}}\tilde{\mathcal{A}}$ whose objects are the left-exact functors from $\mathcal{A}^{\mathrm{op}}$ to Mod-$\mathbf{Z}$, cf. definition 2.5.1. Recall from theorem 2.5.2 that $\mathrm{GQL}(\mathcal{A})$ is an abelian category.

The functor $\mathrm{Y}_{\mathcal{A}} \colon \mathcal{A} \to \mathrm{GQL}(\mathcal{A})$ is a closed immersion, i.e. it is full, faithful and exact, it detects exactness and the essential image $\mathrm{Im}_{\mathrm{ess}}(\mathrm{Y}_{\mathcal{A}})$ is closed under extensions, cf. definition 1.5.8.

Proof. The functor $\mathrm{Y}_{\mathcal{A}}$ is full, faithful and additive since it is the restriction of the full, faithful and additive Yoneda functor ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}} \colon \mathcal{A} \to {}_{\mathbf{Z}}\hat{\mathcal{A}}$. Cf. definition 1.2.3.(b).

We show that $\mathrm{Y}_{\mathcal{A}}$ is exact and that it detects exactness.

Suppose given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$.

Suppose that the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is pure short exact in $\mathcal{A}$.

The sequence

$$\Big(\mathrm{Y}_{\mathcal{A}}(A) \xrightarrow{\mathrm{Y}_{\mathcal{A}}(f)} \mathrm{Y}_{\mathcal{A}}(B) \xrightarrow{\mathrm{Y}_{\mathcal{A}}(g)} \mathrm{Y}_{\mathcal{A}}(C) \Big) = \Big({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(g)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(C) \Big)$$

is left-exact in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$, cf. definition 1.2.3.(b). Therefore this sequence is left-exact in $\mathrm{GQL}(\mathcal{A})$ as well, cf. remark 2.5.5. We have to show that it is exact in $\mathrm{GQL}(\mathcal{A})$. It suffices to prove that ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(g)$ is epimorphic in $\mathrm{GQL}(\mathcal{A})$. Thus we have to show that the cokernel C_g° of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(g)$ in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ is effaceable, cf. remark 2.5.7.

Suppose given $X \xrightarrow{h} C$ in $\mathcal{A}$.

Choose a pullback

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "u"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "h"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

in $\mathcal{A}$. So p is a pure epimorphism in $\mathcal{A}$ with $ph = ug$.

We conclude that C_p° is effaceable. Thus ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(g)$ is epimorphic in $\mathrm{GQL}(\mathcal{A})$ and, consequently, the sequence $\big(\mathrm{Y}_{\mathcal{A}}(A) \xrightarrow{\mathrm{Y}_{\mathcal{A}}(f)} \mathrm{Y}_{\mathcal{A}}(B) \xrightarrow{\mathrm{Y}_{\mathcal{A}}(g)} \mathrm{Y}_{\mathcal{A}}(C)\big)$ is exact in $\mathrm{GQL}(\mathcal{A})$.

Conversely, suppose that the sequence

$$\Big(\ \mathrm{Y}_{\mathcal{A}}(A) \xrightarrow{\mathrm{Y}_{\mathcal{A}}(f)} \mathrm{Y}_{\mathcal{A}}(B) \xrightarrow{\mathrm{Y}_{\mathcal{A}}(g)} \mathrm{Y}_{\mathcal{A}}(C)\ \Big) = \Big({}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(f)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \xrightarrow{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(g)} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(C)\Big)$$

is exact in $\mathrm{GQL}(\mathcal{A})$. So it is left-exact in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$, cf. remark 2.5.3. Thus f is a kernel of g in $\mathcal{A}$, cf. definition 1.2.3.(b). It suffices to show that g is a pure epimorphism. By remark 2.5.7, the cokernel C_g° of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(g)$ in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ is effaceable. So for $C \xrightarrow{1_C} C$ in $\mathcal{A}$, we may choose $P \xrightarrow{u} B$ and a pure epimorphism $P \xrightarrow{p} C$ in $\mathcal{A}$ such that $p = p1_C = ug$. By the obscure axiom 1.5.13.(b), we conclude that g is purely epimorphic. Thus the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is purely short exact in $\mathcal{A}$.

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "u"'] & C \arrow[d, "1_C"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

We show that $\mathrm{Im}_{\mathrm{ess}}(\mathrm{Y}_{\mathcal{A}})$ is closed under extensions.

Suppose given $A, B \in \mathrm{Ob}\,\mathcal{A}$ and a short exact sequence ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A) \xrightarrow{f} F \xrightarrow{g} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B)$ in $\mathrm{GQL}(\mathcal{A})$. We have to find $X \in \mathrm{Ob}\,\mathcal{A}$ such that $F \cong {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(X)$ in $\mathrm{GQL}(\mathcal{A})$.

For $D \in \mathrm{Ob}\,\mathcal{A}$, we have the usual cokernel $\mathrm{Hom}_{\mathcal{A}}(D,B) \xrightarrow{c_D} \mathrm{Hom}_{\mathcal{A}}(D,B)/\operatorname{Im}(g_D)$ of f_D in Mod-$\mathbf{Z}$, where c_D is the projection morphism and $\operatorname{Im}(g_D)$ is the set-theoretic image of the map g_D . We abbreviate $\overline{h} := h + \operatorname{Im}(g_D)$ for $h \in \mathrm{Hom}_{\mathcal{A}}(D,B)$ if unambiguous.

Remark 1.1.9.(b) yields the functor $C\colon \mathcal{A}^{\mathrm{op}} \to$ Mod-$\mathbf{Z}$ and the transformation $c\colon {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \Rightarrow C$ such that c is a cokernel of g in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$. For $E \xrightarrow{k} D$ in $\mathcal{A}$ and $h \in \mathrm{Hom}_{\mathcal{A}}(D,B)$, we have $C(D) = \mathrm{Hom}_{\mathcal{A}}(D,B)/\operatorname{Im}(g_D)$ and $\overline{h}\,C(k^{\mathrm{op}}) = \overline{kh} \in \mathrm{Hom}_{\mathcal{A}}(E,B)/\operatorname{Im}(g_E)$.

Since g is epimorphic in $\mathrm{GQL}(\mathcal{A})$, the cokernel C of g in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$ is effaceable, cf. remark 2.5.7. So for $\overline{1_B} \in \mathrm{Hom}_{\mathcal{A}}(B,B)/\operatorname{Im}(g_B)$, we may choose a pure short exact sequence $I \xrightarrow{i} P \xrightarrow{p} B$ in $\mathcal{A}$ such that

$$\overline{p} = (\overline{1_B})\,C(p^{\mathrm{op}}) = 0 \in \mathrm{Hom}_{\mathcal{A}}(P,B)/\operatorname{Im}(g_P).$$

Thus we may choose $x \in F(P)$ with $(x)g_P = p$.

Let $\alpha := (x)_{\mathbf{Z}}\gamma^{-1}_{F,P} \colon {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \Rightarrow F$, cf. lemma 1.2.2.(b) and definition 1.2.3.(b). Note that $(1_P)\alpha_P = x$ and $\alpha g = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(p)$.

Choose a pullback

$$\begin{tikzcd} Q \arrow[r, "g'"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(p)"] \\ F \arrow[r, "g"] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \end{tikzcd}$$

in $\mathrm{GQL}(\mathcal{A})$. By the kernel-cokernel-criterion, lemma 1.1.2, the morphism g' is epimorphic in $\mathrm{GQL}(\mathcal{A})$ and we may choose a commutative diagram

$$\begin{tikzcd} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A) \arrow[r, "f'"] \arrow[d, "1"'] & Q \arrow[r, "g'"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(p)"] \\ {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A) \arrow[r, "f"] & F \arrow[r, "g"] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \end{tikzcd}$$

in $\mathrm{GQL}(\mathcal{A})$ such that f' is a kernel of g'.

For $1_{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P)}$ and α, we may choose ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \xrightarrow{u} Q$ in $\mathrm{GQL}(\mathcal{A})$ such that $ug' = 1_{{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P)}$ and $uq = \alpha$.

$$\begin{tikzcd} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \arrow[dr, "u"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "\alpha"'] & & \\ & Q \arrow[r, "g'"] \arrow[d, "q"] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(p)"] \\ & F \arrow[r, "g"] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \end{tikzcd}$$

So the sequence ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A) \xrightarrow{f'} Q \xrightarrow{g'} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P)$ is split short exact. Since ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}$ is additive, we may choose an isomorphism ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A \oplus P) \xrightarrow{v} Q$ in $\mathrm{GQL}(\mathcal{A})$ such that $vg' = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.
Cf. convention 38.

Let $r := vq$. We obtain the pullback

$$\begin{tikzcd} {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A \oplus P) \arrow[r, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d, "r"'] \arrow[dr, phantom, "\ulcorner", very near start] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(p)"] \\ F \arrow[r, "g"] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B) \end{tikzcd}$$

in $\mathrm{GQL}(\mathcal{A})$. Since ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}$ is exact, ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(p)$ is epimorphic. Hence r is epimorphic.

Again by the kernel-cokernel-criterion, lemma 1.1.2, and by the fullness of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}$, we may choose a commutative diagram

$$\begin{tikzcd}
{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(I) \arrow[r, "1"] \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(\,a\ b\,)"'] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(I) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(i)"] \\
{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(A \oplus P) \arrow[r, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d, "r"'] \arrow[dr, phantom, "\ulcorner"] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(P) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(p)"] \\
F \arrow[r, "g"] & {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(B)
\end{tikzcd}$$

in $\mathrm{GQL}(\mathcal{A})$ such that ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\,(\,a\ b\,)$ is a kernel of r. Note that ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(b) = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\,(\,a\ b\,) \cdot {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(i)$, so $b = i$ by the faithfulness of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}$.

Moreover, we have

$$\Big(\, I \xrightarrow{(\,a\ i\,)} A \oplus P \,\Big) = \Big(\, I \xrightarrow{(\,0\ 1\,)} A \oplus I \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)} A \oplus I \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)} A \oplus P \,\Big)$$

in $\mathcal{A}$, where $(\,0\ 1\,)$ is purely monomorphic, $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$ is isomorphic with inverse $\left(\begin{smallmatrix} 1 & 0 \\ -a & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ is purely monomorphic by lemma 1.5.11.(b).

Thus $(\,a\ i\,)$ is purely monomorphic as composite of pure monomorphisms. Choose a cokernel $A \oplus P \xrightarrow{\left(\begin{smallmatrix} d \\ e \end{smallmatrix}\right)} X$ of $(\,a\ i\,)$ in $\mathcal{A}$. Since $\mathrm{Y}_{\mathcal{A}}$ is exact, ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\left(\begin{smallmatrix} d \\ e \end{smallmatrix}\right)$ is a cokernel of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\,(\,a\ i\,)$. Since r is also a cokernel of ${}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}\,(\,a\ b\,)$, we conclude that $F \cong {}_{\mathbf{Z}}\mathrm{y}_{\mathcal{A}}(X)$ in $\mathrm{GQL}(\mathcal{A})$. $\square$

2.5.3 An elementary description of cokernels in $\mathrm{GQL}(\mathcal{A})$

We shall give a description of cokernels in the Gabriel-Quillen-Laumon category $\mathrm{GQL}(\mathcal{A})$ without using the language of sheaves. This is possible but probably not very helpful.

We will not use the results of section 2.5.3 in the sequel.

For a pure epimorphism $P \xrightarrow{p} A$ in $\mathcal{A}$, let $K_p \xrightarrow{k_p} P$ be a kernel of p.

Lemma/Definition 2.5.9. Suppose given an additive $\mathbf{Z}$-presheaf $F \colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}\mathbf{Z}$.

Suppose given $C \in \operatorname{Ob} \mathcal{A}$.

Let $F_0^{\maltese}(C)$ denote the set of all pairs $(x,\, P \xrightarrow{p} C)$, where p is a pure epimorphism in $\mathcal{A}$ and $x \in F(P)$ such that $(x)\, F(k_p^{\mathrm{op}}) = 0$.

For $(x,\, P \xrightarrow{p} C), (x',\, P' \xrightarrow{p'} C) \in F_0^{\maltese}(C)$, let $(x,\, P \xrightarrow{p} C) \sim (x',\, P' \xrightarrow{p'} C)$ if and only if there exists a commutative diagram

$$\begin{tikzcd} & P \arrow[dr, "p"] & \\ Q \arrow[ur, "q"] \arrow[dr, "q'"'] & & C \\ & P' \arrow[ur, "p'"'] & \end{tikzcd}$$

in $\mathcal{A}$ such that $qp = q'p'$ is a purely epimorphic and such that $(x)\,F(q^{\mathrm{op}}) = (x')\,F(q'^{\,\mathrm{op}})$.

This defines an equivalence relation on $F_0^{\maltese}(C)$. Let $F^{\maltese}(C) := F_0^{\maltese}(C)/\!\sim$ denote the factor set.

Let $[x,\, P \xrightarrow{p} C] \in F^{\maltese}(C)$ denote the equivalence class of $(x,\, P \xrightarrow{p} C) \in F_0^{\maltese}(C)$.

We shall construct mutually inverse maps

$$\begin{tikzcd} F^{+}(C) \arrow[r, shift left, "u_C^F"] & F^{\maltese}(C) \arrow[l, shift left, "v_C^F"] \end{tikzcd}.$$

Suppose given $[\xi, S] \in F^{+}(C)$, cf. remark 2.3.3. Choose a pure epimorphism $\big(P \xrightarrow{p} C\big) \in S$ and let

$$[\xi, S]u_C^F := [(p)\xi_P\,,\, P \xrightarrow{p} C].$$

This definition is independent of the choice of the pure epimorphism.

Conversely, suppose given $[x,\, P \xrightarrow{p} C] \in F^{\maltese}(C)$. For $X \in \operatorname{Ob}\mathcal{A}$, let

$${}^{p}_{x}\xi_X \colon\ \mathrm{S}_p \cap \operatorname{Hom}_{\mathcal{A}}(X, C) \to F(X) \colon \big(X \xrightarrow{g} P \xrightarrow{p} C\big) \mapsto (x)\,F(g^{\mathrm{op}}).$$

Let ${}^{p}_{x}\xi := ({}^{p}_{x}\xi_X)_{X \in \operatorname{Ob}\mathcal{A}}$ and let $[x,\, P \xrightarrow{p} C]v_C^F := [{}^{p}_{x}\xi, \mathrm{S}_p]$.

Suppose given $D \xrightarrow{f} C$ in $\mathcal{A}$. We shall define $F^{\maltese}(f^{\mathrm{op}}) \colon F^{\maltese}(C) \to F^{\maltese}(D)$.

Suppose given $[x,\, P \xrightarrow{p} C] \in F^{\maltese}(C)$.

Choose a commutative diagram

$$\begin{tikzcd} P' \arrow[r, "p'"] \arrow[d, "f'"'] & D \arrow[d, "f"] \\ P \arrow[r, "p"] & C \end{tikzcd}$$

in $\mathcal{A}$ such that p' is a pure epimorphism. This is possible since we may e.g. form the pullback of f and p.

Let $[x, P \xrightarrow{p} C] F^{\maltese}(f^{\mathrm{op}}) := [(x)\, F(f'^{\,\mathrm{op}}), P' \xrightarrow{p'} D]$. This definition is independent of the choice of the commutative diagram.

By defining the **Z**-module structure on $F^{\maltese}(C)$ via u^F_C for $C \in \mathrm{Ob}\,\mathcal{A}$, the maps u^F_C, v^F_C and $F^{\maltese}(f^{\mathrm{op}})$ become **Z**-linear for $D \xrightarrow{f} C$ in $\mathcal{A}$.

We obtain a **Z**-presheaf $F^{\maltese}\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}\mathbf{Z}$ and an isotransformation
$u^F := (u^F_C)_{C \in \mathrm{Ob}\,\mathcal{A}}\colon F^{+} \Rightarrow F^{\maltese}$.

Proof.

1. We show that $\sim$ defines an equivalence relation on $F^{\maltese}_0(C)$.

   Using identities, we see that $\sim$ is reflexive. By construction, $\sim$ is symmetric.

   Suppose given $(x, P \xrightarrow{p} C), (x', P' \xrightarrow{p'} C), (x'', P'' \xrightarrow{p''} C) \in F^{\maltese}_0(C)$ such that

   $$(x, P \xrightarrow{p} C) \sim (x', P' \xrightarrow{p'} C) \text{ and } (x', P' \xrightarrow{p'} C) \sim (x'', P'' \xrightarrow{p''} C).$$

   We may choose commutative diagrams

   $$\begin{tikzcd} & P \arrow[dr, "p", "|" marking] & \\ Q \arrow[ur, "q"] \arrow[dr, "q'"'] & & C \\ & P' \arrow[ur, "p'"', "|" marking] & \end{tikzcd} \qquad \begin{tikzcd} & P' \arrow[dr, "p'", "|" marking] & \\ R \arrow[ur, "r'"] \arrow[dr, "r''"'] & & C \\ & P'' \arrow[ur, "p''"', "|" marking] & \end{tikzcd}$$

   in $\mathcal{A}$ such that $qp = q'p'$ and $r'p' = r''p''$ are pure epimorphisms and such that $(x)\, F(q^{\mathrm{op}}) = (x')\, F(q'^{\,\mathrm{op}})$ and $(x')\, F(r'^{\,\mathrm{op}}) = (x'')\, F(r''^{\,\mathrm{op}})$.

   Choose a pullback

   $$\begin{tikzcd} & Q \arrow[dr, "q'p'", "|" marking] & \\ S \arrow[ur, "s", "|" marking] \arrow[dr, "t"', "|" marking] \arrow[r, phantom, "<"] & {} & C \\ & R \arrow[ur, "r'p'"', "|" marking] & \end{tikzcd}$$

   in $\mathcal{A}$. Note that s and t are purely epimorphic.

   The diagram

   $$\begin{tikzcd} & P \arrow[dr, "p", "|" marking] & \\ S \arrow[ur, "sq"] \arrow[dr, "tr''"'] & & C \\ & P'' \arrow[ur, "p''"', "|" marking] & \end{tikzcd}$$

commutes in $\mathcal{A}$ since $sqp = sq'p' = tr'p' = tr''p''$, which is, moreover, purely epimorphic. Moreover, we have $(sq' - tr')p' = sq'p' - tr'p' = 0$. So there exists $S \xrightarrow{u} K_{p'}$ in $\mathcal{A}$ such that $uk_{p'} = sq' - tr'$ since $k_{p'}$ is a kernel of p'.

We have

$$\begin{aligned}
(x)\, F((sq)^{\mathrm{op}}) - (x'')\, F((tr'')^{\mathrm{op}}) &= (x)\, F(q^{\mathrm{op}})\, F(s^{\mathrm{op}}) - (x'')\, F(r''^{\mathrm{op}})\, F(t^{\mathrm{op}}) \\
&= (x')\, F(q'^{\mathrm{op}})\, F(s^{\mathrm{op}}) - (x')\, F(r'^{\mathrm{op}})\, F(t^{\mathrm{op}}) \\
&= (x')\, F((sq' - tr')^{\mathrm{op}}) \\
&= (x') F((uk_{p'})^{\mathrm{op}}) \\
&= (x')\, F(k_{p'}^{\mathrm{op}})\, F(u^{\mathrm{op}}) \\
&= (0)\, F(u^{\mathrm{op}}) \\
&= 0,
\end{aligned}$$

thus $(x)\, F((sq)^{\mathrm{op}}) = (x'')\, F((tr'')^{\mathrm{op}})$.

We conclude that $\sim$ is transitive.

$$\begin{tikzcd}
 & & P \arrow[dddr, "p"] & \\
 & Q \arrow[ur, "q"] \arrow[dr, "q'"'] & & \\
 & & P' \arrow[dr, "p'"'] & \\
S \arrow[uuur, "s"] \arrow[ddr, "t"'] & & & C \\
 & & P' \arrow[ur, "p'"] & \\
 & R \arrow[ur, "r'"] \arrow[dr, "r''"'] & & \\
 & & P'' \arrow[uuur, "p''"'] &
\end{tikzcd}$$

2. We show that u_C^F is a well-defined map.

   Suppose given $[\xi, S] \in F^+(C)$. Choose a pure epimorphism $\left(P \xrightarrow{p} C\right) \in S$. We have

$$\begin{aligned}
(p)\xi_P\, F(k_p^{\mathrm{op}}) &= (p)\ \mathrm{F}_S(k_p^{\mathrm{op}})\, \xi_{K_p} = (k_p p)\xi_{K_p} = (0_{K_p,C})\xi_{K_p} \\
&= (0_{K_p,C})\ \mathrm{F}_S(0_{K_p}^{\mathrm{op}})\, \xi_{K_p} = (0_{K_p,C})\, \xi_{K_p} F(0_{K_p}^{\mathrm{op}}) \\
&= 0
\end{aligned}$$

since F is additive.

Suppose given another pure epimorphism $\big(P' \xrightarrow{p'} C\big) \in S$.

Choose a pure epimorphism $q \in \mathrm{S}_p \cap \mathrm{S}_{p'}$. So there exist $Q \xrightarrow{a} P$ and $Q \xrightarrow{a'} P'$ such that $ap = q = a'p'$. So the diagram

$$\begin{tikzcd} & P \arrow[dr, "p"] & \\ Q \arrow[ur, "a"] \arrow[rr, "q"] \arrow[dr, "a'"'] & & C \\ & P' \arrow[ur, "p'"'] & \end{tikzcd}$$

commutes, the morphism $q = ap = a'p'$ is purely epimorphic and we have

$$\begin{aligned}(p)\xi_P\, F(a^{\mathrm{op}}) &= (p)\ \mathrm{F}_S(a^{\mathrm{op}})\,\xi_Q = (ap)\xi_Q = (q)\xi_Q \\ &= (a'p')\xi_Q = (p')\ \mathrm{F}_S(a'^{\,\mathrm{op}})\,\xi_Q \\ &= (p')\xi_{P'}\, F(a'^{\,\mathrm{op}}).\end{aligned}$$

Suppose given $[\xi', S'] \in F^+(C)$ such that $[\xi, S] = [\xi', S']$. We may choose a pure epimorphism $\big(P \xrightarrow{p} C\big) \in S \cap S'$. Since we already know that the definition is independent of the choice of the epimorphism, we conclude that it is also independent of the choice of the representative:

$$[\xi, S]u^F_C = [(p)\xi_P\,,\, P \xrightarrow{p} C] = [\xi', S']u^F_C\,.$$

3. We show that v^F_C is a well-defined map.

Suppose given $[x, P \xrightarrow{p} C] \in F^{✠}(C)$. Suppose given $X \overset{g}{\underset{h}{\rightrightarrows}} P$ in $\mathcal{A}$ such that $gp = hp$. So $(g-h)p = 0$ and thus we may choose $X \xrightarrow{u} K_p$ in $\mathcal{A}$ such that $uk_p = g - h$ since k_p is a kernel of p. We have

$$\begin{aligned}(x)\,F(g^{\mathrm{op}}) - (x)\,F(h^{\mathrm{op}}) &= (x)(F(g^{\mathrm{op}}) - F(h^{\mathrm{op}})) = (x)\,F((g-h)^{\mathrm{op}}) = (x)F((uk_p)^{\mathrm{op}}) \\ &= (x)\,F(k_p^{\mathrm{op}})\,F(u^{\mathrm{op}}) = (0)\,F(u^{\mathrm{op}}) = 0.\end{aligned}$$

We conclude that $(x)\,F(g^{\mathrm{op}}) = (x)\,F(h^{\mathrm{op}})$.

Thus ${}^{p}_{x}\xi_X\colon\ \mathrm{S}_p \cap \mathrm{Hom}_{\mathcal{A}}(X, C) \to F(X)$ is well-defined.

Suppose given $Y \xrightarrow{h} X$ in $\mathcal{A}$. For $X \xrightarrow{g} P$ in $\mathcal{A}$, we have

$$(gp)\,{}^{p}_{x}\xi_X\, F(h^{\mathrm{op}}) = (x)\,F(g^{\mathrm{op}})\,F(h^{\mathrm{op}}) = (x)\,F((hg)^{\mathrm{op}}) = (hg)\,{}^{p}_{x}\xi_Y = (g)\ \mathrm{F}_{\mathrm{S}_p}(h^{\mathrm{op}})\,{}^{p}_{x}\xi_Y\,.$$

Thus ${}^{p}_{x}\xi$ is natural.

$$\begin{tikzcd}
\mathrm{F}_{\mathrm{S}_p}(X) \arrow[r, "{}^{p}_{x}\xi_X"] \arrow[d, "\mathrm{F}_{\mathrm{S}_p}(h^{\mathrm{op}})"'] & F(X) \arrow[d, "F(h^{\mathrm{op}})"] \\
\mathrm{F}_{\mathrm{S}_p}(Y) \arrow[r, "{}^{p}_{x}\xi_Y"] & F(Y)
\end{tikzcd}$$

Suppose $[x,\, P \xrightarrow{p} C] = [x',\, P' \xrightarrow{p'} C]$. Then we have a commutative diagram

$$\begin{tikzcd}
 & P \arrow[dr, "p"] & \\
Q \arrow[ur, "q"] \arrow[dr, "q'"'] & & C \\
 & P' \arrow[ur, "p'"'] &
\end{tikzcd}$$

in $\mathcal{A}$ such that p, p' are purely epimorphic, such that $qp = q'p'$ is purely epimorphic and such that $(x)\, F(q^{\mathrm{op}}) = (x')\, F(q'^{\,\mathrm{op}})$. Note that $\mathrm{S}_{qp} \subseteq \mathrm{S}_p \cap \mathrm{S}_{p'}$. For $R \xrightarrow{r} Q$ in $\mathcal{A}$, we have

$$\begin{aligned}
(rqp)\, {}^{p}_{x}\xi_R &= (x)\, F((rq)^{\mathrm{op}}) = (x)\, F(q^{\mathrm{op}})\, F(r^{\mathrm{op}}) = (x')\, F(q'^{\,\mathrm{op}})\, F(r^{\mathrm{op}}) = (x')\, F((q'r)^{\mathrm{op}}) \\
&= (rq'p')\, {}^{p'}_{x'}\xi_R = (rqp)\, {}^{p'}_{x'}\xi_R\,.
\end{aligned}$$

We conclude that $[{}^{p}_{x}\xi, \mathrm{S}_p] = \big[{}^{p'}_{x'}\xi, \mathrm{S}_{p'}\big]$.

So the definition is independent of the choice of the representative.

4. We show that u^F_C and v^F_C are mutually inverse.

Suppose given $[\xi, S] \in F^+(C)$. Choose a pure epimorphism $\big(P \xrightarrow{p} C\big) \in S$.

We have

$$(([\xi, S])u^F_C)v^F_C = ([(p)\xi_P,\, P \xrightarrow{p} C])v^F_C = \Big[{}^{\;p}_{(p)\xi_P}\xi, \mathrm{S}_p\Big].$$

Note that $\mathrm{S}_p \subseteq S$. For $X \xrightarrow{g} P$ in $\mathcal{A}$, we have

$$(gp)\, {}^{\;p}_{(p)\xi_P}\xi_X = (p)\xi_P\, F(g^{\mathrm{op}}) = (p)\ \mathrm{F}_S(g^{\mathrm{op}})\, \xi_X = (gp)\xi_X\,.$$

Thus $[\xi, S] = \Big[{}^{\;p}_{(p)\xi_P}\xi, \mathrm{S}_p\Big]$ and, consequently, $u^F_C \cdot v^F_C = 1$.

Conversely, suppose given $[x,\, P \xrightarrow{p} C] \in F^{\maltese}(C)$. We have

$$(([x,\, P \xrightarrow{p} C])v^F_C)u^F_C = ([{}^{p}_{x}\xi, \mathrm{S}_p])u^F_C = [(p)\, {}^{p}_{x}\xi_P,\, P \xrightarrow{p} C]$$

and $(p)\, {}^{p}_{x}\xi_P = (x)\, F(1^{\mathrm{op}}) = x$.

Thus $[(p)\, {}^{p}_{x}\xi_P,\, P \xrightarrow{p} C] = [x,\, P \xrightarrow{p} C]$ and, consequently, $v^F_C \cdot u^F_C = 1$.

5. We show that $F^{\maltese}(f^{\mathrm{op}})$ is a well-defined map.

   Suppose given $D \xrightarrow{f} C$ in $\mathcal{A}$ and $[x, P \xrightarrow{p} C] \in F^{\maltese}(C)$.

   Choose a pullback

$$\begin{tikzcd}
Q \arrow[r, "q", "\shortmid" marking] \arrow[d, "r"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "f"] \\
P \arrow[r, "p", "\shortmid" marking] & C
\end{tikzcd}$$

   in $\mathcal{A}$. For a commutative diagram

$$\begin{tikzcd}
P' \arrow[r, "p'", "\shortmid" marking] \arrow[d, "f'"'] & D \arrow[d, "f"] \\
P \arrow[r, "p", "\shortmid" marking] & C
\end{tikzcd}$$

   in $\mathcal{A}$ such that p' is a pure epimorphism, we obtain $P' \xrightarrow{u} Q$ in $\mathcal{A}$ such that $uq = p'$ and $ur = f'$. Thus the diagram

$$\begin{tikzcd}
& P' \arrow[dr, "p'", "\shortmid" marking] & \\
P' \arrow[ur, "1"] \arrow[dr, "u"'] & & D \\
& Q \arrow[ur, "q"', "\shortmid" marking] &
\end{tikzcd}$$

   commutes in $\mathcal{A}$ and $uq = p'$ is a pure epimorphism. We have

$$(x)\,F(f'^{\,\mathrm{op}})\,F(1^{\mathrm{op}}) = (x)\,F((ur)^{\mathrm{op}}) = (x)\,F(r^{\mathrm{op}})\,F(u^{\mathrm{op}})$$

   and, consequently, $[(x)\,F(f'^{\,\mathrm{op}}), P' \xrightarrow{p'} D] = [(x)\,F(r^{\mathrm{op}}), Q \xrightarrow{q} D]$. We conclude that this definition is independent of the choice of the commutative diagram.

   Moreover, let $K_{p'} \xrightarrow{\ell} K_p$ be the induced morphism between the kernels such that the diagram

$$\begin{tikzcd}
K_{p'} \arrow[r, "k_{p'}", "\bullet" marking] \arrow[d, "\ell"'] & P' \arrow[r, "p'", "\shortmid" marking] \arrow[d, "f'"'] & D \arrow[d, "f"] \\
K_p \arrow[r, "k_p", "\bullet" marking] & P \arrow[r, "p", "\shortmid" marking] & C
\end{tikzcd}$$

   commutes. We obtain

$$(x)F(f'^{\,\mathrm{op}})F(k_{p'}^{\mathrm{op}}) = (x)F((k_{p'}f')^{\mathrm{op}}) = (x)F((\ell k_p)^{\mathrm{op}}) = (x)F(k_p^{\mathrm{op}})F(\ell^{\mathrm{op}}) = (0)F(\ell^{\mathrm{op}}) = 0.$$

So indeed $[(x)\,F(f'^{\,\mathrm{op}}),\, P' \xrightarrow{p'} D] \in F^{\maltese}(D)$.

Now suppose given $[x',\, P' \xrightarrow{p'} C] \in F^{\maltese}(C)$ such that $[x,\, P \xrightarrow{p} C] = [x',\, P' \xrightarrow{p'} C]$. There exists a commutative diagram

$$\begin{tikzcd} & P \arrow[dr, "p"] & \\ Q \arrow[ur, "q"] \arrow[dr, "q'"'] & & C \\ & P' \arrow[ur, "p'"'] & \end{tikzcd}$$

in $\mathcal{A}$ such that $qp = q'p'$ is purely epimorphic.

Choose a pullback

$$\begin{tikzcd} R \arrow[r, "r"] \arrow[d, "s"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "f"] \\ Q \arrow[r, "qp"] & C \end{tikzcd}$$

in $\mathcal{A}$. Note that r is purely epimorphic. We obtain the commutative diagrams

$$\begin{tikzcd} R \arrow[r, "r"] \arrow[d, "sq"'] & D \arrow[d, "f"] \\ P \arrow[r, "p"] & C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} R \arrow[r, "r"] \arrow[d, "sq'"'] & D \arrow[d, "f"] \\ P' \arrow[r, "p'"] & C \end{tikzcd}$$

in $\mathcal{A}$. Thus $[(x)\,F((sq)^{\mathrm{op}}),\, R \xrightarrow{r} D] = [(x')\,F((sq')^{\mathrm{op}}),\, R \xrightarrow{r} D]$.

We conclude that this definition is independent of the choice of a representative.

6. We show that u^F is natural.

   Suppose given $D \xrightarrow{f} C$ in $\mathcal{A}$ and $[\xi, S] \in F^{+}(C)$. Choose a pure epimorphism $\big(P \xrightarrow{p} C\big) \in S$ and a pullback

$$\begin{tikzcd} P' \arrow[r, "p'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "f"] \\ P \arrow[r, "p"] & C \end{tikzcd}$$

   in $\mathcal{A}$. Note that p' is purely epimorphic and that $p' \in f^*(S)$ since $p'f = f'p \in S$. We have

$$\begin{aligned} [\xi, S]\, u^F_C\, F^{\maltese}(f^{\mathrm{op}}) &= [(p)\xi_P\,,\, P \xrightarrow{p} C]\, F^{\maltese}(f^{\mathrm{op}}) = [(p)\xi_P\, F(f'^{\,\mathrm{op}}),\, P' \xrightarrow{p'} D] \\ &= [(p)\ \mathrm{F}_S(f'^{\,\mathrm{op}})\, \xi_{P'}\,,\, P' \xrightarrow{p'} D] = [(f'p)\xi_{P'}\,,\, P' \xrightarrow{p'} D] \\ &= [(p'f)\xi_{P'}\,,\, P' \xrightarrow{p'} D] = [(p')\xi^f_{P'}\,,\, P' \xrightarrow{p'} D] \\ &= [\xi^f, f^*(S)]u^F_C = [\xi, S]\, F^{+}(f^{\mathrm{op}})\, u^F_C\,, \end{aligned}$$

cf. definition 2.3.7.

Thus $u^F_C \, F^{\maltese}(f^{\mathrm{op}}) = F^+(f^{\mathrm{op}}) \, u^F_C$.

$$\begin{tikzcd}
F^+(C) \arrow[r, "u^F_C"] \arrow[d, "F^+(f^{\mathrm{op}})"'] & F^{\maltese}(C) \arrow[d, "F^{\maltese}(f^{\mathrm{op}})"] \\
F^+(D) \arrow[r, "u^F_D"] & F^{\maltese}(D)
\end{tikzcd}$$

□

Lemma/Definition 2.5.10. Suppose given an additive $\mathbf{Z}$-presheaf $F\colon \mathcal{A}^{\mathrm{op}} \to \text{Mod-}\mathbf{Z}$.

Suppose given $C \in \operatorname{Ob} \mathcal{A}$.

Let $\breve{F}_0(C)$ denote the set of all pairs $(x,\, Q \xrightarrow{q} P \xrightarrow{p} C)$, where p and q are pure epimorphisms in $\mathcal{A}$ and $x \in F(Q)$ such that the following two conditions hold. Let r denote the induced morphism between the kernels K_{qp} and K_p such that the diagram

$$\begin{tikzcd}
K_{qp} \arrow[r, "k_{qp}", "\bullet" marking] \arrow[d, "r"'] & Q \arrow[r, "qp", "\shortmid" marking] \arrow[d, "q", "\shortmid" marking] & C \arrow[d, "1"] \\
K_p \arrow[r, "k_p", "\bullet" marking] & P \arrow[r, "p", "\shortmid" marking] & C
\end{tikzcd}$$

commutes in $\mathcal{A}$.

- $(x)\, F(k_q^{\mathrm{op}}) = 0$
- There exists $V \xrightarrow{v} K_{qp}$ in $\mathcal{A}$ such that vr is a pure epimorphism and such that $(x)\, F((vk_{qp})^{\mathrm{op}}) = 0$.

For $(x,\, Q \xrightarrow{q} P \xrightarrow{p} C), (x',\, Q' \xrightarrow{q'} P' \xrightarrow{p'} C) \in \breve{F}_0(C)$, let
$(x,\, Q \xrightarrow{q} P \xrightarrow{p} C) \sim (x',\, Q' \xrightarrow{q'} P' \xrightarrow{p'} C)$ if and only if there exists a commutative diagram

$$\begin{tikzcd}
& Q \arrow[dr, "q", "\shortmid" marking] & & \\
& & P \arrow[dr, "p", "\shortmid" marking] & \\
M \arrow[uur, "m"] \arrow[ddr, "m'"'] & & & C \\
& & P' \arrow[ur, "p'"', "\shortmid" marking] & \\
& Q' \arrow[ur, "q'"', "\shortmid" marking] & &
\end{tikzcd}$$

in $\mathcal{A}$ such that $mqp = m'q'p'$ is purely epimorphic and such that $(x)\, F(m^{\mathrm{op}}) = (x')\, F(m'^{\mathrm{op}})$.

This defines an equivalence relation on $\breve{F}_0(C)$. Let $\breve{F}(C) := \breve{F}_0(C)/\sim$ denote the factor set. Let $[x, Q \xrightarrow{q} P \xrightarrow{p} C] \in \breve{F}(C)$ denote the equivalence class of $(x, Q \xrightarrow{q} P \xrightarrow{p} C) \in \breve{F}_0(C)$. The map

$$\begin{aligned} F^{\maltese\maltese}(C) &\xrightarrow{s_C^F} \breve{F}(C) \\ [[x, Q \xrightarrow{q} P], P \xrightarrow{p} C] &\longmapsto [x, Q \xrightarrow{q} P \xrightarrow{p} C] \end{aligned}$$

is a bijection with inverse

$$\begin{aligned} \breve{F}(C) &\xrightarrow{t_C^F} F^{\maltese\maltese}(C) \\ [x, Q \xrightarrow{q} P \xrightarrow{p} C] &\longmapsto [[x, Q \xrightarrow{q} P], P \xrightarrow{p} C]. \end{aligned}$$

By defining the $\mathbf{Z}$-module structure on $\breve{F}(C)$ via s_C^F , we obtain $F^{\maltese\maltese}(C) \cong \breve{F}(C)$ in Mod-$\mathbf{Z}$. By lemma 2.5.9, we have the isotransformations

$$F^{++} \xrightarrow[\sim]{(u^F)^+} (F^{\maltese})^+ \xrightarrow[\sim]{u^{F^{\maltese}}} F^{\maltese\maltese}.$$

Thus

$$\tilde{F}(C) = F^{++}(C) \cong (F^{\maltese})^+(C) \cong F^{\maltese\maltese}(C) \cong \breve{F}(C)$$

in Mod-$\mathbf{Z}$.

Proof.

1. We show that $\sim$ defines an equivalence relation on $\breve{F}_0(C)$.

   Using identities, we see that $\sim$ is reflexive.

   By construction, $\sim$ is symmetric.

   Suppose given

   $(x, Q \xrightarrow{q} P \xrightarrow{p} C), (x', Q' \xrightarrow{q'} P' \xrightarrow{p'} C), (x'', Q'' \xrightarrow{q''} P'' \xrightarrow{p''} C) \in \breve{F}_0(C)$

   such that $(x, Q \xrightarrow{q} P \xrightarrow{p} C) \sim (x', Q' \xrightarrow{q'} P' \xrightarrow{p'} C)$ and

   $(x', Q' \xrightarrow{q'} P' \xrightarrow{p'} C) \sim (x'', Q'' \xrightarrow{q''} P'' \xrightarrow{p''} C)$.

We may choose commutative diagrams

$$
\begin{tikzcd}
 & Q \arrow[dr, "q"] & & \\
 & & P \arrow[dr, "p"] & \\
M \arrow[uur, "m"] \arrow[ddr, "m'"'] & & & C \\
 & & P' \arrow[ur, "p'"'] & \\
 & Q' \arrow[ur, "q'"'] & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & Q' \arrow[dr, "q'"] & & \\
 & & P' \arrow[dr, "p'"] & \\
L \arrow[uur, "\ell'"] \arrow[ddr, "\ell''"'] & & & C \\
 & & P'' \arrow[ur, "p''"'] & \\
 & Q'' \arrow[ur, "q''"'] & &
\end{tikzcd}
$$

in $\mathcal{A}$ such that mqp and $\ell' q' p'$ are purely epimorphic and such that
$(x)\, F(m^{\mathrm{op}}) = (x')\, F(m'^{\,\mathrm{op}})$ and $(x')\, F(\ell'^{\,\mathrm{op}}) = (x'')\, F(\ell''^{\,\mathrm{op}})$.

Let r denote the induced morphism between the kernels $K_{q'p'}$ and $K_{p'}$ such that the diagram

$$
\begin{tikzcd}
K_{q'p'} \arrow[r, "k_{q'p'}"] \arrow[d, "r"'] & Q \arrow[r, "q'p'"] \arrow[d, "q'"] & C \arrow[d, "1"] \\
K_{p'} \arrow[r, "k_{p'}"] & P' \arrow[r, "p'"] & C
\end{tikzcd}
$$

commutes in $\mathcal{A}$. There exists $V \xrightarrow{v} K_{q'p'}$ in $\mathcal{A}$ such that vr is a pure epimorphism and such that $(x')\, F((vk_{q'p'})^{\mathrm{op}}) = 0$.

Choose a pullback

$$
\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d, "m'q'p'"] \\
L \arrow[r, "\ell' q' p'"] & C
\end{tikzcd}
$$

in $\mathcal{A}$. Note that s and t are purely epimorphic. We have $(sm'q' - t\ell' q')p' = 0$. So we may choose $S \xrightarrow{j} K_{p'}$ in $\mathcal{A}$ such that $sm'q' - t\ell' q' = jk_{p'}$.

Choose a pullback

$$
\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, "z"'] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d, "j"] \\
V \arrow[r, "vr"] & K_{p'}
\end{tikzcd}
$$

in $\mathcal{A}$. Note that y is purely epimorphic. We have

$$
\begin{aligned}
(ysm' - yt\ell' - zvk_{q'p'})q' &= ysm'q' - yt\ell' q' - zvk_{q'p'}q' = y(sm'q' - t\ell' q') - zvrk_{p'} \\
&= yjk_{p'} - yjk_{p'} = 0.
\end{aligned}
$$

So we may choose $Y \xrightarrow{k} K_{q'}$ in $\mathcal{A}$ such that $ysm' - yt\ell' - zvk_{q'p'} = kk_{q'}$, i.e. $ysm' - yt\ell' = zvk_{q'p'} + kk_{q'}$.

We obtain

$$\begin{aligned}(x')\,F((ysm')^{\mathrm{op}}) - (x')\,F((yt\ell')^{\mathrm{op}}) &= (x')\,F((ysm' - yt\ell')^{\mathrm{op}}) = (x')\,F((zvk_{q'p'} + kk_{q'})^{\mathrm{op}}) \\ &= (x')\,F((vk_{q'p'})^{\mathrm{op}})\,F(z^{\mathrm{op}}) + (x')\,F(k_{q'}^{\mathrm{op}})\,F(k^{\mathrm{op}}) = 0\end{aligned}$$

and thus

$$\begin{aligned}(x)\,F((ysm)^{\mathrm{op}}) &= (x)\,F(m^{\mathrm{op}})\,F((ys)^{\mathrm{op}}) = (x')\,F(m'^{\,\mathrm{op}})\,F((ys)^{\mathrm{op}}) \\ &= (x')\,F((ysm')^{\mathrm{op}}) = (x')\,F((yt\ell')^{\mathrm{op}}) = (x')\,F(\ell'^{\,\mathrm{op}})\,F((yt)^{\mathrm{op}}) \\ &= (x'')\,F(\ell''^{\,\mathrm{op}})\,F((yt)^{\mathrm{op}}) = (x'')\,F((yt\ell'')^{\mathrm{op}}).\end{aligned}$$

Moreover, the diagram

$$\begin{tikzcd}
 & Q \arrow[dr, "q"] & & \\
 & & P \arrow[dr, "p"] & \\
Y \arrow[uur, "ysm"] \arrow[ddr, "yt\ell''"'] & & & C \\
 & & P'' \arrow[ur, "p''"'] & \\
 & Q'' \arrow[ur, "q''"'] & &
\end{tikzcd}$$

commutes in $\mathcal{A}$ and $ysmqp$ is purely epimorphic.

We conclude that $\sim$ is transitive.

2. We want to show that s_C^F and t_C^F are well-defined maps.

Suppose given $Q \xrightarrow{q} P \xrightarrow{p} C$ in $\mathcal{A}$ such that q and p are purely epimorphic. Suppose given $x \in F(Q)$ such that $(x)\,F(k_q^{\mathrm{op}}) = 0$.

Let r denote the induced morphism between the kernels k_{qp} and k_p such that the diagram

$$\begin{tikzcd}
K_{qp} \arrow[r, "k_{qp}"] \arrow[d, "r"'] & Q \arrow[r, "qp"] \arrow[d, "q"] & C \arrow[d, "1"] \\
K_p \arrow[r, "k_p"] & P \arrow[r, "p"] & C
\end{tikzcd}$$

commutes in $\mathcal{A}$. By lemma 1.1.2.(a), the diagram

$$\begin{tikzcd}
K_{qp} \arrow[r, "k_{qp}"] \arrow[d, "r"'] & Q \arrow[d, "q"] \\
K_p \arrow[r, "k_p"] & P
\end{tikzcd}$$

is a pullback and thus r is purely epimorphic.

We have $[x, Q \xrightarrow{q} P]\, F^{\maltese}(k_p^{\mathrm{op}}) = [(x)\, F(k_{qp}^{\mathrm{op}}),\, K_{qp} \xrightarrow{r} K_p]$.

So $[x, Q \xrightarrow{q} P]\, F^{\maltese}(k_p^{\mathrm{op}}) = 0$ if and only if there exists $V \xrightarrow{v} K_{qp}$ in $\mathcal{A}$ such that vr is purely epimorphic and such that $(x)\, F((vk_{qp})^{\mathrm{op}}) = 0$.

Thus $[[x, Q \xrightarrow{q} P], P \xrightarrow{p} C] \in F^{\maltese\maltese}(C)$ if and only if $[x, Q \xrightarrow{q} P \xrightarrow{p} C] \in \breve{F}(C)$.

Now suppose given $[[x, Q \xrightarrow{q} P], P \xrightarrow{p} C], [[x', Q' \xrightarrow{q'} P'], P' \xrightarrow{p'} C] \in F^{\maltese\maltese}(C)$ such that $[[x, Q \xrightarrow{q} P], P \xrightarrow{p} C] = [[x', Q' \xrightarrow{q'} P'], P' \xrightarrow{p'} C]$.

So we may choose a commutative diagram

$$\begin{tikzcd}
 & P \arrow[dr, "p", "|" marking] & \\
R \arrow[ur, "r"] \arrow[dr, "r'"'] & & C \\
 & P' \arrow[ur, "p'"', "|" marking] &
\end{tikzcd}$$

in $\mathcal{A}$ such that rp is purely epimorphic and such that

$[x, Q \xrightarrow{q} P]\, F^{\maltese}(r^{\mathrm{op}}) = [x', Q' \xrightarrow{q'} P']\, F^{\maltese}(r'^{\,\mathrm{op}})$.

Choose pullbacks

$$\begin{tikzcd}
S \arrow[r, "s", "|" marking] \arrow[d, "s'"'] \arrow[dr, phantom, "\ulcorner" very near start] & R \arrow[d, "r"] \\
Q \arrow[r, "q", "|" marking] & P
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T \arrow[r, "t", "|" marking] \arrow[d, "t'"'] \arrow[dr, phantom, "\ulcorner" very near start] & R \arrow[d, "r'"] \\
Q' \arrow[r, "q'", "|" marking] & P'
\end{tikzcd}$$

in $\mathcal{A}$. Note that s and t are purely epimorphic. We have

$$\begin{aligned}
[(x)\, F(s'^{\,\mathrm{op}}), S \xrightarrow{s} R] = [x, Q \xrightarrow{q} P]\, F^{\maltese}(r^{\mathrm{op}}) &= [x', Q' \xrightarrow{q'} P'] F^{\maltese}(r'^{\,\mathrm{op}}) \\
&= [(x')\, F(t'^{\,\mathrm{op}}), T \xrightarrow{t} R].
\end{aligned}$$

So we may choose a commutative diagram

$$\begin{tikzcd}
 & S \arrow[dr, "s", "|" marking] & \\
U \arrow[ur, "u"] \arrow[dr, "u'"'] & & R \\
 & T \arrow[ur, "t"', "|" marking] &
\end{tikzcd}$$

in $\mathcal{A}$ such that us is purely epimorphic and such that

$(x)\, F(s'^{\,\mathrm{op}})\, F(u^{\mathrm{op}}) = (x')\, F(t'^{\,\mathrm{op}})\, F(u'^{\,\mathrm{op}})$.

We obtain the commutative diagram

$$\begin{tikzcd}
 & Q \arrow[dr, "q", "\shortmid"{marking}] & & \\
 & & P \arrow[dr, "p", "\shortmid"{marking}] & \\
U \arrow[uur, "us'"] \arrow[ddr, "u't'"'] & & & C \\
 & & P' \arrow[ur, "p'"', "\shortmid"{marking}] & \\
 & Q' \arrow[ur, "q'"', "\shortmid"{marking}] & &
\end{tikzcd}$$

in $\mathcal{A}$ with $us'qp = usrp$ purely epimorphic and $(x)\, F((us')^{\mathrm{op}}) = (x')\, F((u't')^{\mathrm{op}})$.

So $[x,\, Q \overset{q}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P \overset{p}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C] = [x',\, Q' \overset{q'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P' \overset{p'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C]$.

Conversely, suppose given $[x,\, Q \overset{q}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P \overset{p}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C], [x',\, Q' \overset{q'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P' \overset{p'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C] \in \breve{F}(C)$ such that $[x,\, Q \overset{q}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P \overset{p}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C] = [x',\, Q' \overset{q'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P' \overset{p'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C]$.
We may choose a commutative diagram

$$\begin{tikzcd}
 & Q \arrow[dr, "q", "\shortmid"{marking}] & & \\
 & & P \arrow[dr, "p", "\shortmid"{marking}] & \\
M \arrow[uur, "m"] \arrow[ddr, "m'"'] & & & C \\
 & & P' \arrow[ur, "p'"', "\shortmid"{marking}] & \\
 & Q' \arrow[ur, "q'"', "\shortmid"{marking}] & &
\end{tikzcd}$$

in $\mathcal{A}$ such that $mqp = m'q'p'$ is purely epimorphic and such that $(x)\, F(m^{\mathrm{op}}) = (x')\, F(m'^{\,\mathrm{op}})$.

We have $[x,\, Q \overset{q}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P]\, F^{\maltese}((mq)^{\mathrm{op}}) = [(x)\, F(m^{\mathrm{op}}),\, M \xrightarrow{1} M]$ and, similarly,
$[x',\, Q' \overset{q'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P']\, F^{\maltese}((m'q')^{\mathrm{op}}) = [(x')\, F(m'^{\,\mathrm{op}}),\, M \xrightarrow{1} M]$.

Thus $[[x,\, Q \overset{q}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P],\, P \overset{p}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C] = [[x',\, Q' \overset{q'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} P'],\, P' \overset{p'}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C]$.

3. The maps s_C^F and t_C^F are mutually inverse by construction. $\square$

Chapter 3

Localisation

Suppose given an abelian category $\mathcal{A}$.

From after remark 3.1.2 on, we will suppose given a thick subcategory $\mathcal{N}$ of $\mathcal{A}$.

3.1 Thick subcategories

3.1.1 Definition and basic properties

Definition 3.1.1. A full non-empty subcategory $\mathcal{N}$ of $\mathcal{A}$ is called *thick in* $\mathcal{A}$ if for each short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$, we have $B \in \operatorname{Ob}\mathcal{N}$ if and only if $A, C \in \operatorname{Ob}\mathcal{N}$.

Remark 3.1.2. A full non-empty subcategory $\mathcal{N}$ of $\mathcal{A}$ is thick in $\mathcal{A}$ if and only if it is closed under extensions, subobjects and factor objects. Cf. convention 40.

Suppose given a thick subcategory $\mathcal{N}$ of $\mathcal{A}$ for the remainder of this chapter 3.

Remark 3.1.3.
We use the term thick in the sense of [13, defn. 19.5.4]. Some use the 'Serre subcategory' instead.

Lemma 3.1.4. Suppose given $n \in \mathbf{N}_0$ and an exact sequence

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} A_{n-1} \xrightarrow{a_{n-1}} A_n$$

in $\mathcal{A}$.

Suppose that $i \in [1, n-1]$ and that $A_{i-1}, A_{i+1} \in \operatorname{Ob}\mathcal{N}$. Then $A_i \in \operatorname{Ob}\mathcal{N}$ as well.

Proof. Choose images

$$\begin{tikzcd}
A_{i-1} \arrow[rr, "a_i"] \arrow[dr, "p_i"'] & & A_i \arrow[rr, "a_{i+1}"] \arrow[dr, "p_{i+1}"'] & & A_{i+1} \\
& I_i \arrow[ur, "j_i"'] & & I_{i+1} \arrow[ur, "j_{i+1}"'] &
\end{tikzcd}$$

in $\mathcal{A}$. We have $I_i \in \operatorname{Ob}\mathcal{N}$ since $A_{i-1} \in \operatorname{Ob}\mathcal{N}$ and, similarly, $I_{i+1} \in \operatorname{Ob}\mathcal{N}$ since $A_{i+1} \in \operatorname{Ob}\mathcal{N}$. Thus $A_i \in \operatorname{Ob}\mathcal{N}$ since $I_i, I_{i+1} \in \operatorname{Ob}\mathcal{N}$. □

Lemma 3.1.5. The thick subcategory $\mathcal{N}$ of $\mathcal{A}$ is a full additive subcategory of $\mathcal{A}$ which is closed under isomorphisms.

Proof. Choose an object $A \in \operatorname{Ob}\mathcal{N}$. The sequence $0_{\mathcal{A}} \xrightarrow{0} A \xrightarrow{1} A$ is short exact in $\mathcal{A}$ and, consequently, we have $0_{\mathcal{A}} \in \operatorname{Ob}\mathcal{N}$.

Suppose given objects $A, B \in \operatorname{Ob}\mathcal{N}$. The sequence $A \xrightarrow{(1\ 0)} A \oplus B \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} B$ is short exact in $\mathcal{A}$ so that we have $A \oplus B \in \operatorname{Ob}\mathcal{N}$. Thus $\mathcal{N}$ is a full additive subcategory of $\mathcal{A}$.

Suppose given an isomorphism $A \xrightarrow{f} B$ in $\mathcal{A}$ with $A \in \operatorname{Ob}\mathcal{N}$. The sequence $0_{\mathcal{A}} \xrightarrow{0} A \xrightarrow{f} B$ is short exact in $\mathcal{A}$ and, consequently, we have $B \in \operatorname{Ob}\mathcal{N}$. □

Remark 3.1.6. Suppose given $A \xrightarrow{f} B$ in $\mathcal{N}$. Suppose given a kernel $K \xrightarrow{k} A$ of f in $\mathcal{A}$ and a cokernel $B \xrightarrow{c} C$ of f in $\mathcal{A}$. Then k is a kernel of f in $\mathcal{N}$ and c is a cokernel of f in $\mathcal{N}$ as well.

Thus $\mathcal{N}$ is an abelian category and the inclusion functor from $\mathcal{N}$ to $\mathcal{A}$ is exact.

Remark 3.1.7. The subcategory $\mathcal{N}^{\mathrm{op}}$ of $\mathcal{A}^{\mathrm{op}}$ is a thick subcategory of $\mathcal{A}^{\mathrm{op}}$. So we may use duality arguments.

Corollary 3.1.8. Suppose given an abelian category $\mathcal{B}$ and an exact functor $F\colon \mathcal{A} \to \mathcal{B}$. Recall that the kernel of F is the full subcategory $\operatorname{Ker}(F)$ of $\mathcal{A}$ defined by

$$\operatorname{Ob}(\operatorname{Ker}(F)) := \{A \in \operatorname{Ob}\mathcal{A} \colon F(A) \cong 0_{\mathcal{B}} \text{ in } \mathcal{B}\},$$

cf. convention 41.

The kernel $\operatorname{Ker}(F)$ of F is a thick subcategory of $\mathcal{A}$.

Cf. remark 3.2.19 below.

Proof. Since F is additive, we have $F(0_{\mathcal{A}}) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$. Thus $\operatorname{Ker}(F)$ is non-empty.

Suppose given a short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. Since F is exact, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is short exact in $\mathcal{B}$.
We have $F(B) \cong 0_{\mathcal{B}}$ if and only if $F(A), F(C) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$ by lemma 1.1.7.

Thus $B \in \operatorname{Ob}(\operatorname{Ker}(F))$ if and only if $A, C \in \operatorname{Ob}(\operatorname{Ker}(F))$. □

3.1.2 Generating thick subcategories

We will not use the results of this section 3.1.2 in the remaining sections 3.2 and 3.3 of chapter 3.

Suppose given a full subcategory $\mathcal{M}$ of $\mathcal{A}$.

3.1.2.1 Definition and basic properties

Remark 3.1.9. An arbitrary intersection of thick subcategories of $\mathcal{A}$ is thick. More precisely, suppose given a set I and thick subcategories $\mathcal{L}_i$ of $\mathcal{A}$ for $i \in I$. Then $\bigcap\{\mathcal{L}_i \colon i \in I\}$ is a thick subcategory of $\mathcal{A}$.

Definition 3.1.10. Let

$$_{\mathcal{A}}\langle\mathcal{M}\rangle := \bigcap\{\mathcal{L} \colon \mathcal{L} \text{ is a thick subcategory of } \mathcal{A} \text{ such that } \mathcal{M} \subseteq \mathcal{L}\}$$

denote the *thick subcategory generated by* $\mathcal{M}$ *in* $\mathcal{A}$. We write $\langle\mathcal{M}\rangle := {}_{\mathcal{A}}\langle\mathcal{M}\rangle$ if unambiguous. Note that $\langle\mathcal{M}\rangle$ is the smallest thick subcategory of $\mathcal{A}$ containing $\mathcal{M}$. Cf. remark 3.1.9.

We present two ways of describing thick subcategories generated by a full subcategory in the sections 3.1.2.2 and 3.1.2.3.

Remark 3.1.11. Suppose given an abelian category $\mathcal{B}$ and an exact functor $F \colon \mathcal{A} \to \mathcal{B}$. If $\mathcal{M} \subseteq \mathrm{Ker}(F)$, then $\langle\mathcal{M}\rangle \subseteq \mathrm{Ker}(F)$.

So if $F(M) \cong 0_{\mathcal{B}}$ for $M \in \mathrm{Ob}\,\mathcal{M}$, then $F(A) \cong 0_{\mathcal{B}}$ for $A \in \mathrm{Ob}(\langle\mathcal{M}\rangle)$.

Proof. By corollary 3.1.8, the kernel $\mathrm{Ker}(F)$ of F is a thick subcategory of $\mathcal{A}$. So if $\mathcal{M} \subseteq \mathrm{Ker}(F)$, then $\langle\mathcal{M}\rangle \subseteq \mathrm{Ker}(F)$ since $\langle\mathcal{M}\rangle$ is the smallest thick subcategory of $\mathcal{A}$ containing $\mathcal{M}$. □

3.1.2.2 Recursive description

Definition 3.1.12. Suppose given a full subcategory $\mathcal{L}$ of $\mathcal{A}$.

An object $A \in \mathrm{Ob}\,\mathcal{A}$ is called $\mathcal{L}$-*thick* if one of the following three statements holds.

- We have $A \in \mathrm{Ob}\,\mathcal{L}$.
- There exists a short exact sequence $L \rightarrowtail A \twoheadrightarrow L'$ in $\mathcal{A}$ with $L, L' \in \mathrm{Ob}\,\mathcal{L}$.
- There exists a short exact sequence $B \rightarrowtail L \twoheadrightarrow B'$ in $\mathcal{A}$ with $L \in \mathrm{Ob}\,\mathcal{L}$ and $A \in \{B, B'\}$.

Proposition 3.1.13. Let $\mathcal{M}_0$ be the full subcategory of $\mathcal{A}$ defined by

$$\mathrm{Ob}(\mathcal{M}_0) := \{A \in \mathrm{Ob}\,\mathcal{A}\colon A \cong M \text{ in } \mathcal{A} \text{ for some } M \in \mathrm{Ob}\,\mathcal{M} \cup \{0_{\mathcal{A}}\}\}.$$

For $i \in \mathbf{Z}_{>0}$, let $\mathcal{M}_i$ be the full subcategory of $\mathcal{A}$ defined recursively by

$$\mathrm{Ob}(\mathcal{M}_i) := \{A \in \mathrm{Ob}\,\mathcal{A}\colon A \text{ is } (\mathcal{M}_{i-1})\text{-thick}\}.$$

We have

$$\langle \mathcal{M} \rangle = \bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i\,.$$

Proof. By construction, $\mathcal{M}_0$ and, consequently, $\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i$ are non-empty and we have $\mathcal{M} \subseteq \mathcal{M}_0 \subseteq \bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i$. Note that $\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$ for $j \in \mathbf{Z}_{\geq 0}$.

Suppose given a short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$.

Suppose that $B \in \mathrm{Ob}\left(\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i\right)$. So there exists $j \in \mathbf{Z}_{\geq 0}$ such that $B \in \mathrm{Ob}(\mathcal{M}_j)$. By definition, A and C are $\mathcal{M}_j$-thick and thus $A, C \in \mathrm{Ob}(\mathcal{M}_{j+1}) \subseteq \mathrm{Ob}\left(\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i\right)$.

Suppose that $A, C \in \mathrm{Ob}\left(\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i\right)$. So there exists $j \in \mathbf{Z}_{\geq 0}$ such that $A, C \in \mathrm{Ob}(\mathcal{M}_j)$. By definition, B is $\mathcal{M}_j$-thick and thus $B \in \mathrm{Ob}(\mathcal{M}_{j+1}) \subseteq \mathrm{Ob}\left(\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i\right)$.

We conclude that $\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i$ is thick.

Suppose given a thick subcategory $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{L}$.

By lemma 3.1.5, we have $\mathcal{M}_0 \subseteq \mathcal{L}$.

Now suppose that $\mathcal{M}_j \subseteq \mathcal{L}$ for some $j \in \mathbf{Z}_{\geq 0}$. Suppose given an $\mathcal{M}_j$-thick object $A \in \mathrm{Ob}\,\mathcal{A}$. Then $A \in \mathrm{Ob}\,\mathcal{L}$ since $\mathcal{L}$ is a thick subcategory of $\mathcal{L}$. Thus $\mathcal{M}_{j+1} \subseteq \mathcal{L}$.

By induction, we conclude that $\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i \subseteq \mathcal{L}$.

So $\bigcup_{i \in \mathbf{Z}_{\geq 0}} \mathcal{M}_i$ is the smallest thick subcategory containing $\mathcal{M}$. □

3.1.2.3 Description using filtrations

We proceed in two steps. At first, we construct a subcategory $\langle \mathcal{M} \rangle_{\mathrm{sf}}$ of $\mathcal{A}$ that is closed under subobjects and factor objects. Then we use filtrations to construct a subcategory $\langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}}$ of $\mathcal{A}$ that is also closed under extensions and show that $\langle \mathcal{M} \rangle = \langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}}$ in case $\mathcal{M}$ is non-empty. In case $\mathcal{M}$ is empty, the objects of $\langle \mathcal{M} \rangle$ are precisely of the zero objects of $\mathcal{A}$.

Definition 3.1.14. Suppose given $M, C \in \mathrm{Ob}\,\mathcal{A}$. We say that C is a *subfactor of* M *in* $\mathcal{A}$ if there exists a diagram

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "b"] \arrow[dr, "c"'] & M \\ & & C \end{tikzcd}$$

in $\mathcal{A}$ such that $A \xrightarrow{a} B \xrightarrow{c} C$ is short exact.

Lemma 3.1.15. Suppose given $M, C \in \operatorname{Ob} \mathcal{A}$. The following three statements are equivalent.

(a) C is a subfactor of M in $\mathcal{A}$, i.e. there exists a diagram

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "b"] \arrow[dr, "c"'] & M \\ & & C \end{tikzcd}$$

in $\mathcal{A}$ such that $A \xrightarrow{a} B \xrightarrow{c} C$ is short exact.

(b) There exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "b"] \arrow[dr, "c"'] & M \arrow[r, "x"] & X \arrow[r, "y"] & Y \\ & & C \arrow[ur, "d"'] & & \end{tikzcd}$$

in $\mathcal{A}$ such that $A \xrightarrow{a} B \xrightarrow{c} C$ and $C \xrightarrow{d} X \xrightarrow{y} Y$ are short exact.

(c) There exists a diagram

$$\begin{tikzcd} M \arrow[r, "x"] & X \arrow[r, "y"] & Y \\ C \arrow[ur, "d"'] & & \end{tikzcd}$$

in $\mathcal{A}$ such that $C \xrightarrow{d} X \xrightarrow{y} Y$ is short exact.

In particular, C is a subfactor of M in $\mathcal{A}$ if and only if C is a subfactor of M in $\mathcal{A}^{\mathrm{op}}$.

Proof. Ad (a)$\Leftrightarrow$(b). Note that (b) implies (a). Conversely, choose a cokernel $M \xrightarrow{x} X$ of ab and a cokernel $M \xrightarrow{z} Y$ of b. By lemma 1.1.1, we obtain an exact sequence

$$0 \longrightarrow C \xrightarrow{d} X \xrightarrow{y} Y \longrightarrow 0$$

in $\mathcal{A}$ such that $bx = cd$.

Ad (b)$\Leftrightarrow$(c). This is dual to (a)$\Leftrightarrow$(b). $\square$

Lemma/Definition 3.1.16. Let ${}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}$ be the full subcategory of $\mathcal{A}$ defined by

$$\mathrm{Ob}({}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}) := \{C \in \mathrm{Ob}\,\mathcal{A} \colon \text{there exists } M \in \mathrm{Ob}\,\mathcal{M} \text{ such that } C \text{ is a subfactor of } M \text{ in } \mathcal{A}\}.$$

We have $\mathcal{M} \subseteq {}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}$ and $({}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}})^{\mathrm{op}} = {}_{\mathcal{A}^{\mathrm{op}}}\langle\mathcal{M}^{\mathrm{op}}\rangle_{\mathrm{sf}}$. Moreover, the subcategory ${}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}$ of $\mathcal{A}$ is closed under subobjects and factor objects.

Suppose given a thick subcategory $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{L}$. Then ${}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}} \subseteq \mathcal{L}$.

We abbreviate $\langle\mathcal{M}\rangle_{\mathrm{sf}} := {}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}$ if unambiguous.

Proof. Suppose given $M \in \mathrm{Ob}\,\mathcal{M}$. The diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "1"] \arrow[dr, "1"'] & M \\ & & M \end{tikzcd}$$

in $\mathcal{A}$ shows that M is a subfactor of M. Thus $M \in \mathrm{Ob}({}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}})$.
We conclude that $\mathcal{M} \subseteq {}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}$.

Suppose given $C \in \mathrm{Ob}\,\mathcal{A}$ and $M \in \mathrm{Ob}\,\mathcal{M}$. By lemma 3.1.15, C is a subfactor of M in $\mathcal{A}$ if and only if C is a subfactor of M in $\mathcal{A}^{\mathrm{op}}$. Thus $({}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}})^{\mathrm{op}} = {}_{\mathcal{A}^{\mathrm{op}}}\langle\mathcal{M}^{\mathrm{op}}\rangle_{\mathrm{sf}}$.

Suppose given $M \in \mathrm{Ob}\,\mathcal{M}$ and $C \in \mathrm{Ob}\,\mathcal{A}$ such that C is a subfactor of M. We may choose a diagram

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "b"] \arrow[dr, "c"'] & M \\ & & C \end{tikzcd}$$

in $\mathcal{A}$ such that $A \xrightarrow{a} B \xrightarrow{c} C$ is short exact.

Suppose given $C \xrightarrow{d} D$ in $\mathcal{A}$, i.e. a factor object of C. Choose a kernel $Z \xrightarrow{z} B$ of cd. So we obtain the diagram

$$\begin{tikzcd} Z \arrow[r, "z"] & B \arrow[r, "b"] \arrow[dr, "cd"'] & M \\ & & D \end{tikzcd}$$

in $\mathcal{A}$ and $Z \xrightarrow{z} B \xrightarrow{cd} D$ is short exact. So D is a subfactor of M as well.

Thus ${}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}$ is closed under factor objects.

Dually, we conclude that ${}_{\mathcal{A}}\langle\mathcal{M}\rangle_{\mathrm{sf}}$ is closed under subobjects.

Now suppose given a thick subcategory $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{L}$.

Suppose given $M \in \operatorname{Ob} \mathcal{M}$ and $C \in \operatorname{Ob} \mathcal{A}$ such that C is a subfactor of M. We may choose a diagram

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, "b"] \arrow[dr, "c"'] & M \\ & & C \end{tikzcd}$$

in $\mathcal{A}$ such that $A \xrightarrow{a} B \xrightarrow{c} C$ is short exact.

Since $\mathcal{L}$ is closed under subobjects, we have $B \in \operatorname{Ob} \mathcal{L}$. Since $\mathcal{L}$ is closed under factor objects, we conclude that $C \in \operatorname{Ob} \mathcal{L}$.

Thus ${}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\mathrm{sf}} \subseteq \mathcal{L}$. $\square$

Definition 3.1.17. Suppose given $X \in \operatorname{Ob} \mathcal{A}$. For $n \in \mathbf{Z}_{\geq 1}$, a sequence

$$X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \cdots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} X_n \xrightarrow{x_n} X_{n+1}$$

in $\mathcal{A}$ is called an *$\mathcal{M}$-filtration of X in $\mathcal{A}$* if the following two conditions hold.

- $X_1 \in \operatorname{Ob} \mathcal{M}$ and $X_{n+1} = X$.
- For $i \in [1, n]$, there exists a cokernel $X_{i+1} \xrightarrow{m_i} M_i$ of x_i in $\mathcal{A}$ such that $M_i \in \operatorname{Ob} \mathcal{M}$.

The object X is called *$\mathcal{M}$-filtering in $\mathcal{A}$* if there exists an $\mathcal{M}$-filtration of X in $\mathcal{A}$.

Lemma 3.1.18. Suppose given $X \in \operatorname{Ob} \mathcal{A}$.
The object X is $\mathcal{M}$-filtering in $\mathcal{A}$ if and only if X is $\mathcal{M}^{\mathrm{op}}$-filtering in $\mathcal{A}^{\mathrm{op}}$.

Proof. Suppose that X is $\mathcal{M}$-filtering in $\mathcal{A}$.
By duality, it suffices to show that X is $\mathcal{M}^{\mathrm{op}}$-filtering in $\mathcal{A}^{\mathrm{op}}$.

We may choose $n \in \mathbf{Z}_{\geq 1}$, an $\mathcal{M}$-filtration

$$X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \cdots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} X_n \xrightarrow{x_n} X_{n+1}$$

of X in $\mathcal{A}$ and cokernels $X_{i+1} \xrightarrow{m_i} M_i$ of x_i such that $M_i \in \operatorname{Ob} \mathcal{M}$ for $i \in [1, n]$.

We want to construct an $\mathcal{M}^{\mathrm{op}}$-filtration of X in $\mathcal{A}^{\mathrm{op}}$.

Let $Y_0 := X$, $Y_n := M_n$ and $q_n := m_n$. Note that $X \xrightarrow{q_n} Y_n$ is a cokernel of x_n .

For $i \in [1, n-1]$, we choose a cokernel $X \xrightarrow{q_i} Y_i$ of $x_i x_{i+1} \cdots x_n$. Let $y_1 := q_1$.

By lemma 1.1.1, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & M_{i-1} \arrow[ddr, "\ell_{i-1}"] & \\
& X_i \arrow[ur, "m_{i-1}"] \arrow[dr, "x_i \cdots x_n"] & & \\
X_{i-1} \arrow[ur, "x_{i-1}"] \arrow[rr, "x_{i-1} \cdots x_n"'] & & X \arrow[r, "q_{i-1}"] \arrow[dr, "q_i"'] & Y_{i-1} \arrow[d, "y_i"] \\
& & & Y_i
\end{tikzcd}$$

in $\mathcal{A}$ such that the sequence $M_{i-1} \xrightarrow{\ell_{i-1}} Y_{i-1} \xrightarrow{y_i} Y_i$ is short exact for $i \in [2, n]$.

We obtain the sequence

$$Y_n \xrightarrow{y_n^{\mathrm{op}}} Y_{n-1} \xrightarrow{y_{n-1}^{\mathrm{op}}} Y_{n-2} \xrightarrow{y_{n-2}^{\mathrm{op}}} \cdots \xrightarrow{y_3^{\mathrm{op}}} Y_2 \xrightarrow{y_2^{\mathrm{op}}} Y_1 \xrightarrow{y_1^{\mathrm{op}}} Y_0$$

in $\mathcal{A}^{\mathrm{op}}$. We have $Y_n = M_n \in \operatorname{Ob} \mathcal{M}$ and $Y_0 = X$. Moreover, $X_1 \xrightarrow{(x_1 x_2 \cdots x_n)^{\mathrm{op}}} Y_0$ is a cokernel of $y_1^{\mathrm{op}} = q_1^{\mathrm{op}}$ in $\mathcal{A}^{\mathrm{op}}$ such that $X_1 \in \operatorname{Ob} \mathcal{M}$. For $i \in [2, n]$, $Y_{i-1} \xrightarrow{\ell_{i-1}^{\mathrm{op}}} M_{i-1}$ is a cokernel of y_i^{op} in $\mathcal{A}^{\mathrm{op}}$ such that $M_{i-1} \in \operatorname{Ob} \mathcal{M}$.

Thus X is $\mathcal{M}^{\mathrm{op}}$-filtering in $\mathcal{A}^{\mathrm{op}}$. □

Lemma/Definition 3.1.19. Let ${}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\mathrm{filt}}$ be the full subcategory of $\mathcal{A}$ defined by

$$\operatorname{Ob}({}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\mathrm{filt}}) := \{X \in \operatorname{Ob} \mathcal{A} \colon X \text{ is } \mathcal{M}\text{-filtering in } \mathcal{A}\}.$$

We have $({}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\mathrm{filt}})^{\mathrm{op}} = {}_{\mathcal{A}^{\mathrm{op}}}\langle \mathcal{M}^{\mathrm{op}} \rangle_{\mathrm{filt}}$ by lemma 3.1.18.

If $\mathcal{M}$ contains a zero object of $\mathcal{A}$, then $\mathcal{M} \subseteq {}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\mathrm{filt}}$.

Suppose given a thick subcategory $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{L}$. Then ${}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\mathrm{filt}} \subseteq \mathcal{L}$.

We abbreviate $\langle \mathcal{M} \rangle_{\mathrm{filt}} := {}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\mathrm{filt}}$ if unambiguous.

Proof. Suppose that $Z \in \operatorname{Ob} \mathcal{M}$ is a zero object of $\mathcal{A}$. Suppose given $M \in \operatorname{Ob} \mathcal{M}$. The sequence $M \xrightarrow{1} M$ is an $\mathcal{M}$-filtration of M in $\mathcal{A}$ since $M \xrightarrow{0} Z$ is a cokernel of 1_M . Thus $M \in \operatorname{Ob}(\langle \mathcal{M} \rangle_{\mathrm{filt}})$. We conclude that $\mathcal{M} \subseteq \langle \mathcal{M} \rangle_{\mathrm{filt}}$.

Suppose given a thick subcategory $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{L}$.

Suppose given $X \in \operatorname{Ob}(\langle \mathcal{M} \rangle_{\mathrm{filt}})$.

We may choose $n \in \mathbf{Z}_{\geq 1}$, an $\mathcal{M}$-filtration

$$X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \cdots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} X_n \xrightarrow{x_n} X_{n+1}$$

of X in $\mathcal{A}$ and cokernels $X_{i+1} \xrightarrow{m_i} M_i$ of x_i such that $M_i \in \operatorname{Ob} \mathcal{M}$ for $i \in [1, n]$.

By definition, we have $X_1 \in \operatorname{Ob} \mathcal{M} \subseteq \operatorname{Ob} \mathcal{L}$.

Suppose that $X_i \in \operatorname{Ob} \mathcal{L}$ for some $i \in [1, n]$. The sequence $X_i \xrightarrow{x_i} X_{i+1} \xrightarrow{m_i} M_i$ is short exact such that $X_i, M_i \in \operatorname{Ob} \mathcal{L}$. Since $\mathcal{L}$ is closed under extensions, we have $X_{i+1} \in \operatorname{Ob} \mathcal{L}$.

Inductively, we conclude that $X = X_{n+1} \in \operatorname{Ob} \mathcal{L}$.

Thus ${}_{\mathcal{A}}\langle \mathcal{M} \rangle_{\text{filt}} \subseteq \mathcal{L}$. □

Lemma 3.1.20. Suppose that $\mathcal{M}$ is non-empty and that it is closed under subobjects and factor objects. Then $\mathcal{M} \subseteq \langle \mathcal{M} \rangle_{\text{filt}}$ and $\langle \mathcal{M} \rangle_{\text{filt}}$ is closed under subobjects, factor objects and extensions in $\mathcal{A}$. I.e. $\langle \mathcal{M} \rangle_{\text{filt}}$ is a thick subcatgory of $\mathcal{A}$ containing $\mathcal{M}$, cf. remark 3.1.2.

Proof. Since $\mathcal{M}$ is non-empty and closed under subobjects, $\mathcal{M}$ contains the zero objects of $\mathcal{A}$. Therefore $\mathcal{M} \subseteq \langle \mathcal{M} \rangle_{\text{filt}}$, cf. definition 3.1.19.

We will use the kernel-cokernel-criterion lemma 1.1.2 repeatedly without comment.

We show that $\langle \mathcal{M} \rangle_{\text{filt}}$ is closed under subobjects.

Suppose given a monomorphism $W \xrightarrow{z} X$ in $\mathcal{A}$ such that X is $\mathcal{M}$-filtering.

We may choose $n \in \mathbf{Z}_{\geq 1}$, an $\mathcal{M}$-filtration

$$X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \cdots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} X_n \xrightarrow{x_n} X_{n+1}$$

of X in $\mathcal{A}$ and cokernels $X_{i+1} \xrightarrow{m_i} M_i$ of x_i such that $M_i \in \operatorname{Ob} \mathcal{M}$ for $i \in [1, n]$.

We want to construct an $\mathcal{M}$-filtration of W in $\mathcal{A}$.

Let $W_{n+1} := W$ and $z_{n+1} := z$.

Recursively, choose pullbacks

$$\begin{tikzcd} W_i \arrow[r, "w_i"] \arrow[d, "z_i"'] \arrow[dr, phantom, "\ulcorner", very near start] & W_{i+1} \arrow[d, "z_{i+1}"] \\ X_i \arrow[r, "x_i"] & X_{i+1} \end{tikzcd}$$

and cokernels $W_i \xrightarrow{\ell_i} L_i$ of w_i in $\mathcal{A}$ for $i \in [1, n]$. Note that the induced morphism between the cokernels L_i and M_i is monomorphic and thus $L_i \in \operatorname{Ob} \mathcal{M}$ for $i \in [1, n]$. Moreover, we have $W_1 \in \operatorname{Ob} \mathcal{M}$ since $X_1 \in \operatorname{Ob} \mathcal{M}$.

Thus the sequence

$$W_1 \xrightarrow{w_1} W_2 \xrightarrow{w_2} W_3 \xrightarrow{w_3} \cdots \xrightarrow{w_{n-2}} W_{n-1} \xrightarrow{w_{n-1}} W_n \xrightarrow{w_n} W_{n+1}$$

is an $\mathcal{M}$-filtration of W. We conclude that $W \in \operatorname{Ob}(\langle \mathcal{M} \rangle_{\text{filt}})$.

Dually, $\langle \mathcal{M} \rangle_{\mathrm{filt}}$ is closed under factor objects.

We show that $\langle \mathcal{M} \rangle_{\mathrm{filt}}$ is closed under extensions.

Suppose given a short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ such that X and Z are $\mathcal{M}$-filtering.

We may choose $n \in \mathbf{Z}_{\geq 1}$, an $\mathcal{M}$-filtration

$$Z_1 \xrightarrow{z_1} Z_2 \xrightarrow{z_2} Z_3 \xrightarrow{z_3} \cdots \xrightarrow{z_{n-2}} Z_{n-1} \xrightarrow{z_{n-1}} Z_n \xrightarrow{z_n} Z_{n+1}$$

of Z in $\mathcal{A}$ and cokernels $Z_{i+1} \xrightarrow{m_i} M_i$ of z_i such that $M_i \in \operatorname{Ob} \mathcal{M}$ for $i \in [1, n]$.

Let $Y_{n+1} := Y$, $g_{n+1} := g$ and $f_{n+1} := f$.

Recursively, choose pullbacks

$$\begin{tikzcd} Y_i \arrow[r, "y_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\ulcorner"] & Y_{i+1} \arrow[d, "g_{i+1}"] \\ Z_i \arrow[r, "z_i"] & Z_{i+1} \end{tikzcd} ,$$

cokernels $Y_i \xrightarrow{c_i} C_i$ of y_i and kernels $X \xrightarrow{f_i} Y_i$ of g_i in $\mathcal{A}$ for $i \in [1, n]$. Note that the induced morphism between the cokernels C_i and M_i is monomorphic and thus $C_i \in \operatorname{Ob} \mathcal{M}$ for $i \in [1, n]$.

Note that $Y_1 \xrightarrow{g_1} Z_1$ is a cokernel of f_1 such that $Z_1 \in \operatorname{Ob} \mathcal{M}$.

Since X is $\mathcal{M}$-filtering, we may choose $k \in \mathbf{Z}_{\geq 1}$, an $\mathcal{M}$-filtration

$$X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} \cdots \xrightarrow{x_{n-2}} X_{n-1} \xrightarrow{x_{n-1}} X_n \xrightarrow{x_n} X_{n+1}$$

of X in $\mathcal{A}$ and cokernels $X_{i+1} \xrightarrow{\ell_i} L_i$ of x_i such that $L_i \in \operatorname{Ob} \mathcal{M}$ for $i \in [1, n]$.

The sequence

$$X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} X_n \xrightarrow{x_n} X_{n+1} \xrightarrow{f_1} Y_1 \xrightarrow{y_1} Y_2 \xrightarrow{y_2} \cdots \xrightarrow{y_{n-1}} Y_n \xrightarrow{y_n} Y_{n+1}$$

is an $\mathcal{M}$-filtration of Y. We conclude that $Y \in \operatorname{Ob}(\langle \mathcal{M} \rangle_{\mathrm{filt}})$. □

Proposition 3.1.21. Suppose that $\mathcal{M}$ is non-empty. Then we have $\langle \mathcal{M} \rangle = \langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}}$.

Proof. By lemmata 3.1.16 and 3.1.20, $\langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}}$ is a thick subcategory of $\mathcal{A}$ containing $\mathcal{M}$.

Suppose given a thick subcategory $\mathcal{L}$ of $\mathcal{A}$ containing $\mathcal{M}$.

We have $\langle \mathcal{M} \rangle_{\mathrm{sf}} \subseteq \mathcal{L}$, cf. definition 3.1.16 and, consequently, $\langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}} \subseteq \mathcal{L}$, cf. definition 3.1.19.

Thus we obtain

$$\langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}} \subseteq \bigcap \{ \mathcal{L} \colon \mathcal{L} \text{ is a thick subcategory of } \mathcal{A} \text{ such that } \mathcal{M} \subseteq \mathcal{L} \} \subseteq \langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}}$$

so that

$$\langle \langle \mathcal{M} \rangle_{\mathrm{sf}} \rangle_{\mathrm{filt}} = \bigcap \{ \mathcal{L} \colon \mathcal{L} \text{ is a thick subcategory of } \mathcal{A} \text{ such that } \mathcal{M} \subseteq \mathcal{L} \} = \langle \mathcal{M} \rangle .$$ □

3.1.3 $\mathcal{N}$-quasi-isomorphisms

Definition 3.1.22. Suppose given $A \xrightarrow{f} B$ in $\mathcal{A}$.
The morphism f is called an $\mathcal{N}$*-quasi-isomorphism* if and only if there exists a kernel $K \xrightarrow{k} A$ of f and a cokernel $B \xrightarrow{c} C$ of f in $\mathcal{A}$ such that $K, C \in \operatorname{Ob} \mathcal{N}$.

Often we will denote an $\mathcal{N}$-quasi-isomorphism by $A \overset{f}{\rightsquigarrow} B$.

Remark 3.1.23. Suppose given $A \xrightarrow{f} B$ in $\mathcal{A}$. Since $\mathcal{N} \subseteq \mathcal{A}$ is closed under isomorphisms by lemma 3.1.5, the morphism f is an $\mathcal{N}$-quasi-isomorphism if and only if for all kernels $K \xrightarrow{k} A$ of f and for all cokernels $B \xrightarrow{c} C$ of f in $\mathcal{A}$, we have $K, C \in \operatorname{Ob} \mathcal{N}$.

Example 3.1.24. Isomorphisms in $\mathcal{A}$ are $\mathcal{N}$-quasi-isomorphisms since $\mathcal{N}$ contains the zero objects of $\mathcal{A}$ by lemma 3.1.5. In particular, identities are $\mathcal{N}$-quasi-isomorphisms.

A zero morphism $A \xrightarrow{0} B$ in $\mathcal{A}$ is an $\mathcal{N}$-quasi-isomorphism if and only if $A, B \in \operatorname{Ob} \mathcal{N}$.

Lemma 3.1.25. Suppose given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. Suppose that $K \xrightarrow{k} A$ is a kernel of f and that $C \xrightarrow{r} R$ is a cokernel of g. If fg is an $\mathcal{N}$-quasi-isomorphism, then $K, R \in \operatorname{Ob} \mathcal{N}$.

Proof. Choose a kernel $L \xrightarrow{\ell} A$ of fg, a kernel $M \xrightarrow{m} B$ of g, a cokernel $B \xrightarrow{p} P$ of f and a cokernel $C \xrightarrow{q} Q$ of fg in $\mathcal{A}$.

Lemma 1.1.1 yields the commutative diagram

$$\begin{tikzcd}
 & & M \arrow[rr, "c"] \arrow[dr, "m"'] & & P \arrow[ddrr, "d"] & & \\
 & & & B \arrow[ur, "p"'] \arrow[dr, "g"] & & & \\
L \arrow[uurr, "b"] \arrow[rr, "\ell"] & & A \arrow[ur, "f"] \arrow[rr, squiggly, "fg"'] & & C \arrow[rr, "q"] \arrow[dr, "r"'] & & Q \arrow[dl, "e"] \\
 & K \arrow[ul, "a"] \arrow[ur, "k"'] & & & & R &
\end{tikzcd}$$

in $\mathcal{A}$, where the sequence

$$0 \longrightarrow K \xrightarrow{a} L \xrightarrow{b} M \xrightarrow{c} P \xrightarrow{d} Q \xrightarrow{e} R \longrightarrow 0$$

is exact. Since fg is an $\mathcal{N}$-quasi-isomorphism, we have $L, Q \in \operatorname{Ob} \mathcal{N}$. Now $L \in \operatorname{Ob} \mathcal{N}$ implies $K \in \operatorname{Ob} \mathcal{N}$ and $Q \in \operatorname{Ob} \mathcal{N}$ implies $R \in \operatorname{Ob} \mathcal{N}$. $\square$

Lemma 3.1.26. Suppose given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. If two out of the three morphisms f, g and fg are $\mathcal{N}$-quasi-isomorphisms, then so is the third.

In other words:

- If f and g are $\mathcal{N}$-quasi-isomorphisms in $\mathcal{A}$, then so is fg.
- If f and fg are $\mathcal{N}$-quasi-isomorphisms in $\mathcal{A}$, then so is g.
- If g and fg are $\mathcal{N}$-quasi-isomorphisms in $\mathcal{A}$, then so is f.

Proof. Choose a kernel $K \xrightarrow{k} A$ of f, a kernel $L \xrightarrow{\ell} A$ of fg, a kernel $M \xrightarrow{m} B$ of g, a cokernel $B \xrightarrow{p} P$ of f, a cokernel $C \xrightarrow{q} Q$ of fg in $\mathcal{A}$ and a cokernel $C \xrightarrow{r} R$ of g.

Lemma 1.1.1 yields the commutative diagram

$$\begin{tikzcd}
 & & M \arrow[rr, "c"] \arrow[dr, "m"'] & & P \arrow[ddrr, "d"] & & \\
 & & & B \arrow[ur, "p"'] \arrow[dr, "g"] & & & \\
L \arrow[uurr, "b"] \arrow[rr, "\ell"] & & A \arrow[ur, "f"] \arrow[rr, "fg"'] & & C \arrow[rr, "q"] \arrow[dr, "r"'] & & Q \arrow[dl, "e"] \\
 & K \arrow[ul, "a"] \arrow[ur, "k"'] & & & & R &
\end{tikzcd}$$

in $\mathcal{A}$, where the sequence

$$0 \longrightarrow K \xrightarrow{a} L \xrightarrow{b} M \xrightarrow{c} P \xrightarrow{d} Q \xrightarrow{e} R \longrightarrow 0$$

is exact. We will use lemma 3.1.4 repeatedly.

Suppose that f and g are $\mathcal{N}$-quasi-isomorphisms. Then $K, P, M, R \in \operatorname{Ob}\mathcal{N}$.
Now $K, M \in \operatorname{Ob}\mathcal{N}$ implies $L \in \operatorname{Ob}\mathcal{N}$. Moreover, $P, R \in \operatorname{Ob}\mathcal{N}$ implies $Q \in \operatorname{Ob}\mathcal{N}$. Thus fg is an $\mathcal{N}$-quasi-isomorphism.

Suppose that f and fg are $\mathcal{N}$-quasi-isomorphisms. Then $K, P, L, Q \in \operatorname{Ob}\mathcal{N}$.
Now $L, P \in \operatorname{Ob}\mathcal{N}$ implies $M \in \operatorname{Ob}\mathcal{N}$. Moreover, $Q \in \operatorname{Ob}\mathcal{N}$ implies $R \in \operatorname{Ob}\mathcal{N}$. Thus g is an $\mathcal{N}$-quasi-isomorphism.

Suppose that fg and g are $\mathcal{N}$-quasi-isomorphisms. Then $L, Q, M, R \in \operatorname{Ob}\mathcal{N}$.
Now $M, Q \in \operatorname{Ob}\mathcal{N}$ implies $P \in \operatorname{Ob}\mathcal{N}$. Moreover $L \in \operatorname{Ob}\mathcal{N}$ implies $K \in \operatorname{Ob}\mathcal{N}$. Thus f is an $\mathcal{N}$-quasi-isomorphism. □

Lemma 3.1.27. Suppose given an $\mathcal{N}$-quasi-isomorphism $A \overset{f}{\rightsquigarrow} B$ in $\mathcal{A}$.

(a) Suppose given a pullback

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "g"] \\
A \arrow[r, squiggly, "f"] & B
\end{tikzcd}$$

in $\mathcal{A}$. Then f' is an $\mathcal{N}$-quasi-isomorphism as well.

(b) Suppose given a pushout

$$\begin{tikzcd} A \arrow[r, rightsquigarrow, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner" near end] & B' \end{tikzcd}$$

in $\mathcal{A}$. Then f' is an $\mathcal{N}$-quasi-isomorphism as well.

Proof. Ad (a). Choose a kernel $K \xrightarrow{k} A$ of f, a kernel $K' \xrightarrow{k'} A'$ of f', a cokernel $B \xrightarrow{c} C$ of f and a cokernel $B' \xrightarrow{c'} C'$ of f' in $\mathcal{A}$.

By the kernel-cokernel-criterion lemma 1.1.2, there exists an isomorphism $K' \xrightarrow[\sim]{u} K$ and a monomorphism $C' \xrightarrow{v} C$.

Now $K \in \operatorname{Ob}\mathcal{N}$ implies $K' \in \operatorname{Ob}\mathcal{N}$ and $C \in \operatorname{Ob}\mathcal{N}$ implies $C' \in \operatorname{Ob}\mathcal{N}$. Thus f' is an $\mathcal{N}$-quasi-isomorphism.

$$\begin{tikzcd} K' \arrow[r, "k'"] \arrow[d, "u"', "\sim"] & A' \arrow[r, rightsquigarrow, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner" near start] & B' \arrow[r, "v'"] \arrow[d, "g"] & C' \arrow[d, "v"] \\ K \arrow[r, "k"] & A \arrow[r, rightsquigarrow, "f"] & B \arrow[r, "v"] & C \end{tikzcd}$$

Ad (b). This is dual to (a). $\square$

Lemma 3.1.28. Suppose given $A \xrightarrow{f} B$ in $\mathcal{A}$. Suppose given an image

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & I \arrow[ur, "i"'] & \end{tikzcd}$$

of f.

The following three statements are equivalent.

(a) There exists an $\mathcal{N}$-quasi-isomorphism $B \overset{t}{\rightsquigarrow} T$ in $\mathcal{A}$ such that $ft = 0$.

(b) We have $I \in \operatorname{Ob}\mathcal{N}$.

(c) There exists an $\mathcal{N}$-quasi-isomorphism $S \overset{s}{\rightsquigarrow} A$ in $\mathcal{A}$ such that $sf = 0$.

Proof. Ad (a)$\Rightarrow$(b). Choose a kernel $K \xrightarrow{k} B$ of t in $\mathcal{A}$. We have $pit = ft = 0$ and thus $it = 0$ since p is epimorphic. So there exists $I \xrightarrow{u} K$ in $\mathcal{A}$ with $uk = i$. Moreover, u is monomorphic since so is i. Now $K \in \operatorname{Ob}\mathcal{N}$ implies $I \in \operatorname{Ob}\mathcal{N}$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "p"'] & & B \arrow[r, "t"] & T \\ I \arrow[rru, "i"] \arrow[r, "u"'] & K \arrow[ru, "k"'] & & \end{tikzcd}$$

Ad (b)⇒(a). Choose a cokernel $B \xrightarrow{c} C$ of i. So $I \xrightarrow{i} B$ is a kernel of c and $C \xrightarrow{0} 0$ is a cokernel of c. We conclude that c is an $\mathcal{N}$-quasi-isomorphism with $fc = pic = 0$.

Ad (b)⇔(c). This is dual to (b)⇔(a). □

3.2 Construction of the quotient category

3.2.1 Definition and duality

Lemma/Definition 3.2.1. Suppose given $A, B \in \operatorname{Ob} \mathcal{A}$.

Let $\mathrm{ND}(A, B)$ denote the set of all diagrams of the form $A \xrightarrow{f} S \xleftarrow{s} B$ in $\mathcal{A}$ such that s is an $\mathcal{N}$-quasi-isomorphism. Given an element $\big(A \xrightarrow{f} S \xleftarrow{s} B\big) \in \mathrm{ND}(A, B)$, then f is called its *numerator* and s is called its *denominator*. ND is short for numerator-denominator. For $\big(A \xrightarrow{f} S \xleftarrow{s} B\big), \big(A \xrightarrow{g} T \xleftarrow{t} B\big) \in \mathrm{ND}(A, B)$, let $\big(A \xrightarrow{f} S \xleftarrow{s} B\big) \sim \big(A \xrightarrow{g} T \xleftarrow{t} B\big)$ if and only if there exist $\mathcal{N}$-quasi-isomorphisms $S \xrightarrow{u} U$ and $T \xrightarrow{v} U$ in $\mathcal{A}$ such that $fu = gv$ and $su = tv$.

$$\begin{tikzcd}
& S \arrow[d, squiggly, "u"] & \\
A \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & U & B \arrow[ul, squiggly, "s"'] \arrow[dl, squiggly, "t"] \\
& T \arrow[u, squiggly, "v"'] &
\end{tikzcd}$$

This defines an equivalence relation on $\mathrm{ND}(A, B)$. Let $\mathrm{ND}(A, B)/\!\sim$ denote the factor set.

Let f/s denote the equivalence class of $\big(A \xrightarrow{f} S \xleftarrow{s} B\big) \in \mathrm{ND}(A, B)$.

Proof.
Suppose given $\big(A \xrightarrow{f} S \xleftarrow{s} B\big), \big(A \xrightarrow{g} T \xleftarrow{t} B\big)$ and $\big(A \xrightarrow{h} W \xleftarrow{w} B\big) \in \mathrm{ND}(A, B)$.

Using identities, we see that $\sim$ is reflexive:

$$\begin{tikzcd}
& S \arrow[d, squiggly, "1"] & \\
A \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f"'] & S & B \arrow[ul, squiggly, "s"'] \arrow[dl, squiggly, "s"] \\
& S \arrow[u, squiggly, "1"'] &
\end{tikzcd}$$

Cf. example 3.1.24.

By construction, $\sim$ is symmetric.

Suppose that

$$\left(A \xrightarrow{f} S \overset{s}{\twoheadleftarrow\!\!\!\sim} B\right) \sim \left(A \xrightarrow{g} T \overset{t}{\twoheadleftarrow\!\!\!\sim} B\right) \quad \text{and} \quad \left(A \xrightarrow{g} T \overset{t}{\twoheadleftarrow\!\!\!\sim} B\right) \sim \left(A \xrightarrow{h} W \overset{w}{\twoheadleftarrow\!\!\!\sim} B\right).$$

So there exist $\mathcal{N}$-quasi-isomorphism $S \overset{u}{\rightsquigarrow} U$, $T \overset{v}{\rightsquigarrow} U$, $T \overset{x}{\rightsquigarrow} X$ and $T \overset{y}{\rightsquigarrow} X$ in $\mathcal{A}$ such that $fu = gv$, $su = tv$, $gx = hy$ and $tx = wy$.

Choose a pushout

$$\begin{tikzcd}
T \arrow[r, rightsquigarrow, "v"] \arrow[d, rightsquigarrow, "x"'] \arrow[dr, phantom, "\lrcorner" very near end] & U \arrow[d, "x'"] \\
X \arrow[r, "v'"'] & P
\end{tikzcd}$$

in $\mathcal{A}$. The morphisms v' and x' are $\mathcal{N}$-quasi-isomorphisms by lemma 3.1.27.(b). The composites ux' and yv' are $\mathcal{N}$-quasi-isomorphisms by lemma 3.1.26.

We have $fux' = gvx' = gxv' = hyv'$ and $sux' = tvx' = txv' = wyv'$. Thus $\sim$ is transitive.

$$\begin{tikzcd}
 & S \arrow[d, rightsquigarrow, "u"] & & \\
A \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] \arrow[dddr, bend right=40, "h"'] & U \arrow[drr, rightsquigarrow, "x'"] & B \arrow[ul, rightsquigarrow, "s"'] \arrow[dl, rightsquigarrow, "t"] \arrow[ddl, rightsquigarrow, "w"] & \\
 & T \arrow[u, rightsquigarrow, "v"] \arrow[d, rightsquigarrow, "x"] & & P \\
 & X \arrow[urr, rightsquigarrow, "v'"'] & & \\
 & W \arrow[u, rightsquigarrow, "y"] & &
\end{tikzcd}$$

$\square$

Remark 3.2.2. Suppose given $A, B \in \operatorname{Ob} \mathcal{A}$.
Suppose given $\left(A \xrightarrow{f} S \overset{s}{\twoheadleftarrow\!\!\!\sim} B\right), \left(A \xrightarrow{g} T \overset{t}{\twoheadleftarrow\!\!\!\sim} B\right) \in \operatorname{ND}(A, B)$.

For an $\mathcal{N}$-quasi-isomorphism $S \overset{u}{\rightsquigarrow} U$ in $\mathcal{A}$, we have $f/s = fu/su$.

If $\left(A \xrightarrow{f} S \overset{s}{\twoheadleftarrow\!\!\!\sim} B\right) \sim \left(A \xrightarrow{g} T \overset{t}{\twoheadleftarrow\!\!\!\sim} B\right)$, then there exist $\mathcal{N}$-quasi-isomorphisms $S \overset{u}{\rightsquigarrow} U$ and $T \overset{v}{\rightsquigarrow} U$ in $\mathcal{A}$ such that $fu = gv$ and $su = tv$. In this case, we have

$$f/s = fu/su = gv/tv = g/t.$$

Lemma/Definition 3.2.3. The *quotient category* $\mathcal{A}/\!\!/\mathcal{N}$ *of* $\mathcal{A}$ *by* $\mathcal{N}$ shall be defined as follows.

- Let $\operatorname{Ob}(\mathcal{A}/\!\!/\mathcal{N}) := \operatorname{Ob} \mathcal{A}$.
- For $A, B \in \operatorname{Ob} \mathcal{A}$, let $\operatorname{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(A, B) := \operatorname{ND}(A, B)/\!\sim$, cf. definition 3.2.1.

- Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$, representing $A \xrightarrow{f/s} B \xrightarrow{g/t} C$ in $\mathcal{A}/\!\!/\mathcal{N}$. We want to define the composite $f/s \cdot g/t$ in $\mathcal{A}/\!\!/\mathcal{N}$.

  Choose a commutative diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, rightsquigarrow, "s"'] & T \arrow[d, rightsquigarrow, "s'"] \\ S \arrow[r, "g'"] & P \end{tikzcd}$$

  in $\mathcal{A}$ such that s' is an $\mathcal{N}$-quasi-isomorphism. This is possible since we may e.g. form the pushout of g and s, cf. lemma 3.1.27.(b).
  Let $f/s \cdot g/t := fg'/ts'$. This definition is independent of the choice of the commutative diagram and of the choice of the representatives of the equivalence classes.

$$\begin{tikzcd} & & C \arrow[d, rightsquigarrow, "t"] \\ & B \arrow[r, "g"] \arrow[d, rightsquigarrow, "s"'] & T \arrow[d, rightsquigarrow, "s'"] \\ A \arrow[r, "f"] & S \arrow[r, "g'"'] & P \end{tikzcd}$$

  The identitiy morphism of $A \in \mathrm{Ob}(\mathcal{A}/\!\!/\mathcal{N})$ is $\left(A \xrightarrow{1/1} A\right) \in \mathrm{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(A, A)$.

Proof. Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C \xrightarrow{h} U \overset{u}{\leftsquigarrow} D$ in $\mathcal{A}$, representing $A \xrightarrow{f/s} B \xrightarrow{g/t} C \xrightarrow{h/u} D$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Choose a pushout

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, rightsquigarrow, "s"'] \arrow[dr, phantom, "\lrcorner" very near end] & T \arrow[d, rightsquigarrow, "\hat{s}"] \\ S \arrow[r, "\hat{g}"'] & Q \end{tikzcd}$$

in $\mathcal{A}$. For a commutative diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, rightsquigarrow, "s"'] & T \arrow[d, rightsquigarrow, "s'"] \\ S \arrow[r, "g'"] & P \end{tikzcd}$$

in $\mathcal{A}$ such that s' is an $\mathcal{N}$-quasi-isomorphism, there exists $Q \xrightarrow{q} P$ in $\mathcal{A}$ with $\hat{s}q = s'$ and $\hat{g}q = g'$. Note that q is an $\mathcal{N}$-quasi-isomorphism by lemma 3.1.26.

Thus $fg'/ts' = f\hat{g}q/t\hat{s}q = f\hat{g}/t\hat{s}$.

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, squiggly, "s"'] & T \arrow[d, squiggly, "\hat{s}"'] \arrow[ddr, squiggly, bend left, "s'"] & \\
S \arrow[r, "\hat{g}"] \arrow[drr, bend right, "g'"'] & Q \arrow[dr, squiggly, "q"] & \\
 & & P
\end{tikzcd}
$$

So the definition of the composite of f/s an g/t is independent of the choice of the commutative diagram.

Suppose given $A \xrightarrow{f'} S' \xleftarrow{s'} B \xrightarrow{g'} T' \xleftarrow{t'} C$ in $\mathcal{A}$, representing $A \xrightarrow{f'/s'} B \xrightarrow{g'/t'} C$ in $\mathcal{A}/\!\!/\mathcal{N}$ with $f/s = f'/s'$ and $g/t = g'/t'$. So there exist $\mathcal{N}$-quasi-isomorphisms $S \xrightarrow{x} X$, $S' \xrightarrow{x'} X$, $T \xrightarrow{y} Y$ and $T' \xrightarrow{y'} Y$ in $\mathcal{A}$ such that $fx = f'x'$, $sx = s'x'$, $gy = g'y'$ and $ty = t'y'$.

Choose a pushout

$$
\begin{tikzcd}
B \arrow[r, "gy"] \arrow[d, squiggly, "sx"'] & Y \arrow[d, squiggly, "b"] \\
X \arrow[r, "a"'] \arrow[ur, phantom, "\lrcorner", near end] & P
\end{tikzcd}
$$

in $\mathcal{A}$. Note that b is an $\mathcal{N}$-quasi-isomorphism by lemma 3.1.27.(b). Since the diagrams

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, squiggly, "s"'] & T \arrow[d, squiggly, "yb"] \\
S \arrow[r, "xa"] & P
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
B \arrow[r, "g'"] \arrow[d, squiggly, "s'"'] & T' \arrow[d, squiggly, "y'b"] \\
S' \arrow[r, "x'a"] & P
\end{tikzcd}
$$

commute in $\mathcal{A}$, we have $f/s \cdot g/t = fxa/tyb = f'x'a/t'y'b = f'/s' \cdot g'/t'$.

$$
\begin{tikzcd}
 & & P & & \\
 & S \arrow[d, squiggly, "x"'] & & T \arrow[d, squiggly, "y"] & \\
A \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f'"'] & X \arrow[uur, "a"] & B \arrow[ul, squiggly, "s"] \arrow[dl, squiggly, "s'"] \arrow[ur, bend left, "g"] \arrow[dr, bend right, "g'"'] & Y \arrow[uul, squiggly, "b"'] & C \arrow[ul, squiggly, "t"'] \arrow[dl, squiggly, "t'"] \\
 & S' \arrow[u, squiggly, "x'"] & & T' \arrow[u, squiggly, "y'"'] &
\end{tikzcd}
$$

So the definition of the composite of f/s an g/t is independent of the choice of the representatives of the equivalence classes.

We want to show that $(f/s \cdot g/t) \cdot h/u = f/s \cdot (g/t \cdot h/u)$.

Choose commutative diagrams (e.g. pushouts)

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, squiggly, "s"'] & T \arrow[d, squiggly, "s'"] \\
S \arrow[r, "g'"] & P,
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, squiggly, "t"'] & U \arrow[d, squiggly, "t'"] \\
T \arrow[r, "h'"] & Q
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
T \arrow[r, "h'"] \arrow[d, squiggly, "s'"'] & Q \arrow[d, squiggly, "s''"] \\
P \arrow[r, "h''"] & R
\end{tikzcd}
$$

in $\mathcal{A}$. So the diagram

$$
\begin{tikzcd}
 & & & D \arrow[d, squiggly, "u"] \\
 & & C \arrow[r, "h"] \arrow[d, squiggly, "t"'] & U \arrow[d, squiggly, "t'"] \\
 & B \arrow[r, "g"] \arrow[d, squiggly, "s"'] & T \arrow[r, "h'"] \arrow[d, squiggly, "s'"] & Q \arrow[d, squiggly, "s''"] \\
A \arrow[r, "f"] & S \arrow[r, "g'"] & P \arrow[r, "h''"] & R
\end{tikzcd}
$$

commutes in $\mathcal{A}$. Thus we have

$$(f/s \cdot g/t) \cdot h/u = fg'/ts' \cdot h/u = fg'h''/ut's'' = f/s \cdot gh'/ut' = f/s \cdot (g/t \cdot h/u).$$

We want to show that $1_A/1_A \cdot f/s = f/s$ and $f/s \cdot 1_B/1_B = f/s$.

The diagram

$$
\begin{tikzcd}
 & & & B \arrow[d, squiggly, "1"] \\
 & & B \arrow[r, "1"] \arrow[d, squiggly, "s"'] & B \arrow[d, squiggly, "s"] \\
 & A \arrow[r, "f"] \arrow[d, squiggly, "1"'] & S \arrow[r, "1"] \arrow[d, squiggly, "1"] & S \\
A \arrow[r, "1"] & A \arrow[r, "f"] & S &
\end{tikzcd}
$$

commutes in $\mathcal{A}$, thus we have $1_A/1_A \cdot f/s = f/s = f/s \cdot 1_B/1_B$. □

Remark 3.2.4. Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B \xrightarrow{1} B \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$.

We have

$$f/s = f/1_S \cdot 1_S/s, \quad (s/1_S)^{-1} = 1_S/s \quad \text{and} \quad f/s \cdot 1_B/t = f/ts$$

in $\mathcal{A}/\!\!/\mathcal{N}$.

Suppose given $A \xrightarrow{f} B \overset{1}{\leftsquigarrow} B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$, representing $A \xrightarrow{f/1} B \xrightarrow{g/t} C$ in $\mathcal{A}/\!\!/\mathcal{N}$.

We have $f/1_B \cdot g/t = fg/t$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Proof. Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B \xrightarrow{1} B \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$.

The diagram

$$\begin{tikzcd}
 & & B \arrow[d, squiggly, "s"] \\
 & S \arrow[r, "1"] \arrow[d, squiggly, "1"'] & S \arrow[d, squiggly, "1"] \\
A \arrow[r, "f"] & S \arrow[r, "1"] & S
\end{tikzcd}$$

commutes in $\mathcal{A}$. Thus we have $f/s = f/1_S \cdot 1_S/s$.

The diagram

$$\begin{tikzcd}
 & & & S \arrow[d, squiggly, "1"] \\
 & & B \arrow[r, "s"] \arrow[d, squiggly, "s"'] & S \arrow[d, squiggly, "1"] \\
 & S \arrow[r, "1"] \arrow[d, squiggly, "1"'] & S \arrow[r, "1"] \arrow[d, squiggly, "1"] & S \\
B \arrow[r, "s"] & S \arrow[r, "1"] & S &
\end{tikzcd}$$

commutes in $\mathcal{A}$. Thus we have $s/1_S \cdot 1_S/s = s/s = 1_B/1_B$ and $1_S/s \cdot s/1_S = 1_S/1_S$.

The diagram

$$\begin{tikzcd}
 & & C \arrow[d, squiggly, "t"] \\
 & B \arrow[r, "1"] \arrow[d, squiggly, "s"'] & B \arrow[d, squiggly, "s"] \\
A \arrow[r, "f"] & S \arrow[r, "1"] & S
\end{tikzcd}$$

commutes in $\mathcal{A}$. Thus we have $f/s \cdot 1_B/t = f/ts$.

Suppose given $A \xrightarrow{f} B \overset{1}{\leftsquigarrow} B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$.

The diagram

$$\begin{tikzcd}
 & & C \arrow[d, squiggly, "t"] \\
 & B \arrow[r, "g"] \arrow[d, squiggly, "1"'] & T \arrow[d, squiggly, "1"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & T
\end{tikzcd}$$

commutes in $\mathcal{A}$, thus we have $f/1_B \cdot g/t = fg/t$. $\square$

Remark 3.2.5. Suppose given $n \in \mathbf{N}$ and $A \xrightarrow{f_i} S_i \overset{s_i}{\leftsquigarrow} B$ in $\mathcal{A}$, representing $A \xrightarrow{f_i/s_i} B$ in $\mathcal{A}/\!\!/\mathcal{N}$ for $i \in [1, n]$. We may choose $\mathcal{N}$-quasi-isomorphisms $B \overset{x}{\rightsquigarrow} X$ and $S_i \overset{t_i}{\rightsquigarrow} X$ for $i \in [1, n]$ in $\mathcal{A}$ such that $f_i/s_i = f_i t_i/x$ for $i \in [1, n]$.

So for a finite number of morphisms with same domain and codomain in $\mathcal{A}/\!\!/\mathcal{N}$, we may choose representatives such that they have a common denominator.

Proof. We use induction on $n \in \mathbf{N}$.

Suppose given $A \xrightarrow{f_i} S_i \overset{s_i}{\leftsquigarrow} B$ in $\mathcal{A}$ for $i \in [1, n+1]$ and $\mathcal{N}$-quasi-isomorphisms $B \overset{x}{\rightsquigarrow} X$ and $S_i \overset{t_i}{\rightsquigarrow} X$ for $i \in [1, n]$ in $\mathcal{A}$ such that $f_i/s_i = f_i t_i/x$ for $i \in [1, n]$.

Choose a pushout

$$\begin{tikzcd} B \arrow[r, rightsquigarrow, "x"] \arrow[d, rightsquigarrow, "s_{n+1}"'] \arrow[dr, phantom, "\lrcorner" very near end] & X \arrow[d, rightsquigarrow, "t"] \\ S_{n+1} \arrow[r, rightsquigarrow, "z"'] & Y \end{tikzcd}$$

in $\mathcal{A}$. Then $f_{n+1}/s_{n+1} = f_{n+1}z/s_{n+1}z = f_{n+1}z/xt$ and $f_i/s_i = f_i t_i/x = f_i t_i t/xt$ for $i \in [1, n]$. $\square$

Lemma/Definition 3.2.6. The *localisation functor* $\mathrm{L}_{\mathcal{A},\mathcal{N}} \colon \mathcal{A} \to \mathcal{A}/\!\!/\mathcal{N}$ *of* $\mathcal{A}$ *by* $\mathcal{N}$ shall be defined as follows.

- For $A \in \operatorname{Ob} \mathcal{A}$, let $\mathrm{L}_{\mathcal{A},\mathcal{N}}(A) := A$.
- For $\big(A \xrightarrow{f} B\big) \in \operatorname{Mor} \mathcal{A}$, let $\mathrm{L}_{\mathcal{A},\mathcal{N}}(f) := f/1_B$.

We abbreviate $\mathrm{L} := \mathrm{L}_{\mathcal{A},\mathcal{N}}$ if unambiguous.

Proof. For $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$, we have $\mathrm{L}(fg) = fg/1_C = f/1_B \cdot g/1_C = \mathrm{L}(f) \cdot \mathrm{L}(g)$ by remark 3.2.4. Moreover, $\mathrm{L}(1_A) = 1_A/1_A$, cf. definition 3.2.3. $\square$

Lemma/Definition 3.2.7. The *duality functor* $\mathcal{D}_{\mathcal{A},\mathcal{N}} \colon (\mathcal{A}/\!\!/\mathcal{N})^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}/\!\!/\mathcal{N}^{\mathrm{op}}$ *of* $\mathcal{A}$ *by* $\mathcal{N}$ shall be defined as follows.

- For $A \in \operatorname{Ob} \mathcal{A}$, let $\mathcal{D}_{\mathcal{A},\mathcal{N}}(A) := A$.
- Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B$ in $\mathcal{A}$, representing $A \xrightarrow{f/s} B$ in $\mathcal{A}/\!\!/\mathcal{N}$.

  Choose a commutative diagram

$$\begin{tikzcd} S_\circ \arrow[r, "f_\circ"] \arrow[d, rightsquigarrow, "s_\circ"'] & B \arrow[d, rightsquigarrow, "s"] \\ A \arrow[r, "f"] & S \end{tikzcd}$$

in $\mathcal{A}$ such that $s_\circ$ is an $\mathcal{N}$-quasi-isomorphism. This is possible since we may e.g. form the pullback of f and s, cf. lemma 3.1.27.(a).

Let $\mathcal{D}_{\mathcal{A},\mathcal{N}}((f/s)^{\mathrm{op}}) := f_\circ^{\mathrm{op}}/s_\circ^{\mathrm{op}}$. This definition is independent of the choice of the commutative diagram and of the choice of the representative of the equivalence class.

The functor $\mathcal{D}_{\mathcal{A},\mathcal{N}}$ is an isomorphism of categories with inverse $\mathcal{D}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}},\mathcal{N}^{\mathrm{op}}}$.

Moreover, we have $\mathcal{D}_{\mathcal{A},\mathcal{N}}((f/1_S)^{\mathrm{op}}) = f^{\mathrm{op}}/1_A^{\mathrm{op}}$ for $A \xrightarrow{f} S$ in $\mathcal{A}$.

Proof. Suppose given $A \xrightarrow{f} S \xleftarrow{s} B$ in $\mathcal{A}$.

Choose a pullback

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, squiggly, "s'"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, squiggly, "s"] \\ A \arrow[r, "f"] & S \end{tikzcd}$$

in $\mathcal{A}$. For a commutative diagram

$$\begin{tikzcd} S_\circ \arrow[r, "f_\circ"] \arrow[d, squiggly, "s_\circ"'] & B \arrow[d, squiggly, "s"] \\ A \arrow[r, "f"] & S \end{tikzcd}$$

in $\mathcal{A}$ such that $s_\circ$ is an $\mathcal{N}$-quasi-isomorphism, there exists $S_\circ \xrightarrow{u} P$ in $\mathcal{A}$ with $uf' = f_\circ$ and $us' = s_\circ$. Note that u is an $\mathcal{N}$-quasi-isomorphism by lemma 3.1.26.
Thus $f'^{\mathrm{op}}/s'^{\mathrm{op}} = f'^{\mathrm{op}}u^{\mathrm{op}}/s'^{\mathrm{op}}u^{\mathrm{op}} = f_\circ^{\mathrm{op}}/s_\circ^{\mathrm{op}}$.
So the definition of $\mathcal{D}_{\mathcal{A},\mathcal{N}}((f/s)^{\mathrm{op}})$ is independent of the choice of the commutative diagram.

Suppose given $A \xrightarrow{g} T \xleftarrow{t} B$ in $\mathcal{A}$ with $f/s = g/t$. So there exist $\mathcal{N}$-quasi-isomorphisms $S \xrightarrow{x} X$ and $T \xrightarrow{y} X$ in $\mathcal{A}$ such that $fx = gy$ and $sx = ty$.

Choose a pullback

$$\begin{tikzcd} P \arrow[r, "a"] \arrow[d, squiggly, "b"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, squiggly, "s"] \\ A \arrow[r, "f"] & S \end{tikzcd}$$

in $\mathcal{A}$. Note that b is an $\mathcal{N}$-quasi-isomorphism by lemma 3.1.27.(a).

We have $(bg - at)y = bgy - aty = bfx - asx = (bf - as)x = 0$. By lemma 3.1.28, there exists an $\mathcal{N}$-quasi-isomorphism $Q \xrightarrow{q} P$ such that $q(bg - at) = 0$.

Since the diagrams

$$
\begin{tikzcd}
Q \arrow[r, "qa"] \arrow[d, squiggly, "qb"'] & B \arrow[d, squiggly, "s"] \\
A \arrow[r, "f"] & S
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
Q \arrow[r, "qa"] \arrow[d, squiggly, "qb"'] & B \arrow[d, squiggly, "t"] \\
A \arrow[r, "g"] & T
\end{tikzcd}
$$

commute in $\mathcal{A}$, we have $\mathcal{D}_{\mathcal{A},\mathcal{N}}((f/s)^{\mathrm{op}}) = (qa)^{\mathrm{op}}/(qb)^{\mathrm{op}} = \mathcal{D}_{\mathcal{A},\mathcal{N}}((g/t)^{\mathrm{op}})$.

$$
\begin{tikzcd}
Q \arrow[r, squiggly, "q"] & P \arrow[dl, squiggly, "b"'] \arrow[ddr, bend left, "a"] & \\
 & S \arrow[d, squiggly, "x"] & \\
A \arrow[ur, "f"'] \arrow[dr, bend right, "g"'] & X & B \arrow[ul, squiggly, "s"'] \arrow[dl, squiggly, "t"] \\
 & T \arrow[u, squiggly, "y"'] &
\end{tikzcd}
$$

So the definition of $\mathcal{D}_{\mathcal{A},\mathcal{N}}((f/s)^{\mathrm{op}})$ is independent of the choice of the representative of the equivalence class.

Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$.

Choose pullbacks

$$
\begin{tikzcd}
S_\circ \arrow[r, "f_\circ"] \arrow[d, squiggly, "s_\circ"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, squiggly, "s"] \\
A \arrow[r, "f"] & S,
\end{tikzcd}
\qquad
\begin{tikzcd}
T_\circ \arrow[r, "g_\circ"] \arrow[d, squiggly, "t_\circ"'] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, squiggly, "t"] \\
B \arrow[r, "g"] & T
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
Q \arrow[r, "h"] \arrow[d, squiggly, "u"'] \arrow[dr, phantom, "\ulcorner", very near start] & T_\circ \arrow[d, squiggly, "t_\circ"] \\
S_\circ \arrow[r, "f_\circ"] & B
\end{tikzcd}
$$

and a pushout

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, squiggly, "s"'] \arrow[dr, phantom, "\lrcorner", very near end] & T \arrow[d, squiggly, "s'"] \\
S \arrow[r, "g'"'] & P
\end{tikzcd}
$$

in $\mathcal{A}$.

So the diagram

$$
\begin{tikzcd}
Q \arrow[r, "h"] \arrow[d, squiggly, "u"'] & T_\circ \arrow[r, "g_\circ"] \arrow[d, squiggly, "t_\circ"] & C \arrow[d, squiggly, "t"] \\
S_\circ \arrow[r, "f_\circ"] \arrow[d, squiggly, "s_\circ"'] & B \arrow[r, "g"] \arrow[d, squiggly, "s"'] & T \arrow[d, squiggly, "s'"] \\
A \arrow[r, "f"] & S \arrow[r, "g'"] & P
\end{tikzcd}
$$

commutes in $\mathcal{A}$.

We have

$$\mathcal{D}_{\mathcal{A},\mathcal{N}}((g/t)^{\mathrm{op}} \cdot (f/s)^{\mathrm{op}}) = \mathcal{D}_{\mathcal{A},\mathcal{N}}((f/s \cdot g/t)^{\mathrm{op}}) = \mathcal{D}_{\mathcal{A},\mathcal{N}}(fg'/ts') = (hg_\circ)^{\mathrm{op}}/(us_\circ)^{\mathrm{op}}.$$

Since the diagram

$$\begin{tikzcd}
& & A \arrow[d, "s_\circ^{\mathrm{op}}"] \\
& B \arrow[r, "f_\circ^{\mathrm{op}}"] \arrow[d, "t_\circ^{\mathrm{op}}"'] & S_\circ \arrow[d, "u^{\mathrm{op}}"] \\
C \arrow[r, "g_\circ^{\mathrm{op}}"] & T_\circ \arrow[r, "h^{\mathrm{op}}"] & Q
\end{tikzcd}$$

commutes in $\mathcal{A}^{\mathrm{op}}$, we conclude that

$$\mathcal{D}_{\mathcal{A},\mathcal{N}}((g/t)^{\mathrm{op}}) \cdot \mathcal{D}_{\mathcal{A},\mathcal{N}}((f/s)^{\mathrm{op}})) = g_\circ^{\mathrm{op}}/t_\circ^{\mathrm{op}} \cdot f_\circ^{\mathrm{op}}/s_\circ^{\mathrm{op}} = (hg_\circ)^{\mathrm{op}}/(us_\circ)^{\mathrm{op}} = \mathcal{D}_{\mathcal{A},\mathcal{N}}((g/t)^{\mathrm{op}} \cdot (f/s)^{\mathrm{op}}).$$

The diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] & S \arrow[d, "1"] \\
A \arrow[r, "f"] & S
\end{tikzcd}$$

commutes in $\mathcal{A}$, so $\mathcal{D}_{\mathcal{A},\mathcal{N}}((f/1_S)^{\mathrm{op}}) = f^{\mathrm{op}}/1_A^{\mathrm{op}}$.

In particular, we have $\mathcal{D}_{\mathcal{A},\mathcal{N}}((1_A/1_A)^{\mathrm{op}}) = 1_A^{\mathrm{op}}/1_A^{\mathrm{op}}$.

We want to show that $\mathcal{D}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}},\mathcal{N}^{\mathrm{op}}} \circ \mathcal{D}_{\mathcal{A},\mathcal{N}} = 1_{(\mathcal{A}/\!\!/\mathcal{N})^{\mathrm{op}}}$.

Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B$ in $\mathcal{A}$.

Choose a pullback

$$\begin{tikzcd}
S_\circ \arrow[r, "f_\circ"] \arrow[d, squiggly, two heads, "s_\circ"'] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[d, squiggly, two heads, "s"] \\
A \arrow[r, "f"] & S
\end{tikzcd}$$

in $\mathcal{A}$. Since the diagram

$$\begin{tikzcd}
S \arrow[r, "f^{\mathrm{op}}"] \arrow[d, "s^{\mathrm{op}}"'] & A \arrow[d, "s_\circ^{\mathrm{op}}"] \\
B \arrow[r, "f_\circ^{\mathrm{op}}"] & S_\circ
\end{tikzcd}$$

commutes in $\mathcal{A}^{\mathrm{op}}$, we have

$$\big(\mathcal{D}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}},\mathcal{N}^{\mathrm{op}}} \circ \mathcal{D}_{\mathcal{A},\mathcal{N}}\big)((f/s)^{\mathrm{op}}) = \mathcal{D}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}},\mathcal{N}^{\mathrm{op}}}(f_\circ^{\mathrm{op}}/s_\circ^{\mathrm{op}}) = (f/s)^{\mathrm{op}}.$$

Dually, we have $\mathcal{D}^{\mathrm{op}}_{\mathcal{A},\mathcal{N}} \circ \mathcal{D}_{\mathcal{A}^{\mathrm{op}},\mathcal{N}^{\mathrm{op}}} = 1_{(\mathcal{A}^{\mathrm{op}}/\!\!/\mathcal{N}^{\mathrm{op}})^{\mathrm{op}}}$ and thus $\mathcal{D}_{\mathcal{A},\mathcal{N}} \circ \mathcal{D}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}},\mathcal{N}^{\mathrm{op}}} = 1_{\mathcal{A}^{\mathrm{op}}/\!\!/\mathcal{N}^{\mathrm{op}}}$. $\square$

Lemma 3.2.8. Suppose given $A \xrightarrow{f} S \xleftarrow{s} B$ in $\mathcal{A}$, representing $A \xrightarrow{f/s} B$ in $\mathcal{A}/\!\!/\mathcal{N}$.

The following three statements are equivalent.

(a) The morphism f/s is an isomorphism in $\mathcal{A}/\!\!/\mathcal{N}$.

(b) The morphism $f/1_S$ is an isomorphism in $\mathcal{A}/\!\!/\mathcal{N}$.

(c) The morphism f is an $\mathcal{N}$-quasi-isomorphism in $\mathcal{A}$.

In this case, we have $(f/s)^{-1} = s/f$.

Proof. By remark 3.2.4, we have $f/s = f/1_S \cdot 1_S/s$, where $1_S/s$ is an isomorphism in $\mathcal{A}/\!\!/\mathcal{N}$. Thus (a) and (b) are equivalent.

If f is an $\mathcal{N}$-quasi-isomorphism in $\mathcal{A}$, then $f/1_S$ is an isomorphism in $\mathcal{A}/\!\!/\mathcal{N}$ with inverse $1_S/f$ by loc. cit.

Suppose that $f/1_S$ is an isomorphism in $\mathcal{A}/\!\!/\mathcal{N}$ and suppose given $S \xrightarrow{g} T \xleftarrow{t} A$ in $\mathcal{A}$ such that g/t is the inverse of $f/1_S$ in $\mathcal{A}/\!\!/\mathcal{N}$.
Thus $1_A/1_A = f/1_S \cdot g/t = fg/t$ by loc. cit. We conlude that there exist $\mathcal{N}$-quasi-isomorphisms $A \xrightarrow{u} U$ and $T \xrightarrow{v} U$ in $\mathcal{A}$ such that $u = fgv$ and $u = tv$. By lemma 3.1.26, fg is an $\mathcal{N}$-quasi-isomorphism.

Choose a kernel $K \xrightarrow{k} A$ of f and a cokernel $S \xrightarrow{c} C$ of f in $\mathcal{A}$. We have $K \in \operatorname{Ob}\mathcal{N}$ by lemma 3.1.25.

Dually, $(f/1_S)^{\mathrm{op}}$ is an isomorphism in $(\mathcal{A}/\!\!/\mathcal{N})^{\mathrm{op}}$ and, consequently, $\mathcal{D}_{\mathcal{A},\mathcal{N}}((f/1_S)^{\mathrm{op}}) = f^{\mathrm{op}}/1_A^{\mathrm{op}}$ is an isomorphism in $\mathcal{A}^{\mathrm{op}}/\!\!/\mathcal{N}^{\mathrm{op}}$, cf. definition 3.2.7. Since c^{op} is a kernel of f^{op} in $\mathcal{A}^{\mathrm{op}}$, we have $C \in \operatorname{Ob}(\mathcal{N}^{\mathrm{op}}) = \operatorname{Ob}\mathcal{N}$ as seen above.

Therefore f is an $\mathcal{N}$-quasi-isomorphism. $\square$

3.2.2 The quotient category is additive

Lemma/Definition 3.2.9. Suppose given $A, B \in \operatorname{Ob}\mathcal{A}$. The set $\operatorname{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(A, B)$ becomes an abelian group with the following addition.

Suppose given $A \xrightarrow{f} S \xleftarrow{s} B$ and $A \xrightarrow{g} T \xleftarrow{t} B$ in $\mathcal{A}$, representing $A \xrightarrow{f/s} B$ and $A \xrightarrow{g/t} B$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Choose a commutative diagram

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "t"'] & S \arrow[d, "u"] \\ T \arrow[r, "v"] & U \end{tikzcd}$$

in $\mathcal{A}$ such that u and v are $\mathcal{N}$-quasi-isomorphisms. This is possible since we may e.g. form the pushout of t and s, cf. lemma 3.1.27.(b).

Let $f/s + g/t := (fu + gv)/su$. This definition is independent of the choice of the commutative diagram and of the choice of the representatives of the equivalence classes.

Note that $f/s = fu/su$ and $g/t = gv/tv$, so

$$f/s + g/t = fu/su + gv/tv = fu/su + gv/su = (fu + gv)/su.$$

The zero element of $\operatorname{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(A, B)$ is $0_{A,B}/1_B$. We have $0_{A,B}/1_B = 0_{A,W}/w$ for all $\mathcal{N}$-quasi-isomorphisms $B \overset{w}{\rightsquigarrow} W$ in $\mathcal{A}$.

For $A \xrightarrow{\hat{f}} S$ in $\mathcal{A}$, we have $(f + \hat{f})/s = f/s + \hat{f}/s$. In particular, the additive inverse of f/s in $\operatorname{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(A, B)$ is $(-f)/s$.

Note that for a finite number of morphisms from A to B in $\mathcal{A}/\!\!/\mathcal{N}$, we may choose representatives such that they have a common denominator by remark 3.2.5.

Proof. Choose a pushout

$$\begin{tikzcd} B \arrow[r, rightsquigarrow, "s"] \arrow[d, rightsquigarrow, "t"'] & S \arrow[d, rightsquigarrow, "p"] \\ T \arrow[r, rightsquigarrow, "q"'] & P \end{tikzcd}$$

in $\mathcal{A}$. For a commutative diagram

$$\begin{tikzcd} B \arrow[r, rightsquigarrow, "s"] \arrow[d, rightsquigarrow, "t"'] & S \arrow[d, rightsquigarrow, "u"] \\ T \arrow[r, rightsquigarrow, "v"] & U \end{tikzcd}$$

in $\mathcal{A}$ such that u and v are $\mathcal{N}$-quasi-isomorphisms, there exists $P \xrightarrow{r} U$ in $\mathcal{A}$ with $pr = u$ and $qr = v$. Note that r is an $\mathcal{N}$-quasi-isomorphism by lemma 3.1.26.
Thus $(fp + gq)/sp = (fp + gq)r/spr = (fpr + gqr)/su = (fu + gv)/su$.
So the definition of $f/s + g/t$ is independent of the choice of the commutative diagram.

Suppose given $A \xrightarrow{f'} S' \overset{s'}{\leftsquigarrow} B$ and $A \xrightarrow{g'} T' \overset{t'}{\leftsquigarrow} B$ in $\mathcal{A}$ with $f/s = f'/s'$ and $g/t = g'/t'$. So there exist $\mathcal{N}$-quasi-isomorphisms $S \overset{x}{\rightsquigarrow} X$, $S' \overset{x'}{\rightsquigarrow} X$, $T \overset{y}{\rightsquigarrow} Y$ and $T' \overset{y'}{\rightsquigarrow} Y$ in $\mathcal{A}$ such that $fx = f'x'$, $sx = s'x'$, $gy = g'y'$ and $ty = t'y'$.

Choose a pushout

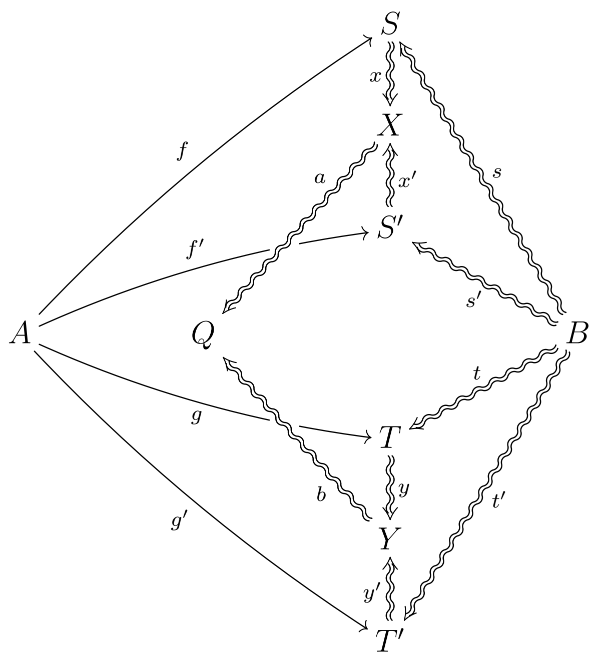
$$\begin{tikzcd} B \arrow[r, rightsquigarrow, "sx"] \arrow[d, rightsquigarrow, "ty"'] & X \arrow[d, rightsquigarrow, "a"] \\ Y \arrow[r, rightsquigarrow, "b"'] & Q \end{tikzcd}$$

in $\mathcal{A}$. Note that a and b are $\mathcal{N}$-quasi-isomorphisms by lemma 3.1.27.(b).

Since the diagrams

$$\begin{tikzcd} B \arrow[r, rightsquigarrow, "s"] \arrow[d, rightsquigarrow, "t"'] & S \arrow[d, rightsquigarrow, "xa"] \\ T \arrow[r, rightsquigarrow, "yb"] & Q \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} B \arrow[r, rightsquigarrow, "s'"] \arrow[d, rightsquigarrow, "t'"'] & S \arrow[d, rightsquigarrow, "x'a"] \\ T \arrow[r, rightsquigarrow, "y'b"] & Q \end{tikzcd}$$

commute in $\mathcal{A}$, we have $f/s + g/t = (fxa + gyb)/sxa = (f'x'a + g'y'b)/s'x'a = f'/s' + g'/t'$.



So the definition of $f/s+g/t$ is independent of the choice of the representatives of the equivalence classes.

For all $\mathcal{N}$-quasi-isomorphisms $B \xrightarrow{w} W$ in $\mathcal{A}$, we have $0_{A,B}/1_B = (0_{A,B} \cdot w)/w = 0_{A,W}/w$.

For $A \xrightarrow{\hat{f}} S$ in $\mathcal{A}$, we have $(f + \hat{f})/s = f/s + \hat{f}/s$ since the diagram

$$\begin{tikzcd} B \arrow[r, rightsquigarrow, "s"] \arrow[d, rightsquigarrow, "s"'] & S \arrow[d, rightsquigarrow, "1"] \\ S \arrow[r, rightsquigarrow, "1"] & S \end{tikzcd}$$

commutes in $\mathcal{A}$.

Suppose given three morphisms from A to B in $\mathcal{A}/\!\!/\mathcal{N}$, which we may assume to be represented by $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B$, $A \xrightarrow{g} S \overset{s}{\leftsquigarrow} B$ and $A \xrightarrow{h} S \overset{s}{\leftsquigarrow} B$ in $\mathcal{A}$ by remark 3.2.5.

We have

$$f/s + g/s = (f + g)/s = (g + f)/s = g/s + f/s,$$

$$\begin{aligned}(f/s+g/s)+h/s &= ((f+g)/s)+h/s = (f+g+h)/s = f/s+(g+h)/s \\ &= f/s+(g/s+h/s),\end{aligned}$$

$$0_{A,S}/s+f/s = (0_{A,S}+f)/s = f/s = (f+0_{A,S})/s = f/s+0_{A,S}/s$$

and

$$f/s+(-f)/s = (f-f)/s = 0_{A,S}/s = (-f+f)/s = (-f)/s+f/s. \qquad \square$$

Lemma 3.2.10. The quotient category $\mathcal{A}/\!\!/\mathcal{N}$ is preadditive.

Proof. Suppose given $A \xrightarrow{f} S \xleftarrow{s} B \overset{g}{\underset{h}{\rightrightarrows}} T \xleftarrow{t} C \xrightarrow{\ell} U \xleftarrow{u} D$ in $\mathcal{A}$, cf. remark 3.2.5. We have to show that $f/s \cdot (g/t+h/t) = f/s \cdot g/t + f/s \cdot h/t$ and that $(g/t+h/t)\cdot \ell/u = g/t \cdot \ell/u + h/t \cdot \ell/u$.

Choose a pushout

$$\begin{tikzcd} C \arrow[r, "\ell"] \arrow[d, "t"'] & U \arrow[d, "v"] \\ T \arrow[r, "\ell'"'] & P \end{tikzcd}$$

in $\mathcal{A}$. We have

$$\begin{aligned}(g/t+h/t)\cdot \ell/u &= (g+h)/t \cdot \ell/u = (g+h)\ell'/uv = (g\ell'+h\ell')/uv = g\ell'/uv + h\ell'/uv \\ &= g/t\cdot \ell/u + h/t \cdot \ell/u.\end{aligned}$$

Write $\mathcal{D} := \mathcal{D}_{\mathcal{A},\mathcal{N}} \colon (\mathcal{A}/\!\!/\mathcal{N})^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}/\!\!/\mathcal{N}^{\mathrm{op}}$. Since $\mathcal{D}$ is an isomorphism of categories, it suffices to show that $\mathcal{D}((f/s\cdot(g/t+h/t))^{\mathrm{op}}) = \mathcal{D}((f/s\cdot g/t + f/s \cdot h/t)^{\mathrm{op}})$. At first, we want to show that $\mathcal{D}((g/t+h/t)^{\mathrm{op}}) = \mathcal{D}((g/t)^{\mathrm{op}}) + \mathcal{D}((h/t)^{\mathrm{op}})$.

Choose pullbacks

$$\begin{tikzcd} P' \arrow[r, "g'"] \arrow[d, "t'"'] & C \arrow[d, "t"] \\ B \arrow[r, "g"'] & T, \end{tikzcd} \qquad \begin{tikzcd} \tilde{P} \arrow[r, "\tilde{h}"] \arrow[d, "\tilde{t}"'] & C \arrow[d, "t"] \\ B \arrow[r, "h"'] & T, \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Q \arrow[r, "x'"] \arrow[d, "\tilde{x}"'] & P' \arrow[d, "t'"] \\ \tilde{P} \arrow[r, "\tilde{t}"'] & B \end{tikzcd}$$

in $\mathcal{A}$. Since the diagrams

$$\begin{tikzcd} Q \arrow[r, "x'g'"] \arrow[d, "x't'"'] & C \arrow[d, "t"] \\ B \arrow[r, "g"'] & T, \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "\tilde{x}\tilde{h}"] \arrow[d, "x't'"'] & C \arrow[d, "t"] \\ B \arrow[r, "h"'] & T \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Q \arrow[rr, "x'g'+\tilde{x}\tilde{h}"] \arrow[d, "x't'"'] & & C \arrow[d, "t"] \\ B \arrow[rr, "g+h"'] & & T \end{tikzcd}$$

commute in $\mathcal{A}$, we have

$$\mathcal{D}((g/t)^{\mathrm{op}}) = (x'g')^{\mathrm{op}}/(x't')^{\mathrm{op}},$$

$$\mathcal{D}((h/t)^{\mathrm{op}}) = (\tilde{x}\tilde{h})^{\mathrm{op}}/(x't')^{\mathrm{op}}$$

and

$$\mathcal{D}((g/t + h/t)^{\mathrm{op}}) = \mathcal{D}(((g+h)/t)^{\mathrm{op}}) = (x'g' + \tilde{x}\tilde{h})^{\mathrm{op}}/(x't')^{\mathrm{op}}.$$

Thus $\mathcal{D}((g/t + h/t)^{\mathrm{op}}) = \mathcal{D}((g/t)^{\mathrm{op}}) + \mathcal{D}((h/t)^{\mathrm{op}})$.

$$\begin{tikzcd}
Q \arrow[r, rightsquigarrow, "x'"] \arrow[d, rightsquigarrow, "\tilde{x}"'] & P' \arrow[dd, rightsquigarrow, "t'"'] \arrow[dr, "g'"] & \\
\tilde{P} \arrow[rr, "\tilde{h}"] \arrow[dr, rightsquigarrow, "\tilde{t}"'] & & C \arrow[d, rightsquigarrow, "t"] \\
& B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & T
\end{tikzcd}$$

We conclude that

$$\begin{aligned}
\mathcal{D}((f/s \cdot (g/t + h/t))^{\mathrm{op}}) &= \mathcal{D}((g/t + h/t)^{\mathrm{op}}) \cdot \mathcal{D}((f/s)^{\mathrm{op}}) \\
&= \big(\mathcal{D}((g/t)^{\mathrm{op}}) + \mathcal{D}((h/t)^{\mathrm{op}})\big) \cdot \mathcal{D}((f/s)^{\mathrm{op}}) \\
&= \mathcal{D}((g/t)^{\mathrm{op}}) \cdot \mathcal{D}((f/s)^{\mathrm{op}}) + \mathcal{D}((h/t)^{\mathrm{op}}) \cdot \mathcal{D}((f/s)^{\mathrm{op}}) \\
&= \mathcal{D}((f/s \cdot g/t)^{\mathrm{op}}) + \mathcal{D}((f/s \cdot h/t)^{\mathrm{op}}) \\
&= \mathcal{D}((f/s \cdot g/t + f/s \cdot h/t)^{\mathrm{op}}).
\end{aligned}$$ $\square$

Lemma 3.2.11. Suppose given a zero object $Z \in \operatorname{Ob}\mathcal{A}$. Then Z is a zero object in $\mathcal{A}/\!\!/\mathcal{N}$ as well.

Suppose given a direct sum $A \underset{p}{\overset{i}{\rightleftarrows}} C \underset{q}{\overset{j}{\leftrightarrows}} B$ in $\mathcal{A}$. Then $A \underset{p/1}{\overset{i/1}{\rightleftarrows}} C \underset{q/1}{\overset{j/1}{\leftrightarrows}} B$ is a direct sum in $\mathcal{A}/\!\!/\mathcal{N}$ as well.

Proof. The object Z is a zero object in $\mathcal{A}/\!\!/\mathcal{N}$ since $1_Z/1_Z = 0_Z/1_Z$.

We have

$$i/1_C \cdot p/1_A = ip/1_A = 1_A/1_A\,,$$

$$j/1_C \cdot q/1_B = jq/1_B = 1_B/1_B$$

and

$$p/1_A \cdot i/1_C + q/1_B \cdot j/1_C = (pi + qj)/1_C = 1_C/1_C\,.$$ $\square$

Proposition 3.2.12. The quotient category $\mathcal{A}/\!\!/\mathcal{N}$ is additive. Moreover, the localisation functor $\mathrm{L} = \mathrm{L}_{\mathcal{A},\mathcal{N}} \colon \mathcal{A} \to \mathcal{A}/\!\!/\mathcal{N}$ is additive.

Proof. The quotient category $\mathcal{A}/\!\!/\mathcal{N}$ is additive by lemmata 3.2.9, 3.2.10 and 3.2.11.
For $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{A}$, we have $\mathrm{L}(f+g) = (f+g)/1_B = f/1_B + g/1_B = \mathrm{L}(f) + \mathrm{L}(g)$, cf. definition 3.2.9. $\square$

Lemma 3.2.13. Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B$ in $\mathcal{A}$, representing $A \xrightarrow{f/s} B$ in $\mathcal{A}/\!\!/\mathcal{N}$. The following four statements are equivalent.

(a) We have $f/s = 0$ in $\mathcal{A}/\!\!/\mathcal{N}$.

(b) There exists an $\mathcal{N}$-quasi-isomorphism $S \overset{u}{\rightsquigarrow} U$ such that $fu = 0$ in $\mathcal{A}$.

(c) We have $f/1_S = 0$ in $\mathcal{A}/\!\!/\mathcal{N}$.

(d) There exists an $\mathcal{N}$-quasi-isomorphism $V \overset{v}{\rightsquigarrow} A$ such that $vf = 0$ in $\mathcal{A}$.

Proof. Ad (a)$\Leftrightarrow$(c). By remark 3.2.4, we have $f/s = f/1_S \cdot 1_S/s$, where $1_S/s$ is an isomorphism in $\mathcal{A}/\!\!/\mathcal{N}$. Thus (a) and (c) are equivalent.

Ad (b)$\Leftrightarrow$(d). This follows from lemma 3.1.28.

Ad (c)$\Leftrightarrow$(b). We have $f/1_S = 0 = 0_{A,S}/1_S$ in $\mathcal{A}/\!\!/\mathcal{N}$ if and only if there exist $\mathcal{N}$-quasi-isomorphisms $S \overset{u}{\rightsquigarrow} U$ and $S \overset{w}{\rightsquigarrow} U$ such that $fu = 0$ and $u = w$, i.e. if and only if there exists an $\mathcal{N}$-quasi-isomorphism $S \overset{u}{\rightsquigarrow} U$ such that $fu = 0$ in $\mathcal{A}$. $\square$

Remark 3.2.14. Suppose given an object $N \in \mathrm{Ob}\,\mathcal{A}$. Then N is a zero object in $\mathcal{A}/\!\!/\mathcal{N}$ if and only if $N \in \mathrm{Ob}\,\mathcal{N}$.

Proof. Suppose that N is a zero object in $\mathcal{A}/\!\!/\mathcal{N}$. Since $1_N/1_N = 0$, lemma 3.2.13 yields an $\mathcal{N}$-quasi-isomorphism $N \overset{s}{\rightsquigarrow} S$ such that $s = 1_N \cdot s = 0$ in $\mathcal{A}$. Therefore $N \in \mathrm{Ob}\,\mathcal{N}$, cf. example 3.1.24.

Conversely, if $N \in \mathrm{Ob}\,\mathcal{N}$, then 0_N is an $\mathcal{N}$-quasi-isomorphism.
Thus $1_N/1_N = 1_N \cdot 0_N/1_N \cdot 0_N = 0_N/0_N$, cf. lemma 3.2.9. $\square$

3.2.3 The quotient category is abelian

Lemma 3.2.15. Suppose given $K \xrightarrow{k} A \xrightarrow{f} B \xrightarrow{c} C$ in $\mathcal{A}$ such that k is a kernel of f and c is a cokernel of f.

(a) The morphism $f/1_B$ is epimorphic in $\mathcal{A}/\!\!/\mathcal{N}$ if and only if $C \in \mathrm{Ob}\,\mathcal{N}$.

(b) The morphism $f/1_B$ is monomorphic in $\mathcal{A}/\!\!/\mathcal{N}$ if and only if $K \in \mathrm{Ob}\,\mathcal{N}$.

Proof. Ad (a). Suppose that $f/1_B$ is epimorphic in $\mathcal{A}/\!\!/\mathcal{N}$.
We have $f/1_B \cdot c/1_C = fc/1_C = 0/1_C$. So $c/1_C = 0$ since $f/1_B$ is epimorphic. By lemma 3.2.13, there exists an $\mathcal{N}$-quasi-isomorphism $C \overset{u}{\rightsquigarrow} U$ such that $cu = 0$ in $\mathcal{A}$. Since c is epimorphic in $\mathcal{A}$, we conclude that $u = 0$. Thus $C \in \operatorname{Ob} \mathcal{N}$, cf. example 3.1.24.

Conversely, suppose that $C \in \operatorname{Ob} \mathcal{N}$. Thus $c/1_C = 0$ in $\mathcal{A}/\!\!/\mathcal{N}$, cf. remark 3.2.14. Suppose given $B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$, representing $B \xrightarrow{g/t} C$ in $\mathcal{A}/\!\!/\mathcal{N}$, such that $fg/t = f/1_B \cdot g/t = 0$. By lemma 3.2.13, there exists an $\mathcal{N}$-quasi-isomorphism $T \overset{u}{\rightsquigarrow} U$ such that $fgu = 0$ in $\mathcal{A}$. Since c is a cokernel of f, there exists $C \xrightarrow{a} U$ in $\mathcal{A}$ such that $gu = ca$. So $g/t = gu/tu = ca/tu = c/1_C \cdot a/tu = 0$. We conclude that $f/1_B$ is epimorphic in $\mathcal{A}/\!\!/\mathcal{N}$.

Ad (b). Note that k^{op} is a cokernel of f^{op} in $\mathcal{A}^{\mathrm{op}}$. The morphism $f/1_B$ is monomorphic if and only if $f^{\mathrm{op}}/1_A^{\mathrm{op}} = \mathcal{D}_{\mathcal{A},\mathcal{N}}((f/1_B)^{\mathrm{op}})$ is epimorphic in $\mathcal{A}^{\mathrm{op}}/\!\!/\mathcal{N}^{\mathrm{op}}$. We have $K \in \operatorname{Ob} \mathcal{N}$ if and only if $K \in \operatorname{Ob}(\mathcal{N}^{\mathrm{op}})$. The result now follows from (a). $\square$

Lemma 3.2.16. Suppose given $K \xrightarrow{k} A \xrightarrow{f} B \xrightarrow{c} C$ in $\mathcal{A}$ such that k is a kernel of f and c is a cokernel of f.

(a) The morphism $c/1_C$ is a cokernel of $f/1_B$ in $\mathcal{A}/\!\!/\mathcal{N}$.

(b) The morphism $k/1_A$ is a kernel of $f/1_B$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Proof. Ad (a). We have $f/1_B \cdot c/1_C = fc/1_C = 0/1_C$.
By lemma 3.2.15.(a), $c/1_C$ is an epimorphism since $C \longrightarrow 0$ is a cokernel of c in $\mathcal{A}$.

Suppose given $B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$ such that $fg/t = f/1_B \cdot g/t = 0$. By lemma 3.2.13, there exists an $\mathcal{N}$-quasi-isomorphism $T \overset{u}{\rightsquigarrow} U$ in $\mathcal{A}$ such that $fgu = 0$. Since c is a cokernel of f in $\mathcal{A}$, there exists $C \xrightarrow{a} U$ in $\mathcal{A}$ such that $gu = ca$. Thus $g/t = gu/tu = ca/tu = c/1_C \cdot a/tu$.

Ad (b). The morphism $k/1_A$ is a kernel of $f/1_B$ in $\mathcal{A}/\!\!/\mathcal{N}$ if and only if $k^{\mathrm{op}}/1_A^{\mathrm{op}} = \mathcal{D}_{\mathcal{A},\mathcal{N}}((k/1_A)^{\mathrm{op}})$ is a cokernel of $f^{\mathrm{op}}/1_A^{\mathrm{op}} = \mathcal{D}_{\mathcal{A},\mathcal{N}}((f/1_B)^{\mathrm{op}})$ in $\mathcal{A}^{\mathrm{op}}/\!\!/\mathcal{N}^{\mathrm{op}}$. Moreover, k^{op} is a cokernel f^{op} in $\mathcal{A}^{\mathrm{op}}$. The result now follows from (a). $\square$

Lemma 3.2.17. Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B$ in $\mathcal{A}$, representing $A \xrightarrow{f/s} B$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Suppose given a kernel-cokernel-factorisation

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[rrr, "f"] \arrow[dr, "p"'] & & & S \arrow[r, "c"] & C \\
& & I \arrow[r, "1"] & I \arrow[ur, "i"'] & &
\end{tikzcd}$$

of f in $\mathcal{A}$. Then

$$\begin{tikzcd}
K \arrow[r, "k/1"] & A \arrow[rrr, "f/s"] \arrow[dr, "p/1"'] & & & B \arrow[r, "sc/1"] & C \\
& & I \arrow[r, "1/1"] & I \arrow[ur, "i/s"'] & &
\end{tikzcd}$$

is a kernel-cokernel-factorisation of f/s in $\mathcal{A}/\!\!/\mathcal{N}$.

Proof. We have $p/1_I \cdot 1_I/1_I \cdot i/s = pi/s = f/s$.

We will use lemma 3.2.16 repeatedly.

The morphism $k/1_A$ is a kernel of $f/1_S$ and, consequently, of $f/s = f/1_S \cdot 1_S/s$ in $\mathcal{A}/\!\!/\mathcal{N}$, cf. remark 3.2.4.

The morphism $p/1_I$ is a cokernel of $k/1_A$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Note that the diagrams

$$\begin{tikzcd} A \arrow[r, "f/1"] \arrow[d, "1/1"'] & S \arrow[r, "c/1"] \arrow[d, "1/s"'] & C \arrow[d, "1/1"] \\ A \arrow[r, "f/s"] & B \arrow[r, "sc/1"] & C \end{tikzcd}$$

and

$$\begin{tikzcd} I \arrow[r, "i/1"] \arrow[d, "1/1"'] & S \arrow[r, "c/1"] \arrow[d, "1/s"'] & C \arrow[d, "1/1"] \\ I \arrow[r, "i/s"] & B \arrow[r, "sc/1"] & C \end{tikzcd}$$

are isomorphisms in $(\mathcal{A}/\!\!/\mathcal{N})^{\Delta_2}$.
Thus $sc/1$ is a cokernel of f/s and i/s is a kernel of $sc/1$ in $\mathcal{A}/\!\!/\mathcal{N}$. □

Theorem 3.2.18. Recall that $\mathcal{A}$ is an abelian category and $\mathcal{N}$ is a thick subcategory of $\mathcal{A}$. Recall that $\mathcal{A}/\!\!/\mathcal{N}$ is the quotient category of $\mathcal{A}$ by $\mathcal{N}$ and $\mathrm{L} = \mathrm{L}_{\mathcal{A},\mathcal{N}} \colon \mathcal{A} \to \mathcal{A}/\!\!/\mathcal{N}$ is the localisation functor. Cf. definitions 3.2.3 and 3.2.6.

The quotient category $\mathcal{A}/\!\!/\mathcal{N}$ is abelian and the localisation functor $\mathrm{L}\colon \mathcal{A} \to \mathcal{A}/\!\!/\mathcal{N}$ is exact. Moreover, $\mathrm{Ker}(\mathrm{L}) = \mathcal{N}$. In particular, $\mathrm{L}(N) \cong 0_{\mathcal{A}/\!\!/\mathcal{N}}$ for $N \in \mathrm{Ob}\,\mathcal{N}$.

Proof. The category $\mathcal{A}/\!\!/\mathcal{N}$ is abelian by proposition 3.2.12 and lemma 3.2.17.

The functor L is additive by proposition 3.2.12.

Suppose given a short exact sequence $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ in $\mathcal{A}$. By lemma 3.2.16, $\mathrm{L}(f) = f/1_B$ is a kernel of $\mathrm{L}(g) = g/1_C$ and $\mathrm{L}(g)$ is a cokernel of $\mathrm{L}(f)$. Thus the sequence $A \xrightarrow{f/1} B \xrightarrow{g/1} C$ is short exact in $\mathcal{A}/\!\!/\mathcal{N}$.

Suppose given $N \in \mathrm{Ob}\,\mathcal{A}$. By remark 3.2.14, $\mathrm{L}(N) = N$ is a zero object in $\mathcal{A}/\!\!/\mathcal{N}$ if and only if $N \in \mathrm{Ob}\,\mathcal{N}$. Thus $\mathrm{Ker}(\mathrm{L}) = \mathcal{N}$. □

Remark 3.2.19. The thick subcategories of $\mathcal{A}$ are precisely the kernels of exact functors from $\mathcal{A}$ to abelian categories. Cf. corollary 3.1.8 and theorem 3.2.18.

Lemma 3.2.20. Each short exact sequence in $\mathcal{A}/\!\!/\mathcal{N}$ is isomorphic to the image of a short exact sequence in $\mathcal{A}$ under L.

More precisely, suppose given $A \xrightarrow{f} S \xleftarrow{s} B \xrightarrow{g} T \xleftarrow{t} D$ in $\mathcal{A}$, representing a short exact sequence $A \overset{f/s}{\rightarrowtail} B \overset{g/t}{\twoheadrightarrow} D$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Choose a kernel-cokernel-factorisation

$$\begin{tikzcd}
K \arrow[r, tail, "k"] & B \arrow[rrr, "g"] \arrow[dr, two heads, "p"'] & & & T \arrow[r, two heads, "c"] & C \\
& & I \arrow[r, "1"] & I \arrow[ur, tail, "i"'] & &
\end{tikzcd}$$

of g in $\mathcal{A}$. Then there exists a morphism $K \xrightarrow{h/u} A$ in $\mathcal{A}/\!\!/\mathcal{N}$ such that

$$\begin{tikzcd}
K \arrow[r, "k/1"] \arrow[d, "h/u"'] & B \arrow[r, "p/1"] \arrow[d, "1/1"] & I \arrow[d, "i/t"] \\
A \arrow[r, "f/s"] & B \arrow[r, "g/t"] & D
\end{tikzcd}$$

is an isomorphism in $(\mathcal{A}/\!\!/\mathcal{N})^{\Delta_2}$.

Proof. The sequence $K \xrightarrow{k/1} B \xrightarrow{p/1} I$ is short exact in $\mathcal{A}/\!\!/\mathcal{N}$ since L is exact, cf. theorem 3.2.18.

The morphism $k/1_B$ is a kernel of $g/t = g/1_T \cdot 1_T/t$ in $\mathcal{A}/\!\!/\mathcal{N}$ by lemma 3.2.16.(b) and remark 3.2.4. Since f/s is a kernel of g/t by assumption, there exists an isomorphism $K \xrightarrow[\sim]{h/u} A$ in $\mathcal{A}/\!\!/\mathcal{N}$ such that $h/u \cdot f/s = k/1_B$.

Since $g/1_T = g/t \cdot t/1_T$ is epimorphic, we have $C \in \operatorname{Ob}\mathcal{N}$ by lemma 3.2.15.(a). Therefore i is an $\mathcal{N}$-quasi-isomorphism in $\mathcal{A}$. So i/t is an isomorphism in $\mathcal{A}/\!\!/\mathcal{N}$, cf. lemma 3.2.8.

Finally, we have $p/1_I \cdot i/t = pi/t = g/t$. $\square$

3.3 The universal property

Theorem 3.3.1. Recall that $\mathcal{A}$ is an abelian category and $\mathcal{N}$ is a thick subcategory of $\mathcal{A}$. Recall that $\mathcal{A}/\!\!/\mathcal{N}$ is the quotient category of $\mathcal{A}$ by $\mathcal{N}$ and $\mathrm{L} = \mathrm{L}_{\mathcal{A},\mathcal{N}} \colon \mathcal{A} \to \mathcal{A}/\!\!/\mathcal{N}$ is the localisation functor. Cf. definitions 3.2.3 and 3.2.6.

The category $\mathcal{A}/\!\!/\mathcal{N}$ is abelian and L is exact with $\operatorname{Ker}(\mathrm{L}) = \mathcal{N}$ by theorem 3.2.18.

Suppose given an abelian category $\mathcal{B}$.

(a) Suppose given an exact functor $F \colon \mathcal{A} \to \mathcal{B}$ satisfying $\mathcal{N} \subseteq \operatorname{Ker}(F)$.

There exists a unique exact functor $\hat{F}\colon \mathcal{A}/\!\!/\mathcal{N} \to \mathcal{B}$ such that $\hat{F} \circ \mathrm{L} = F$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\mathrm{L}"'] & \mathcal{B} \\
\mathcal{A}/\!\!/\mathcal{N} \arrow[ur, "\hat{F}"'] &
\end{tikzcd}$$

For $A \xrightarrow{f/s} B$ in $\mathcal{A}/\!\!/\mathcal{N}$, we have $\hat{F}(f/s) = F(f) \cdot F(s)^{-1}$.

(b) Suppose given exact functors $F, G\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{N} \subseteq \mathrm{Ker}(F)$ and $\mathcal{N} \subseteq \mathrm{Ker}(G)$ and a transformation $\alpha\colon F \Rightarrow G$.

There exists a unique transformation $\hat{\alpha}\colon \hat{F} \Rightarrow \hat{G}$ such that $\hat{\alpha} \star \mathrm{L} = \alpha$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] \arrow[d, "\mathrm{L}"'] & \mathcal{B} \\
\mathcal{A}/\!\!/\mathcal{N} \arrow[ur, "\hat{F}"] \arrow[ur, bend right, "\hat{G}"'] \arrow[ur, phantom, "\hat{\alpha}"] &
\end{tikzcd}$$

For $A \in \mathrm{Ob}\,\mathcal{A} = \mathrm{Ob}(\mathcal{A}/\!\!/\mathcal{N})$, we have $\hat{\alpha}_A = \alpha_A$.

Proof. Ad (a). Suppose given an $\mathcal{N}$-quasi-isomorphism $A \overset{f}{\rightsquigarrow} B$ in $\mathcal{A}$. Choose a kernel $K \xrightarrow{k} A$ of f and a cokernel $B \xrightarrow{c} C$ in $\mathcal{A}$. So $K, C \in \mathrm{Ob}\,\mathcal{N}$. We have $F(K) \cong 0_{\mathcal{B}} \cong F(C)$ by assumption. Since F is exact, $F(k)$ is a kernel of $F(f)$ and $F(c)$ is a cokernel of $F(f)$ in $\mathcal{B}$. We conclude that $F(f)$ is an isomorphism in $\mathcal{B}$.

Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B$ in $\mathcal{A}$.

Given a functor $\tilde{F}\colon \mathcal{A}/\!\!/\mathcal{N} \to \mathcal{B}$ with $\tilde{F} \circ \mathrm{L} = F$, we necessarily have

$$\begin{aligned}
\tilde{F}(f/s) &= \tilde{F}(f/1_S \cdot 1_S/s) = \tilde{F}(f/1_S \cdot (s/1_S)^{-1}) = \tilde{F}(\mathrm{L}(f)) \cdot \tilde{F}(\mathrm{L}(s)^{-1}) = \tilde{F}(\mathrm{L}(f)) \cdot \tilde{F}(\mathrm{L}(s))^{-1} \\
&= F(f) \cdot F(s)^{-1},
\end{aligned}$$

cf. remark 3.2.4.

Let $\hat{F}(A) := F(A)$ and let $\hat{F}(f/s) := F(f) \cdot F(s)^{-1}$.

This is well-defined since for $A \xrightarrow{g} T \overset{t}{\leftsquigarrow} B$ in $\mathcal{A}$ with $f/s = g/t$, there exist $\mathcal{N}$-quasi-isomorphisms $S \overset{u}{\rightsquigarrow} U$ and $T \overset{v}{\rightsquigarrow} U$ in $\mathcal{A}$ such that $fu = gv$ and $su = tv$. Therefore

$$\begin{aligned}
F(f) \cdot F(s)^{-1} &= F(f) \cdot F(u) \cdot F(u)^{-1} \cdot F(s)^{-1} = F(fu) \cdot F(su)^{-1} \\
&= F(gv) \cdot F(tv)^{-1} = F(g) \cdot F(v) \cdot F(v)^{-1} \cdot F(t)^{-1} = F(g) \cdot F(t)^{-1}.
\end{aligned}$$

Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$.

Choose a pushout

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, squiggly, "s"'] & T \arrow[d, squiggly, "s'"] \\ S \arrow[r, "g'"'] & P \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

in $\mathcal{A}$. We have

$$\begin{aligned} \hat{F}(f/s \cdot g/t) &= \hat{F}(fg'/ts') \\ &= F(fg') \cdot F(ts')^{-1} \\ &= F(f) \cdot F(g') \cdot F(ts')^{-1} \\ &= \mathrm{F}(f) \cdot F(s)^{-1} \cdot F(s) \cdot F(g') \cdot F(ts')^{-1} \\ &= \mathrm{F}(f) \cdot F(s)^{-1} \cdot F(sg') \cdot F(ts')^{-1} \\ &= \mathrm{F}(f) \cdot F(s)^{-1} \cdot F(gs') \cdot F(ts')^{-1} \\ &= \hat{F}(f/s) \cdot \hat{F}(gs'/ts') \\ &= \hat{F}(f/s) \cdot \hat{F}(g/t) \end{aligned}$$

and $\hat{F}(1_A/1_A) = F(1_A) \cdot F(1_A)^{-1} = 1_{F(A)}$.

So $\hat{F}$ is in fact a functor.

For $A \xrightarrow{f} B$ in $\mathcal{A}$, we have $\hat{F}(\mathrm{L}(f)) = \hat{F}(f/1_B) = F(f) \cdot F(1_B)^{-1} = F(f)$. Thus $\hat{F} \circ \mathrm{L} = F$.

The functor $\hat{F}$ is additive since

$$\begin{aligned} \hat{F}(f/s + g/s) = \hat{F}((f+g)/s) &= F(f+g) \cdot F(s)^{-1} = (F(f) + F(g)) \cdot F(s)^{-1} \\ &= F(f) \cdot F(s)^{-1} + F(g) \cdot F(s)^{-1} = \hat{F}(f/s) + \hat{F}(g/s) \end{aligned}$$

for $A \overset{f}{\underset{g}{\rightrightarrows}} S \xleftarrow{s} B$ in $\mathcal{A}$.

Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B \xrightarrow{g} T \overset{t}{\leftsquigarrow} C$ in $\mathcal{A}$, representing a short exact sequence $A \overset{f/s}{\rightarrowtail} B \overset{g/t}{\twoheadrightarrow} C$ in $\mathcal{A}/\!\!/\mathcal{N}$. By lemma 3.2.20, there exists a short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathcal{A}$ such that $A \overset{f/s}{\rightarrowtail} B \overset{g/t}{\twoheadrightarrow} C$ and $X \overset{\mathrm{L}(i)}{\rightarrowtail} Y \overset{\mathrm{L}(p)}{\twoheadrightarrow} Z$ are isomorphic in $(\mathcal{A}/\!\!/\mathcal{N})^{\Delta_2}$.

By assumption,

$$\hat{F}\big(X \overset{\mathrm{L}(i)}{\rightarrowtail} Y \overset{\mathrm{L}(p)}{\twoheadrightarrow} Z\big) = \big(X \overset{F(i)}{\rightarrowtail} Y \overset{F(p)}{\twoheadrightarrow} Z\big)$$

is short exact and thus $\hat{F}\big(A \overset{f/s}{\rightarrowtail} B \overset{g/t}{\twoheadrightarrow} C\big)$ is short exact in $\mathcal{B}$ as well since $\hat{F}$ is a functor.

We conclude that $\hat{F}$ is exact.

Ad (b). Given a transformation $\tilde{\alpha}\colon \hat{F} \Rightarrow \hat{G}$ with $\tilde{\alpha}\star\mathrm{L} = \alpha$, we necessarily have $\tilde{\alpha}_A = \tilde{\alpha}_{\mathrm{L}(A)} = \alpha_A$ for $A \in \operatorname{Ob}\mathcal{A}$.

Let $\hat{\alpha}_A := \alpha_A$ for $A \in \operatorname{Ob}\mathcal{A}$. We want to show that $\hat{\alpha} := (\hat{\alpha}_A)_{A\in\operatorname{Ob}\mathcal{A}}\colon \hat{F} \Rightarrow \hat{G}$ is natural.

Suppose given $A \xrightarrow{f} S \overset{s}{\leftsquigarrow} B$ in $\mathcal{A}$. We have

$$\begin{aligned}
\hat{F}(f/s)\cdot\hat{\alpha}_B\cdot G(s) &= F(f)\cdot F(s)^{-1}\cdot\alpha_B\cdot G(s)\\
&= F(f)\cdot F(s)^{-1}\cdot F(s)\cdot\alpha_S\\
&= F(f)\cdot\alpha_S\\
&= \alpha_A\cdot G(f)\\
&= \alpha_A\cdot G(f)\cdot G(s)^{-1}\cdot G(s)\\
&= \hat{\alpha}_A\cdot\hat{G}(f/s)\cdot G(s).
\end{aligned}$$

We conclude that $\hat{F}(f/s)\cdot\hat{\alpha}_B = \hat{\alpha}_A\cdot\hat{G}(f/s)$ since $G(s)$ is an isomorphism in $\mathcal{B}$.

Finally, we have $\hat{\alpha}\star L = \alpha$ by construction. $\square$

Remark 3.3.2. Suppose given an abelian category $\mathcal{B}$, exact functors $F, G\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{N} \subseteq \operatorname{Ker}(F)$ and $\mathcal{N} \subseteq \operatorname{Ker}(G)$ and a transformation $\alpha\colon F \Rightarrow G$. Let $\hat{F}$ and $\hat{G}$ be the unique exact functors with $\hat{F}\circ\mathrm{L} = F$ and $\hat{G}\circ\mathrm{L} = G$ provided by theorem 3.3.1.(a). Let $\hat{\alpha}\colon \hat{F} \Rightarrow \hat{G}$ be the unique transformation satisfying $\hat{\alpha}\star\mathrm{L} = \alpha$ provided by theorem 3.3.1.(b).

If α is an isotransformation, then $\hat{\alpha}$ is an isotransformation as well.

Proof. Let $\beta := \alpha^{-1}$ and let $\hat{\beta}\colon \hat{G} \Rightarrow \hat{F}$ be the unique transformation satisfying $\hat{\beta}\star\mathrm{L} = \beta$ provided by theorem 3.3.1.(b). We have $(\hat{\alpha}\hat{\beta})\star\mathrm{L} = (\hat{\alpha}\star\mathrm{L})(\hat{\beta}\star\mathrm{L}) = \alpha\beta = 1_F = 1_{\hat{F}}\star\mathrm{L}$ and thus $\hat{\alpha}\hat{\beta} = 1_{\hat{F}}$. Similarly, we obtain $\hat{\beta}\hat{\alpha} = 1_{\hat{G}}$. $\square$

Lemma 3.3.3. Suppose given $P \in \operatorname{Ob}\mathcal{A}$ such that ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}\colon \mathcal{A} \to \text{Mod-}\mathbf{Z}$ is exact and such that $\mathcal{N} \subseteq \operatorname{Ker}({}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P})$, cf. definition 1.2.4.(b). Let ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P}\colon \mathcal{A}/\!\!/\mathcal{N} \to \text{Mod-}\mathbf{Z}$ be the unique exact functor with ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P}\circ\mathrm{L} = {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}$ provided by theorem 3.3.1.(a).

Then ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P} \cong {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A}/\!\!/\mathcal{N},P}$ in $(\text{Mod-}\mathbf{Z})^{\mathcal{A}/\!\!/\mathcal{N}}$ and, consequently, ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A}/\!\!/\mathcal{N},P}$ is exact.

In other words: If P is projective in $\mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(P, X) = 0$ for all $X \in \operatorname{Ob}\mathcal{N}$, then P is projective in $\mathcal{A}/\!\!/\mathcal{N}$ as well.

Proof. By remark 3.3.2 and theorem 3.3.1.(a), it suffices to show that ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A}/\!\!/\mathcal{N},P}\circ\mathrm{L} \cong {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}$ in $(\text{Mod-}\mathbf{Z})^{\mathcal{A}}$.

For $X \in \operatorname{Ob}\mathcal{A}$, let

$$\mathrm{L}_{P,X}\colon \operatorname{Hom}_{\mathcal{A}}(P, X) \to \operatorname{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(P, X)\colon h \mapsto \mathrm{L}(h)$$

denote the induced morphism in Mod-$\mathbf{Z}$.

We want to show that $\mathrm{L}_{P,X}$ is bijective.

Suppose given $h \in \mathrm{Hom}_{\mathcal{A}}(P,X)$ such that $h/1 = \mathrm{L}(h) = 0$. By lemma 3.2.13, there exists an $\mathcal{N}$-quasi-isomorphism $X \overset{s}{\rightsquigarrow} S$ in $\mathcal{A}$ such that $hs = 0$. Choose a kernel $K \overset{k}{\rightarrowtail} X$ of s. Note that $K \in \mathrm{Ob}\,\mathcal{N}$. Since $hs = 0$, there exists $P \xrightarrow{u} K$ in $\mathcal{A}$ such that $uk = h$. Since $\mathcal{N} \subseteq \mathrm{Ker}({}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P})$, we have $u = 0$ and thus $h = 0$. We conclude that $\mathrm{L}_{P,X}$ is injective.

Suppose given $P \xrightarrow{g} S \overset{s}{\leftsquigarrow} X$ in $\mathcal{A}$, representing $P \xrightarrow{g/s} X$ in $\mathcal{A}/\!\!/\mathcal{N}$. Choose an image $X \overset{p}{\twoheadrightarrow} I \overset{j}{\rightarrowtail} S$ and a cokernel $S \overset{c}{\twoheadrightarrow} C$ of s in $\mathcal{A}$. Note that $C \in \mathrm{Ob}\,\mathcal{N}$ and that c is a cokernel of j. Since $\mathcal{N} \subseteq \mathrm{Ker}({}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P})$, we have $gc = 0$ and thus there exists $P \xrightarrow{u} I$ in $\mathcal{A}$ such that $uj = g$. Moreover, since P is projective in $\mathcal{A}$, there exists $P \xrightarrow{v} X$ in $\mathcal{A}$ such that $vp = u$ and thus $vs = vpj = uj = g$.

$$\begin{tikzcd}
 & X \arrow[d, two heads, "p"] \arrow[dd, bend left, squiggly, "s"] \\
 & I \arrow[d, tail, "j"] \\
P \arrow[uur, "v"] \arrow[ur, "u"'] \arrow[r, "g"'] & S \arrow[d, two heads, "c"] \\
 & C
\end{tikzcd}$$

We conclude that $\mathrm{L}(v) = v/1 = vs/s = g/s$. Thus $\mathrm{L}_{P,X}$ is surjective.

It remains to show that $\big(\mathrm{L}_{P,X}\big)_{X \in \mathrm{Ob}\,\mathcal{A}} \colon {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P} \Rightarrow {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A}/\!\!/\mathcal{N},P} \circ \mathrm{L}$ is natural.

Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}$. For $w \in \mathrm{Hom}_{\mathcal{A}}(P,X)$, we have

$$\begin{aligned}
(w)\,\mathrm{L}_{P,X}\ {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A}/\!\!/\mathcal{N},P}(\mathrm{L}(f)) &= \mathrm{L}(w)\,{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A}/\!\!/\mathcal{N},P}(\mathrm{L}(f)) = \mathrm{L}(w)\ \mathrm{L}(f) = \mathrm{L}(wf) = (wf)\,\mathrm{L}_{P,Y} \\
&= (w)\,{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}(f)\ \mathrm{L}_{P,Y}\,.
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(P,X) \arrow[r, "\mathrm{L}_{P,X}"] \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}(f)"'] & \mathrm{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(P,X) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A}/\!\!/\mathcal{N},P}(\mathrm{L}(f))"] \\
\mathrm{Hom}_{\mathcal{A}}(P,Y) \arrow[r, "\mathrm{L}_{P,Y}"] & \mathrm{Hom}_{\mathcal{A}/\!\!/\mathcal{N}}(P,Y)
\end{tikzcd}$$

□

Remark 3.3.4. In general, there are objects in $\mathcal{A}$ that become projective in $\mathcal{A}/\!\!/\mathcal{N}$ without satisfying the assumptions of lemma 3.3.3.
Let e.g. $\mathcal{A} = \mathcal{N} = \text{Mod-}\mathbf{Z}$. Then $\mathbf{Z}$ is projective in $\mathcal{A}$, we have $\mathrm{Hom}_{\mathcal{A}}(\mathbf{Z},\mathbf{Z}) \neq 0$ and $\mathbf{Z}$ is also projective in $\mathcal{A}/\!\!/\mathcal{N}$ since $\mathbf{Z} \cong 0$ in $\mathcal{A}/\!\!/\mathcal{N}$.

Chapter 4

The universal abelian category of an exact category

4.1 Adelman's construction for additive categories

Suppose given an additive category $\mathcal{A}$.

We give an overview of Adelman's construction [1] of the universal abelian category $\operatorname{Adel}(\mathcal{A})$ of the additive category $\mathcal{A}$. For details, see e.g. [14, chapters 3 and 4].

4.1.1 The Adelman category

Recall that Δ_2 denotes the poset category of $[0,2] \subseteq \mathbf{Z}$. This category has three objects 0, 1, 2 and three non-identity morphisms $0 \longrightarrow 1$, $1 \longrightarrow 2$, $0 \longrightarrow 2$. Cf. convention 11.

We denote the objects and morphisms of the functor category $\mathcal{A}^{\Delta_2}$ as follows.

We write $A = \big(A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2\big)$ for $A \in \operatorname{Ob}(\mathcal{A}^{\Delta_2})$ and

$$\left(\begin{array}{c} A \\ \downarrow f \\ B \end{array}\right) = \left(\begin{array}{ccccc} A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & A_2 \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 \\ B_0 & \xrightarrow{b_0} & B_1 & \xrightarrow{b_1} & B_2 \end{array}\right)$$

for $A \xrightarrow{f} B$ in $\mathcal{A}^{\Delta_2}$.

The Adelman category $\operatorname{Adel}(\mathcal{A})$ of $\mathcal{A}$ is defined as factor category of $\mathcal{A}^{\Delta_2}$:

$$\operatorname{Adel}(\mathcal{A}) = \mathcal{A}^{\Delta_2} / \operatorname{J}_{\mathcal{A}} .$$

The equivalence class of $f \in \operatorname{Mor}(\mathcal{A}^{\Delta_2})$ in $\operatorname{Adel}(\mathcal{A})$ is denoted by $[f]$ or $[f_0, f_1, f_2]$. For $A \underset{[g]}{\overset{[f]}{\rightrightarrows}} B$

in $\mathrm{Adel}(\mathcal{A})$, we have $[f] = [g]$ if and only if there exist morphisms $s\colon A_1 \to B_0$ and $t\colon A_2 \to B_1$ in $\mathcal{A}$ such that $sb_0 + a_1 t = f_1 - h_1$.

The category $\mathrm{Adel}(\mathcal{A})$ is abelian. Kernels and cokernels of a morphism $A \xrightarrow{[f]} B$ in $\mathrm{Adel}(\mathcal{A})$ are formed as follows.

A kernel of $[f]$ is given by $\mathrm{K}(f) \xrightarrow{[\mathrm{k}(f)]} A$, where

$$\mathrm{K}(f) = \Big(A_0 \oplus B_0 \xrightarrow{\left(\begin{smallmatrix} a_0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} A_1 \oplus B_0 \xrightarrow{\left(\begin{smallmatrix} a_1 & f_1 \\ 0 & -b_0 \end{smallmatrix}\right)} A_2 \oplus B_1 \Big) \text{ and } \mathrm{k}(f) = \Big(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \Big).$$

A cokernel of $[f]$ is given by $B \xrightarrow{[\mathrm{c}(f)]} \mathrm{C}(f)$, where

$$\mathrm{C}(f) = \Big(B_0 \oplus A_1 \xrightarrow{\left(\begin{smallmatrix} b_0 & 0 \\ f_1 & -a_1 \end{smallmatrix}\right)} B_1 \oplus A_2 \xrightarrow{\left(\begin{smallmatrix} b_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} B_2 \oplus A_2 \Big) \text{ and } \mathrm{c}(f) := (\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)).$$

$$\begin{tikzcd}
A_0 \oplus B_0 \arrow[r, "\left(\begin{smallmatrix} a_0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & A_1 \oplus B_0 \arrow[r, "\left(\begin{smallmatrix} a_1 & f_1 \\ 0 & -b_0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & A_2 \oplus B_1 \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] \\
A_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] & A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[d, "f_2"'] \\
B_0 \arrow[r, "b_0"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)"'] & B_1 \arrow[r, "b_1"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)"'] & B_2 \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)"'] \\
B_0 \oplus A_1 \arrow[r, "\left(\begin{smallmatrix} b_0 & 0 \\ f_1 & -a_1 \end{smallmatrix}\right)"'] & B_1 \oplus A_2 \arrow[r, "\left(\begin{smallmatrix} b_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & B_2 \oplus A_2
\end{tikzcd}$$

4.1.2 The inclusion functor and duality

The inclusion functor $\mathrm{I}_{\mathcal{A}}\colon \mathcal{A} \to \mathrm{Adel}(\mathcal{A})$ is defined by

$$\mathrm{I}_{\mathcal{A}}\big(X \xrightarrow{f} Y\big) = \big(0 \longrightarrow X \longrightarrow 0\big) \xrightarrow{[0,f,0]} \big(0 \longrightarrow Y \longrightarrow 0\big)$$

for $X \xrightarrow{f} Y$ in $\mathcal{A}$.

The functor $\mathrm{I}_{\mathcal{A}}$ is full, faithful and additive.

Moreover, we have an isomorphism of categories $\mathrm{D}_{\mathcal{A}}\colon \mathrm{Adel}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Adel}(\mathcal{A}^{\mathrm{op}})$ with

$$\mathrm{D}_{\mathcal{A}}(A) = \Big(A_2 \xrightarrow{a_1^{\mathrm{op}}} A_1 \xrightarrow{a_0^{\mathrm{op}}} A_0 \Big) \text{ and } \mathrm{D}_{\mathcal{A}}([f]^{\mathrm{op}}) = \big[f_2^{\mathrm{op}}, f_1^{\mathrm{op}}, f_0^{\mathrm{op}}\big]$$

for $A \xrightarrow{[f]} B$ in $\mathrm{Adel}(\mathcal{A})$.

Note that $\mathrm{D}_{\mathcal{A}} \circ (\mathrm{I}_{\mathcal{A}})^{\mathrm{op}} = \mathrm{I}_{\mathcal{A}^{\mathrm{op}}}$.

Lemma 4.1.1. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}$.

(a) A kernel of $\mathrm{I}_{\mathcal{A}}(X) \xrightarrow{\mathrm{I}_{\mathcal{A}}(f)} \mathrm{I}_{\mathcal{A}}(Y)$ in $\mathrm{Adel}(\mathcal{A})$ is given by $\big(0 \longrightarrow X \xrightarrow{f} Y\big) \xrightarrow{[0,1,0]} \mathrm{I}_{\mathcal{A}}(X)$.

(b) A cokernel of $\mathrm{I}_{\mathcal{A}}(X) \xrightarrow{\mathrm{I}_{\mathcal{A}}(f)} \mathrm{I}_{\mathcal{A}}(Y)$ in $\mathrm{Adel}(\mathcal{A})$ is given by $\mathrm{I}_{\mathcal{A}}(Y) \xrightarrow{[0,1,0]} \big(X \xrightarrow{f} Y \longrightarrow 0\big)$.

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[d] \\
0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "1"] & 0 \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r] & 0
\end{tikzcd}
$$

Proof. Ad (a). A kernel of $\mathrm{I}_{\mathcal{A}}(f)$ is given by $\mathrm{K}(0,f,0) \xrightarrow{[\mathrm{k}(0,f,0)]} A$, where

$$\mathrm{K}(0,f,0) = \left(0 \oplus 0 \xrightarrow{0} X \oplus 0 \xrightarrow{\left(\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right)} 0 \oplus Y \right) \text{ and } \mathrm{k}(0,f,0) = \left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \right).$$

Using the isomorphisms $0 \oplus 0 \longrightarrow 0$, $X \oplus 0 \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} X$ and $0 \oplus Y \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} Y$ in $\mathcal{A}$, [14, lem. 29] yields the assertion.

Ad (b). This is dual to (a) using the isomorphism of categories $\mathrm{D}_{\mathcal{A}} \colon \mathrm{Adel}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Adel}(\mathcal{A}^{\mathrm{op}})$. $\square$

Lemma 4.1.2. Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

The diagram

$$
\begin{tikzcd}
& \mathrm{I}_{\mathcal{A}}(X) \arrow[rr, "\mathrm{I}_{\mathcal{A}}(f)"] & & \mathrm{I}_{\mathcal{A}}(Y) \arrow[rr, "\mathrm{I}_{\mathcal{A}}(g)"] \arrow[dr, "{[0,1,0]}"] & & \mathrm{I}_{\mathcal{A}}(Z) \arrow[dr, "{[0,1,0]}"] & \\
\big(0 \to X \xrightarrow[f]{} Y\big) \arrow[ur, "{[0,1,0]}"] & & \big(0 \to Y \xrightarrow[g]{} Z\big) \arrow[ur, "{[0,1,0]}"] \arrow[dr, "{[0,1,1]}"'] & & \big(X \xrightarrow[f]{} Y \to 0\big) & & \big(Y \xrightarrow[g]{} Z \to 0\big) \\
& & & \big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \arrow[ur, "{[1,1,0]}"'] & & &
\end{tikzcd}
$$

is a homology diagram of $\mathrm{I}_{\mathcal{A}}(X) \xrightarrow{\mathrm{I}_{\mathcal{A}}(f)} \mathrm{I}_{\mathcal{A}}(Y) \xrightarrow{\mathrm{I}_{\mathcal{A}}(g)} \mathrm{I}_{\mathcal{A}}(Z)$ in $\mathrm{Adel}(\mathcal{A})$, cf. definition 1.6.1.

Proof. This is a consequence of lemma 4.1.1 and [14, th. 42]. $\square$

4.1.3 Adelman's construction for additive functors

Suppose given an additive category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$.

The functor $\mathrm{Adel}(F)\colon\ \mathrm{Adel}(\mathcal{A}) \to \mathrm{Adel}(\mathcal{B})$ is defined by

$$(\mathrm{Adel}(F))(X) := \Big(\ F(X_0) \xrightarrow{F(x_0)} F(X_1) \xrightarrow{F(x_1)} F(X_2)\ \Big)$$

for $X \in \mathrm{Ob}(\mathrm{Adel}(\mathcal{A}))$ and

$$(\mathrm{Adel}(F))([f]) := [F(f_0), F(f_1), F(f_2)]$$

for $[f] \in \mathrm{Mor}(\mathrm{Adel}(\mathcal{A}))$.

The functor $\mathrm{Adel}(F)$ is exact.

4.1.4 The homology functor in the abelian case

Suppose given an abelian category $\mathcal{B}$.

The homology functor $\mathrm{H}_{\mathcal{B}}\colon\ \mathrm{Adel}(\mathcal{B}) \to \mathcal{B}$ was constructed in [14, def. 53 and 55]. The functor $\mathrm{H}_{\mathcal{B}}$ is exact and we have $\mathrm{H}_{\mathcal{B}} \circ \mathrm{I}_{\mathcal{B}} = 1_{\mathcal{B}}$.

We will use the following lemma for calculations involving $\mathrm{H}_{\mathcal{B}}$.

Lemma 4.1.3.

(a) Suppose given $\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \in \mathrm{Ob}(\mathrm{Adel}(\mathcal{B}))$ and a homology diagram

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dr, "d"'] & & Z \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
 & & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

of $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}$, cf. definition 1.6.1.

Then $\mathrm{H}_{\mathcal{B}}\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \cong H$ in $\mathcal{B}$. In particular, we have $\mathrm{H}_{\mathcal{B}}\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \cong 0_{\mathcal{B}}$ if and only if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is semi-exact, cf. definition 1.6.3.

(b) Suppose given $X, M \in \mathrm{Adel}(\mathcal{B})$ such that $x_0 = 0$, $x_1 = 0$ and $m_1 = 0$.

Suppose given $[0, f, 0]\colon X \to M$ in $\mathrm{Adel}(\mathcal{B})$ and a cokernel $M_1 \xrightarrow{d} D$ of m_0 in $\mathcal{B}$.

Then there exist isomorphisms $\mathrm{H}_{\mathcal{B}}(X) \xrightarrow{u} X_1$ and $\mathrm{H}_{\mathcal{B}}(M) \xrightarrow{v} D$ in $\mathcal{B}$ such that $\mathrm{H}_{\mathcal{B}}([0, f, 0])v = ufd$.

In particular, $\mathrm{H}_{\mathcal{B}}([0, f, 0]) = 0$ if and only if $fd = 0$.

$$\begin{tikzcd}
 & & X_1 \arrow[drr, "1"] \arrow[dddr, "f"] & & & \\
X_1 \arrow[urr, "1"] \arrow[drr, "1"'] \arrow[dddr, "f"'] & & & & X_1 \arrow[dddr, "fd"] & \\
 & & X_1 \arrow[urr, "1"'] \arrow[dddr, "fd"] & & & \\
 & & & M_1 \arrow[drr, "d"] & & \\
 & M_1 \arrow[urr, "1"] \arrow[drr, "d"'] & & & & D \\
 & & & D \arrow[urr, "1"'] & &
\end{tikzcd}$$

Proof. Cf. [14, def. 53 and 55]. □

4.1.5 The universal property

Theorem 4.1.4. Recall that $\mathcal{A}$ is an additive category. Suppose given an abelian category $\mathcal{B}$.

(a) Suppose given an additive functor $F \colon \mathcal{A} \to \mathcal{B}$.

We set $\hat{F} := \mathrm{H}_{\mathcal{B}} \circ \mathrm{Adel}(F) \colon \mathrm{Adel}(\mathcal{A}) \to \mathcal{B}$.

The functor $\hat{F}$ is exact with $\hat{F} \circ \mathrm{I}_{\mathcal{A}} = F$.

Suppose given exact functors $G, \tilde{G} \colon \mathrm{Adel}(\mathcal{A}) \to \mathcal{B}$ and an isotransformation $\sigma \colon G \circ \mathrm{I}_{\mathcal{A}} \Rightarrow \tilde{G} \circ \mathrm{I}_{\mathcal{A}}$. Then there exists an isotransformation $\tau \colon G \Rightarrow \tilde{G}$ with $\tau \star \mathrm{I}_{\mathcal{A}} = \sigma$. In particular, this holds for $G \circ \mathrm{I}_{\mathcal{A}} = \tilde{G} \circ \mathrm{I}_{\mathcal{A}}$ and $\sigma = 1_{G \circ \mathrm{I}_{\mathcal{A}}}$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\mathrm{I}_{\mathcal{A}}"'] & \mathcal{B} \\
\mathrm{Adel}(\mathcal{A}) \arrow[ur, "\hat{F}"'] &
\end{tikzcd}$$

(b) Suppose given additive functors $F, G \colon \mathcal{A} \to \mathcal{B}$ and a transformation $\alpha \colon F \Rightarrow G$.

There exists a unique transformation $\hat{\alpha}\colon \hat{F} \Rightarrow \hat{G}$ such that $\hat{\alpha} \star \mathrm{I}_{\mathcal{A}} = \alpha$.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left=10, "F"{name=F}] \arrow[rr, bend right=10, "G"'{name=G}] \arrow[d, "\mathrm{I}_{\mathcal{A}}"'] \arrow[from=F, to=G, Rightarrow, "\alpha"'] & & \mathcal{B} \\
\mathrm{Adel}(\mathcal{A}) \arrow[urr, bend left=15, "\hat{F}"{name=Fh}] \arrow[urr, bend right=15, "\hat{G}"'{name=Gh}] \arrow[from=Fh, to=Gh, Rightarrow, "\hat{\alpha}"'] & &
\end{tikzcd}$$

Proof. See [14, th. 62]. $\square$

Lemma 4.1.5. Suppose given $P \in \mathrm{Ob}\,\mathcal{A}$. Note that ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}\colon \mathcal{A} \to \text{Mod-}\mathbf{Z}$ is additive, cf. definition 1.2.4.(b). Let ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P}\colon \mathrm{Adel}(\mathcal{A}) \to \text{Mod-}\mathbf{Z}$ be the exact functor with ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P} \circ \mathrm{I}_{\mathcal{A}} = {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}$ provided by theorem 4.1.4.(a).

Then ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P} \cong {}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}_{\mathcal{A}}(P)}$ in $(\text{Mod-}\mathbf{Z})^{\mathrm{Adel}(\mathcal{A})}$ and, consequently, ${}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}_{\mathcal{A}}(P)}$ is exact, i.e. $\mathrm{I}_{\mathcal{A}}(P)$ is projective in $\mathrm{Adel}(\mathcal{A})$, cf. [14, exa. 40].

Proof. By theorem 4.1.4.(a), it suffices to show that ${}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}_{\mathcal{A}}(P)} \circ \mathrm{I}_{\mathcal{A}} \cong {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}$ in $(\text{Mod-}\mathbf{Z})^{\mathcal{A}}$.

For $X \in \mathrm{Ob}\,\mathcal{A}$, let

$$\mathrm{I}_{P,X}\colon \mathrm{Hom}_{\mathcal{A}}(P,X) \to \mathrm{Hom}_{\mathrm{Adel}(\mathcal{A})}(\mathrm{I}_{\mathcal{A}}(P), \mathrm{I}_{\mathcal{A}}(X))\colon h \mapsto \mathrm{I}_{\mathcal{A}}(h)$$

denote the induced isomorphism in Mod-$\mathbf{Z}$. It remains to show that $\big(\mathrm{I}_{P,X}\big)_{X \in \mathrm{Ob}\,\mathcal{A}}$ is natural. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}$. For $h \in \mathrm{Hom}_{\mathcal{A}}(P,X)$, we have

$$\begin{aligned}
(h)\,\mathrm{I}_{P,X}\ {}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}_{\mathcal{A}}(P)}(\mathrm{I}_{\mathcal{A}}(f)) &= \mathrm{I}_{\mathcal{A}}(h)\,{}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}_{\mathcal{A}}(P)}(\mathrm{I}_{\mathcal{A}}(f)) = \mathrm{I}_{\mathcal{A}}(h)\ \mathrm{I}_{\mathcal{A}}(f) = \mathrm{I}_{\mathcal{A}}(hf) \\
&= (hf)\,\mathrm{I}_{P,Y} = (h)\,{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}(f)\ \mathrm{I}_{P,Y}\,.
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(P,X) \arrow[r, "\mathrm{I}_{P,X}"] \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}(f)"'] & \mathrm{Hom}_{\mathrm{Adel}(\mathcal{A})}(\mathrm{I}_{\mathcal{A}}(P), \mathrm{I}_{\mathcal{A}}(X)) \arrow[d, "{}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}_{\mathcal{A}}(P)}(\mathrm{I}_{\mathcal{A}}(f))"] \\
\mathrm{Hom}_{\mathcal{A}}(P,Y) \arrow[r, "\mathrm{I}_{P,Y}"] & \mathrm{Hom}_{\mathrm{Adel}(\mathcal{A})}(\mathrm{I}_{\mathcal{A}}(P), \mathrm{I}_{\mathcal{A}}(Y))
\end{tikzcd}$$

$\square$

4.2 Construction of the universal abelian category

Suppose given an exact category $\mathcal{A} = (\mathcal{A}, \mathcal{E})$.

4.2.1 $\mathcal{E}$-homologies and exact functors

Definition 4.2.1. An object $A \in \operatorname{Adel}(\mathcal{A})$ is called an *$\mathcal{E}$-homology* if there exists a pure short exact sequence $\big(X \overset{f}{-\!\!\bullet\!\!\rightarrow} Y \overset{g}{-\!\!+\!\!\rightarrow} Z\big) \in \mathcal{E}$ such that

- $A = \big(0 \longrightarrow X \overset{f}{-\!\!\bullet\!\!\rightarrow} Y\big)$ or
- $A = \big(Y \overset{g}{-\!\!+\!\!\rightarrow} Z \longrightarrow 0\big)$ or
- $A = \big(X \overset{f}{-\!\!\bullet\!\!\rightarrow} Y \overset{g}{-\!\!+\!\!\rightarrow} Z\big)$.

Definition 4.2.2. Let $\mathcal{H}_{\mathcal{A},\mathcal{E}}$ be the full subcategory of $\operatorname{Adel}(\mathcal{A})$ defined by

$$\operatorname{Ob}(\mathcal{H}_{\mathcal{A},\mathcal{E}}) := \{A \in \operatorname{Ob}(\operatorname{Adel}(\mathcal{A}))\colon A \text{ is an } \mathcal{E}\text{-homology}\}$$

We abbreviate $\mathcal{H} := \mathcal{H}_{\mathcal{E}} := \mathcal{H}_{\mathcal{A},\mathcal{E}}$ if unambiguous.

Lemma 4.2.3. Suppose given an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$. Let $\hat{F}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{B}$ be the exact functor with $\hat{F} \circ \mathrm{I}_{\mathcal{A}} = F$ provided by theorem 4.1.4.(a). Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$. The sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is semi-exact in $\mathcal{B}$ if and only if $\hat{F}\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$.

Proof. The diagram

$$\begin{tikzcd}
 & \mathrm{I}_{\mathcal{A}}(X) \arrow[rr, "\mathrm{I}_{\mathcal{A}}(f)"] & & \mathrm{I}_{\mathcal{A}}(Y) \arrow[dr, "{[0,1,0]}"] \arrow[rr, "\mathrm{I}_{\mathcal{A}}(g)"] & & \mathrm{I}_{\mathcal{A}}(Z) \arrow[dr, "{[0,1,0]}"] & \\
\big(0 \to X \xrightarrow[f]{} Y\big) \arrow[ur, "{[0,1,0]}"] & & \big(0 \to Y \xrightarrow[g]{} Z\big) \arrow[ur, "{[0,1,0]}"] \arrow[dr, "{[0,1,1]}"'] & & \big(X \xrightarrow[f]{} Y \to 0\big) & & \big(Y \xrightarrow[g]{} Z \to 0\big) \\
 & & & \big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \arrow[ur, "{[1,1,0]}"'] & & &
\end{tikzcd}$$

is a homology diagram of $\mathrm{I}_{\mathcal{A}}(X) \xrightarrow{\mathrm{I}_{\mathcal{A}}(f)} \mathrm{I}_{\mathcal{A}}(Y) \xrightarrow{\mathrm{I}_{\mathcal{A}}(g)} \mathrm{I}_{\mathcal{A}}(Z)$ in $\operatorname{Adel}(\mathcal{A})$ by lemma 4.1.2. Thus the sequence

$$\big(F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)\big) = \big((\hat{F} \circ \mathrm{I}_{\mathcal{A}})(X) \xrightarrow{(\hat{F}\circ\mathrm{I}_{\mathcal{A}})(f)} (\hat{F} \circ \mathrm{I}_{\mathcal{A}})(Y) \xrightarrow{(\hat{F}\circ\mathrm{I}_{\mathcal{A}})(g)} (\hat{F} \circ \mathrm{I}_{\mathcal{A}})(Z)\big)$$

is semi-exact in $\mathcal{B}$ if and only if $\hat{F}\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \cong 0_{\mathcal{B}}$ in $\mathcal{B}$ by corollary 1.6.9.(a). $\square$

Lemma 4.2.4. Suppose given an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$. Let $\hat{F}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{B}$ be the exact functor with $\hat{F} \circ \mathrm{I}_{\mathcal{A}} = F$ provided by theorem 4.1.4.(a).

Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ such that $fg = 0$. The following three statements are equivalent.

(a) The sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is short exact in $\mathcal{B}$.

(b) There exists a homology diagram

$$\begin{tikzcd}
& F(X) \arrow[rr, "F(f)"] & & F(Y) \arrow[rr, "F(g)"] \arrow[dr, "d"'] & & F(Z) \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & L \arrow[ur, "\ell"'] \arrow[dr, "p"'] & & D & & C \\
& & & H \arrow[ur, "j"'] & & &
\end{tikzcd}$$

of $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$ such that $K \cong 0_{\mathcal{B}}$, $H \cong 0_{\mathcal{B}}$ and $C \cong 0_{\mathcal{B}}$.

(c) We have

$$\hat{F}\big(0 \longrightarrow X \xrightarrow{f} Y\big) \cong 0_{\mathcal{B}}\,,\ \hat{F}\big(Y \xrightarrow{g} Z \longrightarrow 0\big) \cong 0_{\mathcal{B}} \text{ and } \hat{F}\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \cong 0_{\mathcal{B}}$$

in $\mathcal{B}$.

Proof. The statements (a) and (b) are equivalent by corollary 1.6.6.

Ad (a)$\Leftrightarrow$(c). The diagram

$$\begin{tikzcd}
& \mathrm{I}_{\mathcal{A}}(X) \arrow[rr, "\mathrm{I}_{\mathcal{A}}(f)"] & & \mathrm{I}_{\mathcal{A}}(Y) \arrow[rr, "\mathrm{I}_{\mathcal{A}}(g)"] \arrow[dr, "{[0,1,0]}"] & & \mathrm{I}_{\mathcal{A}}(Z) \arrow[dr, "{[0,1,0]}"] & \\
\big(0 \to X \xrightarrow[f]{} Y\big) \arrow[ur, "{[0,1,0]}"] & & \big(0 \to Y \xrightarrow[g]{} Z\big) \arrow[ur, "{[0,1,0]}"] \arrow[dr, "{[0,1,1]}"'] & & \big(X \xrightarrow[f]{} Y \to 0\big) & & \big(Y \xrightarrow[g]{} Z \to 0\big) \\
& & & \big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \arrow[ur, "{[1,1,0]}"'] & & &
\end{tikzcd}$$

is a homology diagram of $\mathrm{I}_{\mathcal{A}}(X) \xrightarrow{\mathrm{I}_{\mathcal{A}}(f)} \mathrm{I}_{\mathcal{A}}(Y) \xrightarrow{\mathrm{I}_{\mathcal{A}}(g)} \mathrm{I}_{\mathcal{A}}(Z)$ in $\operatorname{Adel}(\mathcal{A})$ by lemma 4.1.2.

Thus the sequence

$$\big(F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)\big) = \big((\hat{F} \circ \mathrm{I}_{\mathcal{A}})(X) \xrightarrow{(\hat{F}\circ\mathrm{I}_{\mathcal{A}})(f)} (\hat{F} \circ \mathrm{I}_{\mathcal{A}})(Y) \xrightarrow{(\hat{F}\circ\mathrm{I}_{\mathcal{A}})(g)} (\hat{F} \circ \mathrm{I}_{\mathcal{A}})(Z)\big)$$

is short exact in $\mathcal{B}$ if and only if

$$\hat{F}\big(0 \longrightarrow X \xrightarrow{f} Y\big) \cong 0_{\mathcal{B}}\,,\ \hat{F}\big(Y \xrightarrow{g} Z \longrightarrow 0\big) \cong 0_{\mathcal{B}} \text{ and } \hat{F}\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \cong 0_{\mathcal{B}}$$

in $\mathcal{B}$ by corollary 1.6.9.(b). $\square$

Proposition 4.2.5. Suppose given an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$. Let $\hat{F}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{B}$ be the exact functor with $\hat{F} \circ \mathrm{I}_{\mathcal{A}} = F$ provided by theorem 4.1.4.(a). The functor F is exact if and only if $\mathcal{H} \subseteq \operatorname{Ker}(\hat{F})$.

Proof.
The functor F is exact if and only if for each pure short exact sequence $\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \in \mathcal{E}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is short exact in $\mathcal{B}$. By lemma 4.2.4, this is true if and only if $\mathcal{H} \subseteq \operatorname{Ker}(\hat{F})$. Cf. definition 4.2.2. $\square$

Lemma 4.2.6. Suppose given a thick subcategory $\mathcal{N}$ of $\operatorname{Adel}(\mathcal{A})$.
Recall that $\mathrm{L}_{\operatorname{Adel}(\mathcal{A}),\mathcal{N}}\colon \operatorname{Adel}(\mathcal{A}) \to \operatorname{Adel}(\mathcal{A})/\!\!/\mathcal{N}$ is the localisation functor of $\operatorname{Adel}(\mathcal{A})$ by $\mathcal{N}$. Cf. definition 3.2.6. We abbreviate $\mathrm{L} := \mathrm{L}_{\operatorname{Adel}(\mathcal{A}),\mathcal{N}}$. Let $\widehat{\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}}\colon \operatorname{Adel}(\mathcal{A}) \to \operatorname{Adel}(\mathcal{A})/\!\!/\mathcal{N}$ be the exact functor with $\widehat{\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}} \circ \mathrm{I}_{\mathcal{A}} = \mathrm{L} \circ \mathrm{I}_{\mathcal{A}}$ provided by theorem 4.1.4.(a).

We have $\widehat{\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}} \cong \mathrm{L}$ in $(\operatorname{Adel}(\mathcal{A})/\!\!/\mathcal{N})^{\operatorname{Adel}(\mathcal{A})}$ and $\operatorname{Ker}(\widehat{\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}}) = \operatorname{Ker}(\mathrm{L}) = \mathcal{N}$.

The functor $\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}$ is exact if and only if $\mathcal{H} \subseteq \mathcal{N}$.

Proof. The functor $\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}$ is additive since it is a composite of additive functors, cf. section 4.1.2 and proposition 3.2.12. Moreover, L is exact by theorem 3.2.18. Since $\widehat{\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}} \circ \mathrm{I}_{\mathcal{A}} = \mathrm{L} \circ \mathrm{I}_{\mathcal{A}}$, we have $\widehat{\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}} \cong \mathrm{L}$ in $(\operatorname{Adel}(\mathcal{A})/\!\!/\mathcal{N})^{\operatorname{Adel}(\mathcal{A})}$ by theorem 4.1.4.(a).

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}"] \arrow[d, "\mathrm{I}_{\mathcal{A}}"'] & \operatorname{Adel}(\mathcal{A})/\!\!/\mathcal{N} \\ \operatorname{Adel}(\mathcal{A}) \arrow[ur, "\mathrm{L}"'] & \end{tikzcd}$$

Thus $\operatorname{Ker}(\widehat{\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}}) = \operatorname{Ker} \mathrm{L} = \mathcal{N}$ by lemma 1.1.5 and theorem 3.2.18.

We conclude that $\mathrm{L} \circ \mathrm{I}_{\mathcal{A}}$ is exact if and only if $\mathcal{H} \subseteq \mathcal{N}$ by proposition 4.2.5. $\square$

4.2.2 Characterisations of the thick subcategory generated by $\mathcal{H}$

Lemma 4.2.7. Recall that ${}_{\operatorname{Adel}(\mathcal{A})}\langle \mathcal{H}_{\mathcal{A},\mathcal{E}} \rangle = \langle \mathcal{H} \rangle$ denotes the thick subcategory of $\operatorname{Adel}(\mathcal{A})$ generated by $\mathcal{H}$. Cf. definitions 3.1.10 and 4.2.2.

For an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$, let $\hat{F}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{B}$ be the exact functor with $\hat{F} \circ \mathrm{I}_{\mathcal{A}} = F$ provided by theorem 4.1.4.(a).

Then we have

$$\begin{aligned} \langle \mathcal{H} \rangle &= \operatorname{Ker}(\mathrm{L}_{\operatorname{Adel}(\mathcal{A}),\langle \mathcal{H} \rangle}) \\ &= \bigcap \{\operatorname{Ker}(\hat{F})\colon \mathcal{B} \text{ is an abelian category, } F\colon \mathcal{A} \to \mathcal{B} \text{ is an exact functor}\}. \end{aligned}$$

Cf. definition 4.2.9 below.

Proof. We have $\langle \mathcal{H} \rangle = \operatorname{Ker}(\mathrm{L}_{\mathrm{Adel}(\mathcal{A}),\langle \mathcal{H} \rangle})$ by theorem 3.2.18. Since $\mathrm{L}_{\mathrm{Adel}(\mathcal{A}),\langle \mathcal{H} \rangle}$ is exact by lemma 4.2.6, we have

$$\bigcap \{\operatorname{Ker}(\hat{F}) \colon \mathcal{B} \text{ is an abelian category, } F \colon \mathcal{A} \to \mathcal{B} \text{ is an exact functor}\} \subseteq \operatorname{Ker}(\mathrm{L}_{\mathrm{Adel}(\mathcal{A}),\langle \mathcal{H} \rangle}).$$

By proposition 4.2.5, we have

$$\mathcal{H} \subseteq \bigcap \{\operatorname{Ker}(\hat{F}) \colon \mathcal{B} \text{ is an abelian category, } F \colon \mathcal{A} \to \mathcal{B} \text{ is an exact functor}\}.$$

Note that $\bigcap \{\operatorname{Ker}(\hat{F}) \colon \mathcal{B} \text{ is an abelian category, } F \colon \mathcal{A} \to \mathcal{B} \text{ is an exact functor}\}$ is a thick subcategory of $\mathrm{Adel}(\mathcal{A})$ by remark 3.1.9 and corollary 3.1.8. Thus

$$\langle \mathcal{H} \rangle \subseteq \bigcap \{\operatorname{Ker}(\hat{F}) \colon \mathcal{B} \text{ is an abelian category, } F \colon \mathcal{A} \to \mathcal{B} \text{ is an exact functor}\}.$$

Altogether, we obtain

$$\langle \mathcal{H} \rangle = \bigcap \{\operatorname{Ker}(\hat{F}) \colon \mathcal{B} \text{ is an abelian category, } F \colon \mathcal{A} \to \mathcal{B} \text{ is an exact functor}\}. \qquad \square$$

Remark 4.2.8. Let $\mathcal{P}_{\mathcal{A},\mathcal{E}}$ be the full subcategory of $\mathrm{Adel}(\mathcal{A})$ defined by

$$\mathrm{Ob}(\mathcal{P}_{\mathcal{A},\mathcal{E}}) := \Big\{ \big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \in \mathrm{Ob}(\mathrm{Adel}(\mathcal{A})) \colon X \xrightarrow{f} Y \xrightarrow{g} Z \text{ is } \mathcal{E}\text{-pure exact in } \mathcal{A} \Big\}.$$

Cf. definition 1.6.10. We abbreviate $\mathcal{P} = \mathcal{P}_{\mathcal{A},\mathcal{E}}$ if unambiguous.

We have $\mathcal{H} \subseteq \mathcal{P} \subseteq \langle \mathcal{H} \rangle$ and thus $\langle \mathcal{H} \rangle = \langle \mathcal{P} \rangle$.

Proof. Suppose given a pure short exact sequence $\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \in \mathcal{E}$.
The following $\mathcal{E}$-purity diagrams show that $\mathcal{H} \subseteq \mathcal{P}$.

$$\begin{tikzcd}
& 0 \arrow[rr] \arrow[dr] & & X \arrow[rr, "f"] \arrow[dr, "1"'] & & Y \arrow[dr, "g"] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & X \arrow[ur, "f"'] & & Z
\end{tikzcd}$$

$$\begin{tikzcd}
& Y \arrow[rr, "g"] \arrow[dr, "g"'] & & Z \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & \\
X \arrow[ur, "f"] & & Z \arrow[ur, "1"'] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[rr, "f"] \arrow[dr, "1"'] & & Y \arrow[rr, "g"] \arrow[dr, "g"'] & & Z \arrow[dr] & \\
0 \arrow[ur] & & X \arrow[ur, "f"'] & & Z \arrow[ur, "1"'] & & 0
\end{tikzcd}$$

Suppose given an $\mathcal{E}$-pure exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.

Suppose given an abelian category $\mathcal{B}$ and an exact functor $F\colon \mathcal{A} \to \mathcal{B}$. Let $\hat{F}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{B}$ be the exact functor with $\hat{F} \circ \mathrm{I}_{\mathcal{A}} = F$ provided by theorem 4.1.4.(a). So $\hat{F} = \mathrm{H}_{\mathcal{B}} \circ \operatorname{Adel}(F)$. Note that $\operatorname{Adel}(F)\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) = \big(F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)\big)$.
We have $\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) \in \operatorname{Ob}(\operatorname{Ker}(\hat{F}))$ by lemmata 4.2.3, 1.6.11 and 1.6.4.

We conclude that

$$\mathcal{P} \subseteq \bigcap \{\operatorname{Ker}(\hat{F})\colon \mathcal{B} \text{ is an abelian category, } F\colon \mathcal{A} \to \mathcal{B} \text{ is an exact functor}\} = \langle \mathcal{H} \rangle,$$

cf. lemma 4.2.7. $\square$

4.2.3 Definition of the universal abelian category

Definition 4.2.9. The *universal abelian category* $\mathcal{U}(\mathcal{A},\mathcal{E})$ *of* $(\mathcal{A},\mathcal{E})$ is defined as the quotient category of $\operatorname{Adel}(\mathcal{A})$ by ${}_{\operatorname{Adel}(\mathcal{A})}\langle \mathcal{H}_{\mathcal{A},\mathcal{E}} \rangle$, i.e.

$$\mathcal{U}(\mathcal{A},\mathcal{E}) = \operatorname{Adel}(\mathcal{A}) /\!\!/ {}_{\operatorname{Adel}(\mathcal{A})}\langle \mathcal{H}_{\mathcal{A},\mathcal{E}} \rangle = \operatorname{Adel}(\mathcal{A}) /\!\!/ \langle \mathcal{H} \rangle.$$

Cf. definition 3.2.3.

We write $\mathrm{L}_{(\mathcal{A},\mathcal{E})} := \mathrm{L}_{\operatorname{Adel}(\mathcal{A}),\langle \mathcal{H} \rangle}$ for the localisation functor of $\operatorname{Adel}(\mathcal{A})$ by $\langle \mathcal{H} \rangle$, cf. definition 3.2.6.

The *universal functor* $\mathfrak{I}_{(\mathcal{A},\mathcal{E})}\colon \mathcal{A} \to \mathcal{U}(\mathcal{A},\mathcal{E})$ *of* $(\mathcal{A},\mathcal{E})$ is defined as the composite of the inclusion functor $\mathrm{I}_{\mathcal{A}}\colon \mathcal{A} \to \operatorname{Adel}(\mathcal{A})$ of $\mathcal{A}$ and the localisation functor $\mathrm{L}_{(\mathcal{A},\mathcal{E})}\colon \operatorname{Adel}(\mathcal{A}) \to \mathcal{U}(\mathcal{A},\mathcal{E})$ of $\operatorname{Adel}(\mathcal{A})$ by $\langle \mathcal{H} \rangle$, i.e.

$$\mathfrak{I}_{(\mathcal{A},\mathcal{E})} = \mathrm{L}_{(\mathcal{A},\mathcal{E})} \circ \mathrm{I}_{\mathcal{A}}\ .$$

Cf. section 4.1.2.

The functor $\mathfrak{I}_{(\mathcal{A},\mathcal{E})}$ is exact by lemma 4.2.6.
We will show in theorem 4.4.4 below that it is also full and faithful and that it detects exactness.

We abbreviate $\mathcal{U}(\mathcal{A}) = \mathcal{U}(\mathcal{A},\mathcal{E})$, $\mathrm{L} = \mathrm{L}_{(\mathcal{A},\mathcal{E})}$, $\mathrm{I} = \mathrm{I}_{\mathcal{A}}$ and $\mathfrak{I} = \mathfrak{I}_{(\mathcal{A},\mathcal{E})}$ if unambiguous.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{I}"] \arrow[rr, bend right, "\mathfrak{I}"'] & \operatorname{Adel}(\mathcal{A}) \arrow[r, "\mathrm{L}"] & \mathcal{U}(\mathcal{A})
\end{tikzcd}$$

Remark 4.2.10. We have $\operatorname{Ob}(\mathcal{U}(\mathcal{A})) = \operatorname{Ob}(\operatorname{Adel}(\mathcal{A})) = \operatorname{Ob}(\mathcal{A}^{\Delta_2})$, so an object $A \in \operatorname{Ob}(\mathcal{U}(\mathcal{A}))$ is of the form $A = \big(A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2\big)$.

Morphisms in $\mathcal{U}(\mathcal{A})$ are of the form $A \xrightarrow{[f]/[s]} B$, where the diagram $A \xrightarrow{[f]} S \xleftarrow{[s]} B$ is in $\operatorname{Adel}(\mathcal{A})$ such that $[s]$ is an $\langle \mathcal{H} \rangle$-quasi-isomorphism.

4.3 The universal property

Suppose given an exact category $\mathcal{A} = (\mathcal{A}, \mathcal{E})$.

Theorem 4.3.1. Recall that $\mathrm{Adel}(\mathcal{A})$ is the Adelman category of $\mathcal{A}$ and $\mathrm{I}\colon \mathcal{A} \to \mathrm{Adel}(\mathcal{A})$ is the inclusion functor. Cf. section 4.1.
Recall that $\mathcal{U}(\mathcal{A})$ is the universal abelian category of $\mathcal{A} = (\mathcal{A}, \mathcal{E})$, that $\mathrm{L}\colon \mathrm{Adel}(\mathcal{A}) \to \mathcal{U}(\mathcal{A})$ is the localisation functor of $\mathrm{Adel}(\mathcal{A})$ by the thick subcategory $\langle \mathcal{H} \rangle$, which is generated by the category of $\mathcal{E}$-homologies, and that $\mathfrak{I}\colon \mathcal{A} \to \mathcal{U}(\mathcal{A})$ is the universal functor. Cf. definition 4.2.9.

Suppose given an abelian category $\mathcal{B}$.

(a) Suppose given an exact functor $F\colon \mathcal{A} \to \mathcal{B}$.

Let $\hat{F}\colon\ \mathrm{Adel}(\mathcal{A}) \to \mathcal{B}$ be the exact functor with $\hat{F} \circ \mathrm{I}_{\mathcal{A}} = F$ provided by theorem 4.1.4.(a). We have $\mathcal{H} \subseteq \mathrm{Ker}(\hat{F})$, so theorem 3.3.1.(a) yields an exact functor $\tilde{F}\colon \mathcal{U}(\mathcal{A}) \to \mathcal{B}$ such that $\tilde{F} \circ \mathfrak{I} = F$.

Suppose given exact functors $G, \bar{G}\colon \mathcal{U}(\mathcal{A}) \to \mathcal{B}$ and an isotransformation $\sigma\colon G \circ \mathfrak{I} \Rightarrow \bar{G} \circ \mathfrak{I}$. Then there exists an isotransformation $\tau\colon G \Rightarrow \bar{G}$ with $\tau \star \mathfrak{I} = \sigma$. In particular, this holds for $G \circ \mathfrak{I} = \bar{G} \circ \mathfrak{I}$ and $\sigma = 1_{G \circ \mathfrak{I}}$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\mathrm{I}"] \arrow[dd, bend right=60, "\mathfrak{I}"'] & \mathcal{B} \\
\mathrm{Adel}(\mathcal{A}) \arrow[ur, "\hat{F}"'] \arrow[d, "\mathrm{L}"] & \\
\mathcal{U}(\mathcal{A}) \arrow[uur, bend right, "\tilde{F}"'] &
\end{tikzcd}$$

(b) Suppose given exact functors $F, G\colon \mathcal{A} \to \mathcal{B}$ and a transformation $\alpha\colon F \Rightarrow G$.

There exists a unique transformation $\tilde{\alpha}\colon \tilde{F} \Rightarrow \tilde{G}$ such that $\tilde{\alpha} \star \mathfrak{I} = \alpha$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] \arrow[d, "\mathfrak{I}"'] & \mathcal{B} \\
\mathcal{U}(\mathcal{A}) \arrow[ur, bend left, "\tilde{F}"] \arrow[ur, bend right, "\tilde{G}"'] \arrow[ur, phantom, "\tilde{\alpha}"] &
\end{tikzcd}$$

Proof. Ad (a). We have $\mathcal{H} \subseteq \mathrm{Ker}(\hat{F})$ by proposition 4.2.5. This implies $\langle \mathcal{H} \rangle \subseteq \mathrm{Ker}(\hat{F})$ since $\mathrm{Ker}(\hat{F})$ is a thick subcategory of $\mathrm{Adel}(\mathcal{A})$ by corollary 3.1.8.

Suppose given exact functors $G, \bar{G}\colon \mathcal{U}(\mathcal{A}) \to \mathcal{B}$ and an isotransformation $\sigma\colon G \circ \mathfrak{I} \Rightarrow \bar{G} \circ \mathfrak{I}$. So $\sigma\colon (G \circ \mathrm{L}) \circ \mathrm{I} \Rightarrow (\bar{G} \circ \mathrm{L}) \circ \mathrm{I}$. By theorem 4.1.4.(a), there exists an isotransformation $\rho\colon G \circ \mathrm{L} \Rightarrow \bar{G} \circ \mathrm{L}$ such that $\rho \star \mathrm{I} = \sigma$. By theorem 3.3.1.(b), there exists a transformation

$\tau\colon G \Rightarrow \bar{G}$ with $\tau \star \mathrm{L} = \rho$. So $\tau \star \mathfrak{I} = \tau \star (\mathrm{L} \circ \mathrm{I}) = (\tau \star \mathrm{L}) \star \mathrm{I} = \rho \star \mathrm{I} = \sigma$. The transformation τ is an isotransformation by remark 3.3.2.

Ad (b). By theorem 4.1.4.(b), there exists a unique transformation $\hat{\alpha}\colon \hat{F} \Rightarrow \hat{G}$ such that $\hat{\alpha} \star \mathrm{I} = \alpha$. By theorem 3.3.1.(b), there exists a unique transformation $\tilde{\alpha}\colon \tilde{F} \Rightarrow \tilde{G}$ such that $\tilde{\alpha} \star \mathrm{L} = \hat{\alpha}$. So $\tilde{\alpha} \star \mathfrak{I} = (\tilde{\alpha} \star \mathrm{L}) \star \mathrm{I} = \hat{\alpha} \star \mathrm{I} = \alpha$.

Suppose given another transformation $\beta\colon \tilde{F} \Rightarrow \tilde{G}$ with $\beta \star \mathfrak{I} = \alpha$. So $(\beta \star \mathrm{L}) \star \mathrm{I} = \alpha$. Thus $\beta \star \mathrm{L} = \hat{\alpha}$. We conclude that indeed $\beta = \tilde{\alpha}$. □

Remark 4.3.2. Suppose that $\mathcal{A}$ is abelian, equipped with the natural exact structure $\mathcal{E} = \mathcal{E}^{\mathrm{all}}_{\mathcal{A}}$. Then $\mathfrak{I}\colon \mathcal{A} \to \mathcal{U}(\mathcal{A})$ is an equivalence of categories.

Cf. definition 1.5.6.

Proof. Let $\tilde{1}_{\mathcal{A}}\colon \mathcal{U}(\mathcal{A}) \to \mathcal{A}$ be the exact functor with $\tilde{1}_{\mathcal{A}} \circ \mathfrak{I} = 1_{\mathcal{A}}$ provided by theorem 4.3.1.(a).

We have $(\mathfrak{I} \circ \tilde{1}_{\mathcal{A}}) \circ \mathfrak{I} = \mathfrak{I} \circ (\tilde{1}_{\mathcal{A}} \circ \mathfrak{I}) = \mathfrak{I} = 1_{\mathcal{U}(\mathcal{A})} \circ \mathfrak{I}$. Thus $\mathfrak{I} \circ \tilde{1}_{\mathcal{A}} \cong 1_{\mathcal{U}(\mathcal{A})}$ by loc. cit.

We conclude that $\mathfrak{I}$ is an equivalence of categories. □

Remark 4.3.3. Suppose that $\mathcal{A}$ is equipped with the split exact structure $\mathcal{E} = \mathcal{E}^{\mathrm{split}}_{\mathcal{A}}$. Then $\mathrm{L}\colon \mathrm{Adel}(\mathcal{A}) \to \mathcal{U}(\mathcal{A})$ is an equivalence of categories.

Cf. definition 1.5.4.

Proof. Note that I is an exact functor in this case since it is additive, cf. remark 1.5.5.

Let $\hat{\mathrm{I}}\colon \mathrm{Adel}(\mathcal{A}) \to \mathrm{Adel}(\mathcal{A})$ be the exact functor with $\hat{\mathrm{I}} \circ \mathrm{I} = \mathrm{I}$ provided by theorem 4.1.4.(a). Thus $\hat{\mathrm{I}} \cong 1_{\mathrm{Adel}(\mathcal{A})}$ in $\mathrm{Adel}(\mathcal{A})^{\mathrm{Adel}(\mathcal{A})}$.

Let $\tilde{\mathrm{I}}\colon \mathcal{U}(\mathcal{A}) \to \mathrm{Adel}(\mathcal{A})$ be the exact functor with $\tilde{\mathrm{I}} \circ \mathfrak{I} = \mathrm{I}$ provided by theorem 4.3.1.(a). Thus $\tilde{\mathrm{I}} \circ \mathrm{L} = \hat{\mathrm{I}} \cong 1_{\mathrm{Adel}(\mathcal{A})}$.

We have $(\mathrm{L} \circ \tilde{\mathrm{I}}) \circ \mathfrak{I} = \mathrm{L} \circ (\tilde{\mathrm{I}} \circ \mathfrak{I}) = \mathrm{L} \circ \mathrm{I} = \mathfrak{I} = 1_{\mathcal{U}(\mathcal{A})} \circ \mathfrak{I}$. Thus $\mathrm{L} \circ \tilde{\mathrm{I}} \cong 1_{\mathcal{U}(\mathcal{A})}$ in $\mathcal{U}(\mathcal{A})^{\mathcal{U}(\mathcal{A})}$ by loc. cit.

We conclude that L is an equivalence of categories. □

Remark 4.3.4. The universal functor is self-dual in the following sense.

Note that $(\mathfrak{I}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}}\colon \mathcal{A} \to \mathcal{U}(\mathcal{A}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$ is $\mathcal{E}$-exact and that $(\mathfrak{I}_{(\mathcal{A},\mathcal{E})})^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{U}(\mathcal{A}, \mathcal{E})^{\mathrm{op}}$ is $\mathcal{E}^{\mathrm{op}}$-exact.

Let $\tilde{\mathfrak{I}}\colon \mathcal{U}(\mathcal{A}, \mathcal{E}) \to \mathcal{U}(\mathcal{A}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$ be the functor provided by theorem 4.3.1.(a).
So $\tilde{\mathfrak{I}} \circ \mathfrak{I}_{(\mathcal{A},\mathcal{E})} = (\mathfrak{I}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}}$.

Let $\tilde{\mathfrak{I}}^{\circ}\colon \mathcal{U}(\mathcal{A}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{U}(\mathcal{A}, \mathcal{E})$ be the opposite functor of the one provided by theorem 4.3.1.(a) for the exact category $(\mathcal{A}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})$. So $\tilde{\mathfrak{I}}^{\circ} \circ (\mathfrak{I}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} = \mathfrak{I}_{(\mathcal{A},\mathcal{E})}$.

Then $\tilde{\mathfrak{J}} \circ \tilde{\mathfrak{J}}^\circ \cong 1_{\mathcal{U}(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})^{\mathrm{op}}}$ and $\tilde{\mathfrak{J}}^\circ \circ \tilde{\mathfrak{J}} \cong 1_{\mathcal{U}(\mathcal{A},\mathcal{E})}$. In particular, $\tilde{\mathfrak{J}}$ is an equivalence of categories.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "(\mathfrak{J}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}}"] \arrow[d, "\mathfrak{J}_{(\mathcal{A},\mathcal{E})}"'] & \mathcal{U}(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})^{\mathrm{op}} \arrow[dl, bend right, "\tilde{\mathfrak{J}}^\circ"'] \\
\mathcal{U}(\mathcal{A},\mathcal{E}) \arrow[ur, bend right, "\tilde{\mathfrak{J}}"'] &
\end{tikzcd}$$

4.4 Further properties

Suppose given an exact category $\mathcal{A} = (\mathcal{A},\mathcal{E})$.

Recall that $\mathrm{Y}_{(\mathcal{A},\mathcal{E})} \colon \mathcal{A} \to \mathrm{GQL}(\mathcal{A},\mathcal{E})$ is the closed immersion of the exact category $(\mathcal{A},\mathcal{E})$ into its Gabriel-Quillen-Laumon category. Cf. definition 2.5.6 and theorem 2.5.8.

Let $\hat{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})} \colon \mathrm{Adel}(\mathcal{A}) \to \mathrm{GQL}(\mathcal{A},\mathcal{E})$ be the exact functor with $\hat{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})} \circ \mathrm{I}_\mathcal{A} = \mathrm{Y}_{(\mathcal{A},\mathcal{E})}$ provided by theorem 4.1.4.(a).

Let $\tilde{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})} \colon \mathcal{U}(\mathcal{A},\mathcal{E}) \to \mathrm{GQL}(\mathcal{A},\mathcal{E})$ be the exact functor with $\tilde{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})} \circ \mathfrak{J}_{(\mathcal{A},\mathcal{E})} = \mathrm{Y}_{(\mathcal{A},\mathcal{E})}$ provided by theorem 4.3.1.(a).

Thus the following diagram commutes.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{Y}_{(\mathcal{A},\mathcal{E})}"] \arrow[d, "\mathrm{I}_\mathcal{A}"'] \arrow[dd, bend right=60, "\mathfrak{J}_{(\mathcal{A},\mathcal{E})}"'] & \mathrm{GQL}(\mathcal{A},\mathcal{E}) \\
\mathrm{Adel}(\mathcal{A}) \arrow[ur, "\hat{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})}"'] \arrow[d, "\mathrm{L}_{(\mathcal{A},\mathcal{E})}"'] & \\
\mathcal{U}(\mathcal{A},\mathcal{E}) \arrow[uur, bend right, "\tilde{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})}"'] &
\end{tikzcd}$$

Dually, $\mathrm{Y}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})} \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{GQL}(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})$ is the closed immersion of the exact category $(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})$ into its Gabriel-Quillen-Laumon category.
Thus we obtain a closed immersion $(\mathrm{Y}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} \colon \mathcal{A} \to \mathrm{GQL}(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$ of the exact category $(\mathcal{A},\mathcal{E})$ into $\mathrm{GQL}(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$. This immersion is essentially different from $\mathrm{Y}_{(\mathcal{A},\mathcal{E})}$ in general, cf. remark 4.5.5 below.

Abbreviate $\mathrm{Y} = \mathrm{Y}_{(\mathcal{A},\mathcal{E})}$, $\hat{\mathrm{Y}} = \hat{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})}$, $\tilde{\mathrm{Y}} = \tilde{\mathrm{Y}}_{(\mathcal{A},\mathcal{E})}$ and $\mathrm{Y}^\circ = (\mathrm{Y}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}}$.

Moreover, abbreviate $\mathrm{GQL}(\mathcal{A}) = \mathrm{GQL}(\mathcal{A},\mathcal{E})$ and $\mathrm{GQL}(\mathcal{A}^{\mathrm{op}}) = \mathrm{GQL}(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})$.

Let $\hat{\mathrm{Y}}^\circ \colon \mathrm{Adel}(\mathcal{A}) \to \mathrm{GQL}(\mathcal{A})$ be the exact functor with $\hat{\mathrm{Y}}^\circ \circ \mathrm{I}_\mathcal{A} = \mathrm{Y}^\circ$ provided by theorem 4.1.4.(a).

Let $\tilde{\mathrm{Y}}^\circ \colon \mathcal{U}(\mathcal{A},\mathcal{E}) \to \mathrm{GQL}(\mathcal{A})$ be the exact functor with $\tilde{\mathrm{Y}}^\circ \circ \mathfrak{J} = \mathrm{Y}^\circ$ provided by theorem 4.3.1.(a).

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{Y}"] \arrow[d, "\mathrm{I}"'] \arrow[dd, bend right=60, "\mathfrak{I}"'] & \mathrm{GQL}(\mathcal{A}) \\
\mathrm{Adel}(\mathcal{A}) \arrow[ur, "\hat{\mathrm{Y}}"'] \arrow[d, "\mathrm{L}"'] & \\
\mathcal{U}(\mathcal{A}) \arrow[uur, bend right, "\tilde{\mathrm{Y}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{Y}^\circ"] \arrow[d, "\mathrm{I}"'] \arrow[dd, bend right=60, "\mathfrak{I}"'] & \mathrm{GQL}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \\
\mathrm{Adel}(\mathcal{A}) \arrow[ur, "\hat{\mathrm{Y}}^\circ"'] \arrow[d, "\mathrm{L}"'] & \\
\mathcal{U}(\mathcal{A}) \arrow[uur, bend right, "\tilde{\mathrm{Y}}^\circ"'] &
\end{tikzcd}
$$

Remark 4.4.1. Recall the duality functor $\mathrm{D}^{\mathrm{op}}_{\mathcal{A}} \colon \mathrm{Adel}(\mathcal{A}) \to \mathrm{Adel}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$ from section 4.1.2. We have $\hat{\mathrm{Y}}^\circ \cong (\hat{\mathrm{Y}}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} \circ \mathrm{D}^{\mathrm{op}}_{\mathcal{A}}$ in $(\mathrm{GQL}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{Adel}(\mathcal{A})}$ and thus
$\mathrm{Ker}(\hat{\mathrm{Y}}^\circ) = \mathrm{Ker}((\hat{\mathrm{Y}}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} \circ \mathrm{D}^{\mathrm{op}}_{\mathcal{A}})$ as subcategories of $\mathrm{Adel}(\mathcal{A})$.

Proof. We have $\mathrm{D}^{\mathrm{op}}_{\mathcal{A}} \circ \mathrm{I}_{\mathcal{A}} = (\mathrm{I}_{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}}$ and thus

$$(\hat{\mathrm{Y}}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} \circ \mathrm{D}^{\mathrm{op}}_{\mathcal{A}} \circ \mathrm{I}_{\mathcal{A}} = (\hat{\mathrm{Y}}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} \circ (\mathrm{I}_{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} = (\mathrm{Y}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} = \mathrm{Y}^\circ = \hat{\mathrm{Y}}^\circ \circ \mathrm{I}_{\mathcal{A}}\,.$$

So $\hat{\mathrm{Y}}^\circ \cong (\hat{\mathrm{Y}}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} \circ \mathrm{D}^{\mathrm{op}}_{\mathcal{A}}$ by theorem 4.1.4.(a).

We conclude that $\mathrm{Ker}(\hat{\mathrm{Y}}^\circ) = \mathrm{Ker}((\hat{\mathrm{Y}}_{(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})})^{\mathrm{op}} \circ \mathrm{D}^{\mathrm{op}}_{\mathcal{A}})$ by lemma 1.1.5. $\square$

4.4.1 The induced functor from $\mathrm{Adel}(\mathcal{A})$ to $\mathrm{GQL}(\mathcal{A})$

Lemma 4.4.2. Suppose given $\left(X \xrightarrow{f} Y \xrightarrow{g} Z \right) \in \mathrm{Ob}(\mathrm{Adel}(\mathcal{A}))$.

(a) We have $\left(X \xrightarrow{f} Y \xrightarrow{g} Z \right) \in \mathrm{Ker}(\hat{\mathrm{Y}})$ if and only if for all $A \in \mathrm{Ob}\,\mathcal{A}$ and all $A \xrightarrow{h} Y$ in $\mathcal{A}$ with $hg = 0$, there exists a pure epimorphism $P \xrightarrow{p} A$ and a morphism $P \xrightarrow{u} X$ in $\mathcal{A}$ such that $ph = uf$.

In symbols:

$$\left(X \xrightarrow{f} Y \xrightarrow{g} Z \right) \in \mathrm{Ker}(\hat{\mathrm{Y}}) \iff$$
$$\forall A \in \mathrm{Ob}\,\mathcal{A}\ \forall\, A \xrightarrow{h} Y \ \text{ with } hg = 0\ \exists\, P \xrightarrow{p} A\ \ \exists\, P \xrightarrow{u} X \,:\, ph = uf$$

$$
\begin{tikzcd}
& X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
P \arrow[ur, "u"] \arrow[rr, "p"'] & & A \arrow[u, "h"] \arrow[ur, "0"'] &
\end{tikzcd}
$$

(b) We have $\left(X \xrightarrow{f} Y \xrightarrow{g} Z \right) \in \mathrm{Ker}(\hat{\mathrm{Y}}^\circ)$ if and only if for all $A \in \mathrm{Ob}\,\mathcal{A}$ and all $Y \xrightarrow{h} A$ with $fh = 0$ in $\mathcal{A}$, there exists a pure monomorphism $A \xrightarrow{m} M$ and a morphism $Z \xrightarrow{u} M$

in $\mathcal{A}$ such that $hm = gu$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[drr, "0"'] & Y \arrow[r, "g"] \arrow[dr, "h"] & Z \arrow[dr, "u"] & \\
& & A \arrow[r, "m"] & M
\end{tikzcd}$$

Proof. Ad (a). We have

$$\operatorname{Adel}(\hat{\mathrm{Y}})\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big) = \big(\mathrm{Y}(X) \xrightarrow{\mathrm{Y}(f)} \mathrm{Y}(Y) \xrightarrow{\mathrm{Y}(g)} \mathrm{Y}(Z)\big).$$

Choose an image $\mathrm{K}_g \to H \to \mathrm{C}_f$ of $\mathrm{k}_g\, \mathrm{c}_f$, cf. definition 2.5.4. The diagram

$$\begin{tikzcd}
& \mathrm{Y}(X) \arrow[rr, "\mathrm{Y}(f)"] & & \mathrm{Y}(Y) \arrow[rr, "\mathrm{Y}(g)"] \arrow[dr, "\mathrm{c}_f"] & & \mathrm{Y}(Z) \arrow[dr, "\mathrm{c}_g"] & \\
\mathrm{K}_f \arrow[ur, "\mathrm{k}_f"] & & \mathrm{K}_g \arrow[ur, "\mathrm{k}_g"'] \arrow[dr] & & \mathrm{C}_f & & \mathrm{C}_g \\
& & & H \arrow[ur] & & &
\end{tikzcd}$$

is a homology diagram of $\mathrm{Y}(X) \xrightarrow{\mathrm{Y}(f)} \mathrm{Y}(Y) \xrightarrow{\mathrm{Y}(g)} \mathrm{Y}(Z)$ in $\mathrm{GQL}(\mathcal{A})$, cf. remark 2.5.5. Note that in ${}_{\mathbf{Z}}\hat{\mathcal{A}}$, we have $\mathrm{c}_f = \mathrm{c}_f^\circ\, \mathrm{e}_{\mathrm{C}_f^\circ}\, \mathrm{e}_{(\mathrm{C}_f^\circ)^+}$ by loc. cit. and that $\mathrm{e}_{(\mathrm{C}_f^\circ)^+}$ is monomorphic there by remark 2.3.18.(b).

We have $H \cong \hat{\mathrm{Y}}\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big)$, cf. lemma 4.1.3.(a).

Thus the following five statements are equivalent.

- $\big(\, X \xrightarrow{f} Y \xrightarrow{g} Z \,\big) \in \operatorname{Ker}(\hat{\mathrm{Y}})$
- $H \cong 0_{\mathrm{GQL}(\mathcal{A})}$
- $\mathrm{k}_g\ \mathrm{c}_f^\circ\ \mathrm{e}_{\mathrm{C}_f^\circ}\ \mathrm{e}_{(\mathrm{C}_f^\circ)^+} = 0$
- $\mathrm{k}_g\ \mathrm{c}_f^\circ\ \mathrm{e}_{\mathrm{C}_f^\circ} = 0$
- For $A \in \operatorname{Ob}\mathcal{A}$ and $h \in \mathrm{K}_g(A)$, we have $(h)(\mathrm{k}_g)_A\, (\mathrm{c}_f^\circ)_A\, (\mathrm{e}_{\mathrm{C}_f^\circ})_A = 0$.

Suppose given $A \in \operatorname{Ob}\mathcal{A}$ and $h \in \mathrm{K}_g(A)$. So $h \in \operatorname{Hom}_{\mathcal{A}}(A, Y)$ with $hg = 0$. We have

$$\begin{aligned}
(h)(\mathrm{k}_g)_A\, (\mathrm{c}_f^\circ)_A\, (\mathrm{e}_{\mathrm{C}_f^\circ})_A &= (h)(\mathrm{c}_f^\circ)_A\, (\mathrm{e}_{\mathrm{C}_f^\circ})_A \\
&= (h + \operatorname{Hom}_{\mathcal{A}}(A, X)f)\, (\mathrm{e}_{\mathrm{C}_f^\circ})_A \\
&= \big[\beta^{(h+\operatorname{Hom}_{\mathcal{A}}(A,X)f)}, \max(A)\big],
\end{aligned}$$

cf. definition 2.3.16.

By lemma 2.4.3, we have $\left[\beta^{(h+\mathrm{Hom}_{\mathcal{A}}(A,X)f)}, \max(A)\right] = 0$ if and only if there exists a pure epimorphism $P \overset{p}{\longrightarrow\!\!\!\!\!+} A$ in $\mathcal{A}$ such that

$$ph + \mathrm{Hom}_{\mathcal{A}}(P,X)f = (h + \mathrm{Hom}_{\mathcal{A}}(A,X)f)\ \mathrm{C}^{\circ}_{f}(p^{\mathrm{op}}) = (p)\beta_P^{(h+\mathrm{Hom}_{\mathcal{A}}(A,X)f)} = 0,$$

cf. definition 2.3.14.

This is true if and only if there exists $P \xrightarrow{u} X$ in $\mathcal{A}$ such that $ph = uf$.

Ad (b). This is dual to (a) using remark 4.4.1. $\square$

Lemma 4.4.3. Suppose given $X \in \mathrm{Ob}\,\mathcal{A}$ and $M \in \mathrm{Adel}(\mathcal{A})$ with $m_1 = 0$.
Suppose given $[0,f,0]\colon \mathrm{I}_{\mathcal{A}}(X) \to M$ in $\mathrm{Adel}(\mathcal{A})$.

Then $\hat{\mathrm{Y}}([0,f,0]) = 0$ if and only if there exist $P \xrightarrow{u} M_0$ and a pure epimorphism $P \overset{p}{\longrightarrow\!\!\!\!\!+} X$ in $\mathcal{A}$ such that $pf = um_0$.

$$\begin{tikzcd}
 & P \arrow[dl, "u"'] \arrow[d, "p"] & \\
 & X \arrow[d, "f"] & \\
M_0 \arrow[r, "m_0"] & M_1 \arrow[r, "0"] & M_2
\end{tikzcd}$$

Proof. We have $\mathrm{Adel}(\mathrm{Y})([0,f,0]) = [0,\mathrm{Y}(f),0]$. The morphism $\mathrm{Y}(M_1) \xrightarrow{\mathrm{c}_{m_0}} \mathrm{C}_{m_0}$ is a cokernel of $\mathrm{Y}(m_0)$ in $\mathrm{GQL}(\mathcal{A})$ by remark 2.5.5.
Moreover, $\mathrm{c}_{m_0} = \mathrm{c}^{\circ}_{m_0}\ \mathrm{e}_{\mathrm{C}^{\circ}_{m_0}}\ \mathrm{e}_{(\mathrm{C}^{\circ}_{m_0})^+}$ and $\mathrm{e}_{(\mathrm{C}^{\circ}_{m_0})^+}$ is monomorphic by remark 2.3.18.(b).

By lemma 4.1.3.(b), we have $\hat{\mathrm{Y}}([0,f,0]) = 0$ if and only if $\mathrm{Y}(f)\ \mathrm{c}_{m_0} = 0$. This is true if and only if $\mathrm{Y}(f)\ \mathrm{c}^{\circ}_{m_0}\ \mathrm{e}_{\mathrm{C}^{\circ}_{m_0}} = 0$. This in turn is equivalent to $(1_X)\,(\mathrm{Y}(f))_X\left(\mathrm{c}^{\circ}_{m_0}\right)_X\left(\mathrm{e}_{\mathrm{C}^{\circ}_{m_0}}\right)_X = 0$ by the Yoneda lemma 1.2.2.(b).

We calculate

$$\begin{aligned}
(1_X)\,(\mathrm{Y}(f))_X\left(\mathrm{c}^{\circ}_{m_0}\right)_X\left(\mathrm{e}_{\mathrm{C}^{\circ}_{m_0}}\right)_X &= (f)\left(\mathrm{c}^{\circ}_{m_0}\right)_X\left(\mathrm{e}_{\mathrm{C}^{\circ}_{m_0}}\right)_X \\
&= (f + \mathrm{Hom}_{\mathcal{A}}(X,M_0)m_0)\left(\mathrm{e}_{\mathrm{C}^{\circ}_{m_0}}\right)_X \\
&= \left[\beta^{(f+\mathrm{Hom}_{\mathcal{A}}(X,M_0)m_0)}, \max(X)\right],
\end{aligned}$$

cf. definition 2.3.16.

By lemma 2.4.3, we have $\left[\beta^{(f+\mathrm{Hom}_{\mathcal{A}}(X,M_0)m_0)}, \max(X)\right] = 0$ if and only if there exists a pure epimorphism $P \overset{p}{\longrightarrow\!\!\!\!\!+} X$ such that $(p)\,\beta_P^{(f+\mathrm{Hom}_{\mathcal{A}}(X,M_0)m_0)} = 0$. We calculate

$$(p)\,\beta_P^{(f+\mathrm{Hom}_{\mathcal{A}}(X,M_0)m_0)} = (f + \mathrm{Hom}_{\mathcal{A}}(X,M_0)m_0)\ \mathrm{C}^{\circ}_{m_0}(p^{\mathrm{op}}) = pf + \mathrm{Hom}_{\mathcal{A}}(P,M_0)m_0\,.$$

Note that $pf + \mathrm{Hom}_{\mathcal{A}}(P,M_0)m_0 = 0$ if and only if there exists $P \xrightarrow{u} M_0$ in $\mathcal{A}$ such that $pf = um_0$. $\square$

4.4.2 Properties of the universal functor

Theorem 4.4.4. The universal functor $\mathfrak{I}\colon \mathcal{A} \to \mathcal{U}(\mathcal{A})$ is an immersion, i.e. it is full, faithful, exact and detects exactness, cf. definition 1.5.8.

Proof. The functor $\mathfrak{I}$ is exact, cf. definition 4.2.9.

We want to show that it detects exactness.

Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ such that $\mathfrak{I}(X) \xrightarrow{\mathfrak{I}(f)} \mathfrak{I}(Y) \xrightarrow{\mathfrak{I}(g)} \mathfrak{I}(Z)$ is short exact in $\mathcal{U}(\mathcal{A})$. Since $\tilde{\mathrm{Y}}\colon \mathcal{U}(\mathcal{A}) \to \mathrm{GQL}(\mathcal{A})$ is exact, the sequence

$$\big((\tilde{\mathrm{Y}} \circ \mathfrak{I})(X) \xrightarrow{(\tilde{\mathrm{Y}}\circ\mathfrak{I})(f)} (\tilde{\mathrm{Y}} \circ \mathfrak{I})(Y) \xrightarrow{(\tilde{\mathrm{Y}}\circ\mathfrak{I})(g)} (\tilde{\mathrm{Y}} \circ \mathfrak{I})(Z)\big) = \big(\mathrm{Y}(X) \xrightarrow{\mathrm{Y}(f)} \mathrm{Y}(Y) \xrightarrow{\mathrm{Y}(g)} \mathrm{Y}(Z)\big)$$

is short exact in $\mathrm{GQL}(\mathcal{A})$. Now Y detects exactness by theorem 2.5.8 and thus $X \xrightarrow{f} Y \xrightarrow{g} Z$ is pure short exact in $\mathcal{A}$. We conclude that $\mathfrak{I}$ detects exactness as well.

For an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ between preadditive categories $\mathcal{C}$ and $\mathcal{D}$ and objects $X, Y \in \operatorname{Ob}\mathcal{C}$, let

$$F_{X,Y}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X),F(Y))\colon f \mapsto F(f)$$

denote the induced $\mathbf{Z}$-linear map.

We want to show that $\mathfrak{I}$ is full and faithful.

Suppose given $X, Z \in \operatorname{Ob}\mathcal{A}$. We obtain the following commutative diagram in Mod-$\mathbf{Z}$.

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}(X,Z) \arrow[rr, "\mathrm{Y}_{X,Z}"] \arrow[d, "\mathrm{I}_{X,Z}"'] & & \operatorname{Hom}_{\mathrm{GQL}(\mathcal{A})}(\mathrm{Y}(X),\mathrm{Y}(Z)) \\
\operatorname{Hom}_{\mathrm{Adel}(\mathcal{A})}(\mathrm{I}(X),\mathrm{I}(Z)) \arrow[urr, "\hat{\mathrm{Y}}_{\mathrm{I}(X),\mathrm{I}(Z)}"'] \arrow[d, "\mathrm{L}_{\mathrm{I}(X),\mathrm{I}(Z)}"'] & & \\
\operatorname{Hom}_{\mathcal{U}(\mathcal{A})}(\mathfrak{I}(X),\mathfrak{I}(Z)) \arrow[uurr, bend right, "\tilde{\mathrm{Y}}_{\mathfrak{I}(X),\mathfrak{I}(Z)}"'] & &
\end{tikzcd}$$

The functors I and Y are full and faithful, cf. section 4.1.2 and theorem 2.5.8. Thus $\mathrm{I}_{X,Z}$ and $\mathrm{Y}_{X,Z}$ are isomorphisms in Mod-$\mathbf{Z}$. We conclude that $\hat{\mathrm{Y}}_{\mathrm{I}(X),\mathrm{I}(Z)}$ is an isomorphism as well, that $\mathrm{L}_{\mathrm{I}(X),\mathrm{I}(Z)}$ is injective and that $\tilde{\mathrm{Y}}_{\mathfrak{I}(X),\mathfrak{I}(Z)}$ is surjective. We want to show that $\tilde{\mathrm{Y}}_{\mathfrak{I}(X),\mathfrak{I}(Z)}$ is injective since then all morphisms in the diagram above are isomorphisms.

Suppose given $\mathrm{I}(X) \xrightarrow{f} M \overset{s}{\leftsquigarrow} \mathrm{I}(Z)$ in $\mathrm{Adel}(\mathcal{A})$, representing $\mathfrak{I}(X) \xrightarrow{f/s} \mathfrak{I}(Z)$ in $\mathcal{U}(\mathcal{A})$. Suppose further that $\hat{\mathrm{Y}}(f) \cdot \hat{\mathrm{Y}}(s)^{-1} = \tilde{\mathrm{Y}}(f/s) = 0$. So $\hat{\mathrm{Y}}(f) = 0$.

Let $m := [1,1,0]\colon M \to \big(M_0 \xrightarrow{m_0} M_1 \longrightarrow 0\big)$ in $\mathrm{Adel}(\mathcal{A})$. This is a monomorphism in $\mathrm{Adel}(\mathcal{A})$, cf. [14, th. 42.(b)]. Since L is exact by theorem 3.2.18, we conclude that $\mathrm{L}(m)$ is monomorphic in $\mathcal{U}(\mathcal{A})$.

We have $\hat{\mathrm{Y}}(fm) = \hat{\mathrm{Y}}(f) \cdot \hat{\mathrm{Y}}(m) = 0$. Lemma 4.4.3 yields $P \xrightarrow{u} M_0$ and a pure epimorphism $P \xrightarrow{p} X$ in $\mathcal{A}$ such that $pf_1 = um_0$. Note that $\mathfrak{I}(p)$ is epimorphic in $\mathcal{U}(\mathcal{A})$ since $\mathfrak{I}$ is exact.

Moreover, $\mathrm{I}(p) \cdot fm = 0$ in $\mathrm{Adel}(\mathcal{A})$ since we have $P \xrightarrow{u} M_0$ and $0 \longrightarrow M_1$ in $\mathcal{A}$ such that $um_0 + 0 = f_1$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "p"] \arrow[ddl, "u"'] & 0 \arrow[d] \arrow[ddl] \\
0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f_1"'] & 0 \arrow[d] \\
M_0 \arrow[r, "m_0"] & M_1 \arrow[r] & 0
\end{tikzcd}$$

So $\mathfrak{I}(p) \cdot \mathrm{L}(f) \cdot \mathrm{L}(m) = \mathrm{L}(\mathrm{I}(p) \cdot fm) = 0$ in $\mathcal{U}(\mathcal{A})$ and thus $\mathrm{L}(f) = 0$ since $\mathfrak{I}(p)$ is epimorphic and $\mathrm{L}(m)$ is monomorphic.

We obtain $f/s = f/1 \cdot 1/s = \mathrm{L}(f) \cdot 1/s = 0$.

This shows that $\tilde{\mathrm{Y}}_{\mathfrak{I}(X),\mathfrak{I}(Z)}$ is indeed injective. So $\mathfrak{I}_{X,Z} = \mathrm{I}_{X,Z}\ \mathrm{L}_{\mathrm{I}(X),\mathrm{I}(Z)}$ is an isomorphism in Mod-$\mathbf{Z}$.

We conclude that $\mathfrak{I}$ is full and faithful. $\square$

Proposition 4.4.5.
Suppose given a relative projective object $P \in \mathrm{Ob}\,\mathcal{A}$. Then $\mathfrak{I}(P)$ is projective in $\mathcal{U}(\mathcal{A})$.

Proof. By remark 1.2.5, it suffices to show that the functor ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{U}(\mathcal{A}),\mathfrak{I}(P)}$ is exact.

Let ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P} \colon \mathrm{Adel}(\mathcal{A}) \to \text{Mod-}\mathbf{Z}$ be the exact functor with ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P} \circ \mathrm{I} = {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}$ provided by theorem 4.1.4.(a).

Then ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P} \cong {}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}(P)}$ in $(\text{Mod-}\mathbf{Z})^{\mathrm{Adel}(\mathcal{A})}$ by lemma 4.1.5. We have $\langle \mathcal{H} \rangle \subseteq \mathrm{Ker}({}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P}) = \mathrm{Ker}({}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}(P)})$ since P is relative projective, i.e. since ${}_{\mathbf{Z}}\mathrm{y}^{\mathcal{A},P}$ is exact, cf. definition 1.5.9 and lemmata 1.1.5 and 4.2.7.

Let ${}_{\mathbf{Z}}\tilde{\mathrm{y}}^{\mathcal{A},P} \colon \mathcal{U}(\mathcal{A}) \to \text{Mod-}\mathbf{Z}$ be the exact functor with ${}_{\mathbf{Z}}\tilde{\mathrm{y}}^{\mathcal{A},P} \circ \mathrm{L} = {}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathcal{A},P}$ provided by theorem 3.3.1.(a) and let ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}(P)}$ be the exact functor with ${}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}(P)} \circ \mathrm{L} = {}_{\mathbf{Z}}\mathrm{y}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}(P)}$ provided by loc. cit.

We have ${}_{\mathbf{Z}}\tilde{\mathrm{y}}^{\mathcal{A},P} \cong {}_{\mathbf{Z}}\hat{\mathrm{y}}^{\mathrm{Adel}(\mathcal{A}),\mathrm{I}(P)} \cong {}_{\mathbf{Z}}\mathrm{y}^{\mathcal{U}(\mathcal{A}),\mathfrak{I}(P)}$ in $(\text{Mod-}\mathbf{Z})^{\mathcal{U}(\mathcal{A})}$ by remark 3.3.2 and lemma 3.3.3. $\square$

Proposition 4.4.6. Recall the notion of a closed immersion from definition 1.5.8.

(a) Suppose that $\mathcal{A}$ has enough relative projectives. Then the immersion $\mathfrak{I} \colon \mathcal{A} \to \mathcal{U}(\mathcal{A})$ is closed, cf. definition 1.5.8.

(b) Suppose that $\mathcal{A}$ has enough relative injectives. Then the immersion $\mathfrak{I} \colon \mathcal{A} \to \mathcal{U}(\mathcal{A})$ is closed.

Proof. Ad (a).

Suppose given $X', X'' \in \operatorname{Ob}\mathcal{A}$ and a short exact sequence $\mathfrak{I}(X') \xrightarrow{f} Z \xrightarrow{g} \mathfrak{I}(X'')$ in $\mathcal{U}(\mathcal{A})$. We have to find $C \in \operatorname{Ob}\mathcal{A}$ such that $Z \cong \mathfrak{I}(C)$ in $\mathcal{U}(\mathcal{A})$.

Since $\mathcal{A}$ has enough relative projectives, we may choose pure short exact sequences

$$K' \xrightarrow{k'} P' \xrightarrow{p'} X',\ K'' \xrightarrow{k''} P'' \xrightarrow{p''} X'',\ L' \xrightarrow{\ell'} Q' \xrightarrow{q'} K' \text{ and } L'' \xrightarrow{\ell''} Q'' \xrightarrow{q''} K''$$

in $\mathcal{A}$. Since $\mathfrak{I}$ maps relative projectives to projectives by proposition 4.4.5, the proof of the horseshoe lemma (cf. [16, lem. 2.2.8] and [2, th. 12.8]) yields the commutative diagram

$$\begin{tikzcd}
\mathfrak{I}(L') \arrow[r, "\mathfrak{I}(\ell')"] \arrow[d, "f''"] & \mathfrak{I}(Q') \arrow[r, "\mathfrak{I}(q')"] \arrow[d, "\mathfrak{I}(1\ 0)"] & \mathfrak{I}(K') \arrow[r, "\mathfrak{I}(k')"] \arrow[d, "f'"] & \mathfrak{I}(P') \arrow[r, "\mathfrak{I}(p')"] \arrow[d, "\mathfrak{I}(1\ 0)"] & \mathfrak{I}(X') \arrow[d, "f"] \\
N \arrow[r, "n"] \arrow[d, "g''"] & \mathfrak{I}(Q' \oplus Q'') \arrow[r, "s"] \arrow[d, "\mathfrak{I}\binom{0}{1}"] & M \arrow[r, "m"] \arrow[d, "g'"] & \mathfrak{I}(P' \oplus P'') \arrow[r, "r"] \arrow[d, "\mathfrak{I}\binom{0}{1}"] & Z \arrow[d, "g"] \\
\mathfrak{I}(L'') \arrow[r, "\mathfrak{I}(\ell'')"] & \mathfrak{I}(Q'') \arrow[r, "\mathfrak{I}(q'')"] & \mathfrak{I}(K'') \arrow[r, "\mathfrak{I}(k'')"] & \mathfrak{I}(P'') \arrow[r, "\mathfrak{I}(p'')"] & \mathfrak{I}(X'')
\end{tikzcd}$$

in $\mathcal{U}(\mathcal{A})$ such that that the sequences $(f'', g''), (f', g'), (n, s)$ and (m, r) are short exact.

Moreover, since $\mathfrak{I}$ is full and faithful, we may write $sm = \mathfrak{I}(a)$ for a uniquely determined $Q' \oplus Q'' \xrightarrow{a} P' \oplus P''$ in $\mathcal{A}$. Note that Z is a cokernel of $\mathfrak{I}(a)$ in $\mathcal{U}(\mathcal{A})$.

We want to use that we know that the essential image of Y is closed under extensions. Therefore we apply the exact functor $\tilde{\mathrm{Y}}$ to the diagram above and obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Y}(L') \arrow[r, "\mathrm{Y}(\ell')"] \arrow[d, "\tilde{\mathrm{Y}}(f'')"] & \mathrm{Y}(Q') \arrow[r, "\mathrm{Y}(q')"] \arrow[d, "\mathrm{Y}(1\ 0)"] & \mathrm{Y}(K') \arrow[r, "\mathrm{Y}(k')"] \arrow[d, "\tilde{\mathrm{Y}}(f)"] & \mathrm{Y}(P') \arrow[r, "\mathrm{Y}(p')"] \arrow[d, "\mathrm{Y}(1\ 0)"] & \mathrm{Y}(X') \arrow[d, "\tilde{\mathrm{Y}}(f)"] \\
\tilde{\mathrm{Y}}(N) \arrow[r, "\tilde{\mathrm{Y}}(n)"] \arrow[d, "\tilde{\mathrm{Y}}(g'')"] & \mathrm{Y}(Q' \oplus Q'') \arrow[r, "\tilde{\mathrm{Y}}(s)"] \arrow[d, "\mathrm{Y}\binom{0}{1}"] & \tilde{\mathrm{Y}}(M) \arrow[r, "\tilde{\mathrm{Y}}(m)"] \arrow[d, "\tilde{\mathrm{Y}}(g')"] & \mathrm{Y}(P' \oplus P'') \arrow[r, "\tilde{\mathrm{Y}}(r)"] \arrow[d, "\mathrm{Y}\binom{0}{1}"] & \tilde{\mathrm{Y}}(Z) \arrow[d, "\tilde{\mathrm{Y}}(g)"] \\
\mathrm{Y}(L'') \arrow[r, "\mathrm{Y}(\ell'')"] & \mathrm{Y}(Q'') \arrow[r, "\mathrm{Y}(q'')"] & \mathrm{Y}(K'') \arrow[r, "\mathrm{Y}(k'')"] & \mathrm{Y}(P'') \arrow[r, "\mathrm{Y}(p'')"] & \mathrm{Y}(X'')
\end{tikzcd}$$

in $\mathrm{GQL}(\mathcal{A})$.

Since the sequences $(\tilde{\mathrm{Y}}(f''), \tilde{\mathrm{Y}}(g'')), (\tilde{\mathrm{Y}}(f'), \tilde{\mathrm{Y}}(g'))$ and $(\tilde{\mathrm{Y}}(f), \tilde{\mathrm{Y}}(g))$ are short exact, we may replace $\tilde{\mathrm{Y}}(N), \tilde{\mathrm{Y}}(M)$ and $\tilde{\mathrm{Y}}(Z)$ isomorphically with the images of objects in $\mathcal{A}$ under Y and obtain the following commutative diagram in $\mathrm{GQL}(\mathcal{A})$.

$$\begin{tikzcd}
\mathrm{Y}(L') \arrow[r, "\mathrm{Y}(\ell')"] \arrow[d] & \mathrm{Y}(Q') \arrow[r, "\mathrm{Y}(q')"] \arrow[d, "\mathrm{Y}(1\ 0)"] & \mathrm{Y}(K') \arrow[r, "\mathrm{Y}(k')"] \arrow[d] & \mathrm{Y}(P') \arrow[r, "\mathrm{Y}(p')"] \arrow[d, "\mathrm{Y}(1\ 0)"] & \mathrm{Y}(X') \arrow[d] \\
\mathrm{Y}(L) \arrow[r, "\mathrm{Y}(\ell)"] \arrow[d] & \mathrm{Y}(Q' \oplus Q'') \arrow[r, "\mathrm{Y}(q)"] \arrow[d, "\mathrm{Y}\binom{0}{1}"] & \mathrm{Y}(K) \arrow[r, "\mathrm{Y}(k)"] \arrow[d] & \mathrm{Y}(P' \oplus P'') \arrow[r, "\mathrm{Y}(p)"] \arrow[d, "\mathrm{Y}\binom{0}{1}"] & \mathrm{Y}(C) \arrow[d] \\
\mathrm{Y}(L'') \arrow[r, "\mathrm{Y}(\ell'')"] & \mathrm{Y}(Q'') \arrow[r, "\mathrm{Y}(q'')"] & \mathrm{Y}(K'') \arrow[r, "\mathrm{Y}(k'')"] & \mathrm{Y}(P'') \arrow[r, "\mathrm{Y}(p'')"] & \mathrm{Y}(X'')
\end{tikzcd}$$

Since Y is full, faithful and detects exactness and since we have $\mathrm{Y}(a) = \tilde{\mathrm{Y}}(sm) = \mathrm{Y}(qk)$, we obtain the commutative diagram

$$\begin{tikzcd}
L \arrow[r, "\ell"] & Q' \oplus Q'' \arrow[rr, "a"] \arrow[dr, "q"'] & & P' \oplus P'' \arrow[r, "p"] & C \\
& & K \arrow[ur, "k"'] & &
\end{tikzcd}$$

in $\mathcal{A}$ such that the sequences (ℓ, q) and (k, p) are pure short exact. Thus a cokernel of $\mathfrak{I}(a)$ in $\mathcal{U}(\mathcal{A})$ is given by $\mathfrak{I}(C)$. We conclude that indeed $Z \cong \mathfrak{I}(C)$ in $\mathcal{U}(\mathcal{A})$.

Ad (b). This is dual to (a) using that $\mathfrak{I}$ is self-dual, cf. remark 4.3.4. $\square$

Remark 4.4.7. I do not know whether the immersion $\mathfrak{I}$ is closed in general. This question remains open, cf. section 4.6.

More precisely, for a short exact sequence $\mathfrak{I}(X') \xrightarrow{f} Z \xrightarrow{g} \mathfrak{I}(X'')$ in $\mathcal{U}(\mathcal{A})$, the sequence $\mathrm{Y}(X') \xrightarrow{\tilde{\mathrm{Y}}(f)} \tilde{\mathrm{Y}}(Z) \xrightarrow{\tilde{\mathrm{Y}}(g)} \mathrm{Y}(X'')$ is short exact in $\mathrm{GQL}(\mathcal{A})$. Thus $\tilde{\mathrm{Y}}(Z) \cong \mathrm{Y}(X) = \tilde{\mathrm{Y}}(\mathfrak{I}(X))$ in $\mathrm{GQL}(\mathcal{A})$ for some $X \in \operatorname{Ob} \mathcal{A}$ since Y is a closed immersion. However, we would need $Z \cong \mathfrak{I}(X)$ in $\mathcal{U}(\mathcal{A})$ but I do not know if this is true in general, cf. proposition 4.5.4.(c).

4.5 A counterexample to some assertions

Let $\mathcal{A} = \text{mod-}\mathbf{Z}$ be the abelian category whose objects are the finitely generated $\mathbf{Z}$-modules. Note that $\mathcal{A}$ has kernels and cokernels since it is abelian.

We will use two different exact structures on $\mathcal{A}$, namely the split exact structure $\mathcal{E}_{\mathcal{A}}^{\text{split}}$ and the natural exact structure $\mathcal{E}_{\mathcal{A}}^{\text{all}}$, cf. definitions 1.5.4 and 1.5.6.

We keep the notation from section 4.4.

4.5.1 Natural exact structure

Let $\mathcal{A}$ be equipped with the natural exact structure $\mathcal{E} = \mathcal{E}_{\mathcal{A}}^{\text{all}}$.

Proposition 4.5.1. The functor $\mathrm{I}\colon \mathcal{A} \to \mathrm{Adel}(\mathcal{A})$ is not left-exact.

Proof. Assume that I is left-exact. For each additive functor $F\colon \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is an abelian category, the induced functor $\hat{F}\colon \mathrm{Adel}(\mathcal{A}) \to \mathcal{B}$ provided by theorem 4.1.4 is exact with $\hat{F} \circ \mathrm{I} = F$. So F has to be left-exact as composite of a left-exact and an exact functor.

So to arrive at a *contradiction*, it suffices to show that there is an additive functor from $\mathcal{A} = \text{mod-}\mathbf{Z}$ to an abelian category that is not left-exact.

The tensor functor $- \otimes \mathbf{Z}/2\colon \text{mod-}\mathbf{Z} \to \text{Mod-}\mathbf{Z}$ is additive and $\big(\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{1} \mathbf{Z}/2\big) \in \mathcal{E}$. Now $\mathbf{Z} \otimes \mathbf{Z}/2 \xrightarrow{2\otimes 1} \mathbf{Z} \otimes \mathbf{Z}/2$ is not injective since it is isomorphic to $\mathbf{Z}/2 \xrightarrow{0} \mathbf{Z}/2$ in $(\text{Mod-}\mathbf{Z})^{\Delta_1}$. We conclude that $- \otimes \mathbf{Z}/2$ is not left-exact. $\square$

4.5.2 Split exact structure

Let $\mathcal{A}$ be equipped with the split exact structure $\mathcal{E} = \mathcal{E}^{\text{split}}_{\mathcal{A}}$.

Let $X := \big(0 \longrightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z}\big) \in \text{Adel}(\mathcal{A})$.

Lemma 4.5.2. We have $X \in \text{Ob}(\text{Ker}(\hat{\text{Y}}))$ but $X \notin \text{Ob}(\text{Ker}(\hat{\text{Y}}^\circ))$.

Proof. We show that $X \in \text{Ob}(\text{Ker}(\hat{\text{Y}}))$ using lemma 4.4.2.(a).

Suppose given $A \xrightarrow{h} \mathbf{Z}$ in $\mathcal{A}$ such that $h \cdot 2 = 0$. Since 2 is monomorphic in $\mathcal{A}$, we obtain $h = 0$. Thus the following diagram commutes.

$$\begin{tikzcd}
 & 0 \arrow[rr] & & \mathbf{Z} \arrow[rr, "2"] & & \mathbf{Z} \\
A \arrow[ur, "0"] \arrow[rr, "1"'] & & A \arrow[ur, "h"] \arrow[urrr, "0"'] & & &
\end{tikzcd}$$

We conclude that $X \in \text{Ob}(\text{Ker}(\hat{\text{Y}}))$.

We show that $X \notin \text{Ob}(\text{Ker}(\hat{\text{Y}}^\circ))$ using lemma 4.4.2.(b). Suppose that $X \in \text{Ob}(\text{Ker}(\hat{\text{Y}}^\circ))$. For $\mathbf{Z} \xrightarrow{1} \mathbf{Z}$ in $\mathcal{A}$, we obtain $\mathbf{Z} \xrightarrow{u} M$ and a split monomorphism $\mathbf{Z} \xrightarrow{m} M$ in $\mathcal{A}$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[drr, "0"'] & \mathbf{Z} \arrow[r, "2"] \arrow[dr, "1"] & \mathbf{Z} \arrow[dr, "u"] & \\
 & & \mathbf{Z} \arrow[r, "m"] & M
\end{tikzcd}$$

Since m is split monomorphic, there exists $M \xrightarrow{n} \mathbf{Z}$ in $\mathcal{A}$ such that $mn = 1$. Thus $2 \cdot un = 1 \cdot mn = 1$. So 2 would be split monomorphic as well, which is not true.

We conclude that $X \notin \text{Ob}(\text{Ker}(\hat{\text{Y}}^\circ))$. $\square$

Lemma 4.5.3. Suppose given additive categories $\mathcal{C}$ and $\mathcal{D}$, an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ and $C \in \text{Ob}\,\mathcal{C}$. Suppose that C is not a zero object in $\mathcal{C}$ and that $C \in \text{Ob}(\text{Ker}(F))$. Then F is not faithful.

Proof. Since C is not a zero object in $\mathcal{C}$, we have $1_C \neq 0_C$. But $C \in \text{Ob}(\text{Ker}(F))$ implies that $F(1_C) = 1_{F(C)} = 0_{F(C)} = F(0_C)$. So F is not faithful. $\square$

Proposition 4.5.4.

(a) We have $\langle \mathcal{H} \rangle \subsetneq \mathrm{Ker}(\hat{\mathrm{Y}})$.

(b) The functor $\hat{\mathrm{Y}}\colon \mathrm{Adel}(\mathcal{A}) \to \mathrm{GQL}(\mathcal{A})$ is not faithful.

(c) The functor $\tilde{\mathrm{Y}}\colon \mathcal{U}(\mathcal{A}) \to \mathrm{GQL}(\mathcal{A})$ is not faithful. In particular, $\tilde{\mathrm{Y}}$ is not an equivalence and thus the construction of Gabriel-Quillen-Laumon is not universal.

Proof. We use lemma 4.5.2 repeatedly.

Ad (a). By lemma 4.2.7, we have $\langle \mathcal{H} \rangle \subseteq \mathrm{Ker}(\hat{\mathrm{Y}})$ and $\langle \mathcal{H} \rangle \subseteq \mathrm{Ker}(\hat{\mathrm{Y}}^\circ)$. Thus $X \in \mathrm{Ob}(\mathrm{Ker}(\hat{\mathrm{Y}}))$ but $X \notin \mathrm{Ob}(\langle \mathcal{H} \rangle)$. We conclude that indeed $\langle \mathcal{H} \rangle \subsetneq \mathrm{Ker}(\hat{\mathrm{Y}})$.

Ad (b). Note that X is not a zero object in $\mathrm{Adel}(\mathcal{A})$ since $X \notin \mathrm{Ob}(\mathrm{Ker}(\hat{\mathrm{Y}}^\circ))$. So $\hat{\mathrm{Y}}$ is not faithful by lemma 4.5.3.

Ad (c). Note that $\mathrm{L}(X)$ is not a zero object in $\mathcal{U}(\mathcal{A})$ since $X \notin \mathrm{Ob}(\langle \mathcal{H} \rangle)$, cf. remark 3.2.14. Moreover, we have $\mathrm{L}(X) \in \mathrm{Ob}(\mathrm{Ker}(\tilde{\mathrm{Y}}))$ since $\tilde{\mathrm{Y}} \circ \mathrm{L} = \hat{\mathrm{Y}}$. Thus $\tilde{\mathrm{Y}}$ is not faithful by lemma 4.5.3. □

Remark 4.5.5.
So for this exact category $(\mathcal{A}, \mathcal{E})$, we have $\mathrm{Ker}(\hat{\mathrm{Y}}) \neq \mathrm{Ker}(\hat{\mathrm{Y}}^\circ)$ and $\langle \mathcal{H} \rangle \subsetneq \mathrm{Ker}(\hat{\mathrm{Y}})$.

I do not know whether $\mathrm{Ker}(\hat{\mathrm{Y}}) \cap \mathrm{Ker}(\hat{\mathrm{Y}}^\circ) = \langle \mathcal{H} \rangle$ in this case and in the case of an arbitrary exact category. This question remains open, cf. section 4.6.

In general, we have $\langle \mathcal{H} \rangle \subseteq \mathrm{Ker}(\hat{\mathrm{Y}}) \cap \mathrm{Ker}(\hat{\mathrm{Y}}^\circ)$.

4.6 Open questions

We keep the notation from section 4.4.

- When does $\mathcal{U}(\mathcal{A})$ have enough projectives/injectives? If $\mathcal{A}$ is abelian and equipped with the natural exact structure, then $\mathfrak{I}$ is an equivalence. So in this case $\mathcal{U}(\mathcal{A})$ has enough projectives/injectives if and only if $\mathcal{A}$ does.

- Is $\mathfrak{I}$ a closed immersion, i.e. is the image of $\mathfrak{I}$ always closed under extensions?
  Cf. remark 4.4.7.

- Does $\mathrm{Ker}(\hat{\mathrm{Y}}) \cap \mathrm{Ker}(\hat{\mathrm{Y}}^\circ)$ equal $\langle \mathcal{H} \rangle$? Cf. remark 4.5.5.

- Are there objects in $\langle \mathcal{H} \rangle$ that are not isomorphic to a pure exact sequence?
  Cf. remark 4.2.8.

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